Asymptotics, Groups, and Measures

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1 Hook

The current problems I'm working on all sort of come out of some work that Nir Avni and Avraham Aizenbud did together over the past few years. Here's the question: given a group G, what can you say about the number of n-dimensional irreps as n grows? We know the answer for some classes of groups (If you're a compact simple Lie group, the number of irreps of dimension n or less grows as $n^{\alpha(L)+o(1)}$ where $\alpha(L) = \frac{\mathrm{rk} L}{|\Phi^+|}$ [in particular, the term on the left is at most 1 and tends to 0 as the dimension of L goes to infinity]) but not too much about the p-adic case: there's a uniform lower bound of $\alpha(G) > 1/15$ and previous work of Larsen and Lubotzky showed that $\alpha(G) \leq 3 \dim G + 1$.

The big theorem is essentially the following:

Theorem 1.1. *(A-A '14) Let G be a semisimple algebraic group defined over $\mathbb{Q}_p$ with Γ a compact open subgroup of $G(\mathbb{Q}_p)$. Then there exists a constant C independent of G such that the number of n-dimensional representations of Γ grows slower than n^C. C may be taken to be $3\dim(E_8)+1 = 745$. These bounds also hold over local fields of large enough characteristic.*

Additionally, the coarse moduli space of G-local systems on a smooth projective curve of genus at least $\lceil C/2 \rceil + 1 = 374$ has rational singularities.

Recent (unpublished) work has C down to 100, and it should probably drop some more (I would guess maybe somewhere in the 90s- this is actually a modified graph-coloring problem [don't ask me to describe the graphs!]). I can give you an exact table if you want:

G	$\mathcal{B}(G)$
A_n	22
BC_n	22
D_n	40
G_2	6
F_4	18
E_6	25
E_7	44
E_8	100

The bounds above are the best this method can do (except on E_7 and E_8).

2 Line

Here's where I tell you a little about how this is proven, connect it up to some concepts we've talked about in this class, and then later I'll tell you about what I actually do (because it's an extension of some of the techniques of the proof).

Definition Deformation Variety. Let Γ be a group and π the fundamental group of a closed orientable surface of genus $n \geq 2$. The deformation space of π inside Γ is the set of homs from π to Γ- we'll call it $\mathrm{Def}_{\Gamma,n}$. For groups with adjectives, this deformation space inherits those adjectives- if Γ is an algebraic/p-adic analytic/topological group, then $\mathrm{Def}_{\Gamma,n}$ is a variety/p-adic analytic variety/topological space.

Let G be an algebraic group defined over a local field F. The geometry of $\mathrm{Def}_{G,n}$ can be connected to the representation theory of compact open subgroups of $G(F)$ and these connections lead to results on the singularities of the deformation variety and the asymptotic representation theory

Basically what we're going to do is we're going to get a souped up version of something Frobenius did:

Theorem 2.1. *Let Γ be a finite group and let $n \geq 1$ an integer. For every $g \in \Gamma$ the number of solutions to $[x_1, y_1] \cdots [x_n, y_n] = g$ is equal to*

$$|\Gamma|^{2n-1} \sum_{\chi \in \mathrm{Irr}(\Gamma)} \frac{\chi(g)}{\chi(1)^{2n-1}}$$

To get from finite to pro-finite groups, we'll need some notation to make things easier on ourselves:

Definition Let Γ a group, $n \geq 1$ an integer. Let $\Phi_{\Gamma,n} : \Gamma^{2n} \to \Gamma$ be the map

$$\Phi(x_1, y_1, \cdots, x_n, y_n) = [x_1, y_1] \cdots [x_n, y_n]$$

Definition Let X, Y be measure spaces with m a measure on X and $f : X \to Y$ a measurable function. Then we call $f_* m$ the pushforward of m along f- it's a measure on Y defined by $f_* m(A) = m(f^{-1}(A))$ for all $A \subset Y$.

With these two definitions in mind, I can tell you the souped up version of our previous theorem of Frobenius:

Theorem 2.2. *(Souped up version of 2.1) Let Γ a finitely generated profinite group with $\Gamma(i)_{i \in \mathbb{N}}$ a decreasing chain of open normal subgroups with trivial intersection. Let $n \geq 2$ an integer. Denote the normalized Haar measure on Γ by λ_Γ and the normalized Haar measure on Γ^{2n} by $\lambda_{\Gamma^2 n}$. TFAE:*

1. *The measure $(\Phi_{\Gamma,n})_* \lambda_{\Gamma^{2n}}$ has continuous density with respect to λ_Γ.*

2. *The series* $\sum_{\chi\in\mathrm{Irr}\Gamma}\chi(1)^{2-2n}$ *converges.*

If either of these hold, the density of $(\Phi_{\Gamma,n})_*\lambda_{\Gamma^{2n}}$ *is given by*

$$g\mapsto\sum_{\chi\in\mathrm{Irr}(\Gamma)}\frac{\chi(g)}{\chi(1)^{2n-1}}$$

Condition 2 above leads you to the asymptotic representation theory results. Here's how: For a topological group Γ, denote the number of non-isomorphich continuous complex irreps with dimension at most n by $R_n(\Gamma)$. If Γ is fg profinite, a necessary an sufficient condition for $R_n(\Gamma)$ being finite for all n is that every finite-index subgroup of Γ has a finite abelianization (this condtion is called FAb). Some examples of these things are compact opens in semi-simple algebraic groups over local fields.

For polynomially bounded $R_n(\Gamma)$, ie there exists a constant C such that for all n we have $R_n(\Gamma)\le Cn^C$ we can introduce the representation zeta function:

$$\zeta_\Gamma(s)=\sum_{\chi\in\mathrm{Irr}\Gamma}\chi(1)^{-s}$$

which is defined for $s\in\mathbb{C}$ with $\mathrm{Re}(s)\ge C+1$.

By a theorem of Jaikin-Zapirain, we have that $\zeta_\Gamma(s)$ has meromorphic continuation to the whole complex plane, and there's a rational number $\alpha(\Gamma)$ such that $R_n(\Gamma)=n^{\alpha(\Gamma)+o(1)}$. This $\alpha(\Gamma)$ is the maximum of the real values of the poles of $\zeta_\Gamma(s)$, but we don't know it's exact value. We can get some bounds on it via studying the deformation variety, though.

Theorem 2.3. *(A-A '14) Let G be semi-simple algebraic group defined over a finitely generated field k of characteristic 0. TFAE:*

1. $(1,\cdots,1)$ *is a rational singularity of* $\mathrm{Def}_{G,n}=(\Phi_{G,n})^{-1}(1)$.
2. $\Phi_{G,n}$ *is flat with rational singularities.*
3. *For every non-archimedean local field F containing k and every compact open $\Gamma\subset G(F)$, we have $\alpha(\Gamma)<2n-2$.*
4. *For every finite extension k'/k, there is a local field F' containing k' and compact open $\Gamma\subset G(F)$ such that $\alpha(\Gamma)<2n-2$.*

Something else that you can use this representation zeta function to do is an analogue of a computation of Witten:

Theorem 2.4. *(A-A '14) Let G a semisimple algebraic group defined over a non-archimedean field F of characteristic 0 with $\Gamma\subset G(F)$ a compact open subgroup and Σ a compact orientable surface. The collection of Γ-local systems on Σ is in bijection with $\mathrm{Def}_{\Gamma,n}/\Gamma$, where Γ acts by conjugation.*

Restricting to the open homomorphisms, ie those $\rho\in\mathrm{Def}_{\Gamma,n}$ such that the closure of $\rho(\pi_1(\Sigma))$ is open in Γ, we obtain an open, dense, Γ-invariant subset of $\mathrm{Def}_{\Gamma,n}$ and the quotient $\mathrm{Def}^{open}_{\Gamma,n}/\Gamma$ is a p-adic analytic manifold with volume

$$|Z(\Gamma)|\cdot(\mu(\Gamma))^{2n-2}\cdot\zeta_\Gamma(2n-2)$$

Another result in the spirit of this class that you can give is the following, having to do with the coarse moduli space of G-bundles on projective curves of high enough genus:

Theorem 2.5. *Let G a semi-simple algebraic group over a field k of characteristic 0 and n an integer such that $n \geq \mathcal{B}(G)/2+1$, where $\mathcal{B}(G) = \max_{\mathfrak{g}|\mathrm{Lie}(G)\otimes\bar{k}}(\mathcal{B}(\mathfrak{g}))$. Then the map $\Phi_{G,n} : G^{2n} \to G$ is flat with rational singularities.*

In particular, the deformation variety, which is equal to $\Phi^{-1}(1)$, has rational singularities if $n \geq \mathcal{B}(G)/2+1$. By a theorem of Boutout, the categorical quotient $\mathrm{Def}_{G,n}/G$ has rational singularities. When $k = \mathbb{C}$, and we take a fixed smooth projective curve Σ of genus n, the Riemann-Hilbert correspondence tells us that analytically, this quotient of our deformation variety is isomorphic to the coarse moduli space of G-principal bundles with a connection on Σ. Since the notion of rational singularities only depends on the underlying analytic variety, I can the specialize to flat connections.

Something else you might care about is that this volume formula gives you some information about topological field theories- particularly, Dijkgraaf-Witten TQFTs (this is essentially the finite-group version of Chern-Simons theory). Let G be a group scheme over $\mathbb{Z}_p$ with a semisimple generic fiber. Consider the D-W TQFTs Z_r with gauge groups $G(\mathbb{Z}/p^r)$ and trivial Lagrangian (ie pick $[0] \in H^2(B(G(\mathbb{Z}/p^r)), \mathbb{R}/\mathbb{Z}))$. For any compact orientable surface Σ we have that $Z_r(\Sigma) = \zeta_{G(\mathbb{Z}/p^r)}(-\chi(\Sigma))$ where $\chi(\Sigma)$ is the Euler characteristic. If Σ has genus at least $\mathcal{B}(G \times_{\mathrm{Spec}\mathbb{Z}_p} \mathrm{Spec}\mathbb{Q}_p)$, then the limit $\lim_{r\to\infty} Z_r(\Sigma)$ exists.

3 Sinker

By now I've thrown up several TFAE statements and I haven't really provided any idea of how you prove any of these statements. The key piece of the puzzle comes in talking about pushforwards of smooth measures. Here's some relevant definitions:

Definition Let X be a smooth d-dimensional variety over a non-archimedean local field F. Let $O \subset F$ be the ring of integers.

1. A measure m on $X(F)$ is called smooth if every F-point has an analytic neighborhood U and analytic diffeomorphism $f : U \to O^d$ such that $f_* m$ is a Haar measure on O^d.

2. A measure on the F-points of X is called Schwartz if it's smooth and compactly supported.

3. We say that a measure on $X(F)$ has continuous density if there's a smooth measure m and a continuous function $f : X(F) \to \mathbb{C}$ such that $\mu = f \cdot m$.

Nir and Rami covered this question for p-adic maps and pushforwards of p-adic measures. The statement is as follows:

Theorem 3.1. *Let X and Y be smooth irreducible varieties over a local field F of characteristic 0, and let $\varphi : X \to Y$.*

1. *Assume that φ is flat with rational singularities. Then, for every Schwartz measure m on $X(F)$, the pushforwards $\varphi_* m$ has continuous density.*

2. *Conversely, assume that for every finite extension F'/F and every Schwartz measure m' on $X(F')$, the measure $\varphi_* m'$ has continuous density. Then φ is flat with rational singularities.*

Additionally, if φ is flat with rational singularities and ω_X is a nowhere-vanishing regular top differential form on X_F with ω_Y a regular and nowhere-vanishing regular top differential form on Y_F with $f : X(F) \to \mathbb{C}$ a Schwartz function, the density of $\varphi_*(f|\omega_X|)$ with respect to $|\omega_Y|$ is given by

$$\frac{\varphi_*(f|\omega_X|)}{|\omega_Y|}(y) = \int_{(\varphi^{-1}(y)\cap X)(F)} f \cdot \left| \frac{\omega_X}{\varphi^* \omega_Y} |_{(\varphi^{-1}(y)\cap X)(F)} \right|$$

(in particular, the integral on the right hand side converges.)

This theorem is obvious enough if φ is smooth- that's certainly a sufficient condition. Finding a necessary condition is difficult- Nir and Rami showed that rational singularities is also sufficient for the p-adic case, and a little stronger. You'd also like to know if the theorem would work over archimedean fields- $\mathbb{R}$ or $\mathbb{C}$. These are the questions I'm thinking about.

If I have time, here's my favorite/an instructional example: consider $f : \mathbb{A}^2 \to \mathbb{A}^1$ given by $(x, y) \mapsto x^2 + y^2$. I'll compute the density function for both $\mathbb{R}$ and $\mathbb{C}$. Actually, I won't quite compute the density function, we'll just investigate the places it could blow up and become non-continuous with respect to the standard measure on $\mathbb{A}^1$ given by $|dx|$.

To do this computation, it's enough to look at the singular points of x^2+y^2. In our case, this is just the origin. Over $\mathbb{R}$, the preimage of any point under the map $x^2 + y^2$ is bounded, so it doesn't really matter how I cut off the measure- I'll use a measure which is equal to the usual one inside the unit square and zero outside of a small open ball containing the unit cube. The density at 0 is given by $\lim_{\varepsilon\to 0} \frac{Vol(0\leq x^2+y^2\leq\varepsilon)}{Vol(-\varepsilon\leq x\leq\varepsilon)} = \lim \frac{\pi\varepsilon}{2\varepsilon} = \frac{\pi}{2}$.

Over $\mathbb{C}$, I can make the coordinate change $z = x + iy$ and $w = x - iy$. This makes my map $(w, z) \mapsto wz$. Again, I'll cut off my measure on $\mathbb{A}^2$ with coordinates w, z to be 0 outside of some small ball around $U = \{(a + ib, c + id)| - 1 \leq a, b, c, d \leq 1\}$ and the standard measure inside of that closed set. The density at 0 is given by $\lim_{\varepsilon\to 0} \frac{Vol(\{-\varepsilon\leq|wz|\leq\varepsilon\}\cap U)}{Vol(|x|<\varepsilon)} = \lim \frac{4\pi^2(4\varepsilon+4\log(\varepsilon))}{\pi\varepsilon} = \infty$ (I might have accidentally dropped a constant or something, but the point is that the log term doesn't go away and causes the whole thing to blow up).

On the one hand, if the archimedean version of Nir/Rami's theorem is true, it's clear that this doesn't have rational singularities, since over $\mathbb{C}$ the density of this measure blows up (there's other reasons to believe it doesn't have rational singularities- it's not normal, etc). But you'd love to know why it doesn't blow up over $\mathbb{R}$ and to be able to explain this reasonably- so you need something more. You'd also love to be able to give an analogue of the integral formula (perhaps with conditions on when it converges), too.

Here's a start for the x^2+y^2 case. To find the differential form $\frac{\omega_X}{\varphi^*\omega_Y}$, we need to find a differential form so that when we wedge with $d(x^2 + y^2) = 2xdx + 2ydy$, we get the standard top form on $\mathbb{A}^2$. Here's a candidate: $\frac{ydx-xdy}{2(x^2+y^2)}$. Now we have to figure out how to integrate this in a reasonable fashion over $x^2 + y^2 = 0$, though- there's several problems here. First of all, the denominator is identically zero on that variety, and second of all you're asking me to integrate a 1-form on a point with 2-d tangent space.

So what's the correct move? Well, in the p-adic rational singularities case, the correct move is to replace the singular locus with a strong resolution of singularities- since you have rational singularities, you know that your form extends to 0 along the exceptional divisor, and you can just do your integration. Here, you'd like to do the same thing- replace $\mathbb{A}^2$ with it's blowup, $Bl_0\mathbb{A}^2$, and then do the computation there.

Something's still not right, though- in this blowup, the strict transform of the variety $x^2+y^2 = 0$ has no real points. If you take the limit of the real points of $x^2 + y^2 = \varepsilon$ as $\varepsilon \to 0$, you see you end up on the $\mathbb{P}^1$- this is bad news, since for any smooth differential form, it's value on the exceptional divisor must be 0. Boo!

But wait- our differential form had log poles. Making the substitutions $x = x$, $y = xz$ (this corresponds to pulling back to the total space of the blowup minus the y-axis) and then sending $x \to 0$, we get the differential form $\frac{dz}{1+z^2}$. Integrating this from $-\infty$ to ∞ we get $\frac{\pi}{2}$, which is exactly what we expect.

On the other hand, why couldn't we have just chosen $\frac{dy}{2x}$? If you do the same procedure, you end up with $\frac{xdz+zdx}{2x} = dz + \frac{zdx}{x}$, which doesn't behave so well with setting $x = 0$ (you might argue that the fractional part cancels and so you only get dz, but this wouldn't really give the correct answer either- you'd get infinity).

The conjecture would be that if you had log-canonical singularities (a limit of rational singularities) and your differential form on the fibers had at most log-poles that weren't visible to your field of definition, the integral would converge. There's major gaps here, though (there's a reduction step in the proof that is vastly more complicated in archimedean cases than non-archimedean cases, for example, and you'd need some results about being able to choose differential forms with log poles in a certain way over log-canonical varities, etc.).

4 Citations

If you're interested in the precise formulation of most of the statements I've talked about above, the paper you want to read is **Representation Growth and Rational Singularities of the Moduli Space of Local Systems** by Avraham Aizenbud and Nir Avni, arXiv:1307.0371. I somehow screwed up BibTex and I need to submit these notes now, so my bad.