Friday, Feb 5, 2016

Quantum $SU(2)$ at finite level

We're now at the half-way point in the Seminar. Next week we'll begin an entirely new topic — geometric quantization — and I'll try to explain eventually how it's really the same. Those of you who haven't spoken and want to (e.g. if you are enrolled you probably should speak) should plan on one lecture each in the last week or two. Treat it like a conference on Chern-Simons Theory: we are currently in the "school" week, and then there are research talks that vaguely relate to the theme of the conference. So your job will be to spend 40 minutes explaining your thesis project, but preface it with 10 minutes on how it relates to the (many!) topics of the class.

In any case, my goal for today is to explain how my original talks are related to the talks you heard from Piotr and Sean. Just to recap, Piotr explained a construction

$$\left\{\begin{matrix}\text{modular tensor}\\ \text{categories}\end{matrix}\right\} \rightsquigarrow \left\{\begin{matrix}\text{invariants of}\\ \text{oriented 3-manifolds}\end{matrix}\right\}$$

due to (Reshetikhin and) Turaev. The invariants were just defined on connected 3-folds, but you can easily extend them to all 3-folds by declaring that disjoint union corresponds to multiplication — this makes sense because they were valued in $\mathbb{C}$.

I. Review of MTC + RT

Let me remind how this works. The first datum of a MTC is a semisimple abelian category / $\mathbb{C}$ with finitely many simples $\{I_0, \ldots, I_n\}$ and all hom-sets finite-dimensional, so a generic object is $\bigoplus_i n_i \cdot I_i$ with $n_i \in \mathbb{N}$, where $n \cdot I = \underbrace{I \oplus I \oplus \ldots \oplus I}_{n \text{ times}}$. The next datum was to make this a monoidal category with unit $I_0 = \mathbb{1}$, which has the property of being rigid, meaning every object has left and right dual objects. You should visualize this, so far, as the category $\mathrm{Rep}_{\mathbb{C}}(G)$ of a finite group G (although in a moment we'll diverge from that picture). Semisimplicity, etc., you know well. If you think about the unitary representation theory, then it also makes sense to ask $\mathcal{C}$ to have a "Hermitian" structure; some people care about that axiom, some don't.

The next datum was to make $\mathcal{C}$ ribbon, which means to choose a braiding  and twist  satisfying some natural axioms. Piotr explained that this data allows $\mathcal{C}$ to ~~interpret~~ interpret ribbon links in S^3 which are oriented and labeled by objects of $\mathcal{C}$, with the axiom that orientation reversal = switching the label for its dual object (= complex conjugation if we have a Hermitian structure).

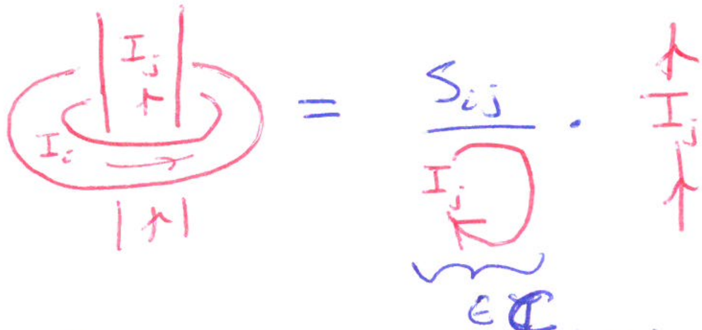
moreover, for closed links, ($\oplus$ of labels) $\mapsto$ (+ of interpretation). So actually we can label by $\mathbb{C}$-linear "direct sums" of objects of $\mathcal{C}$.

Finally, Piotr explained that $\mathcal{C}$ is <u>modular</u> if the $(k+1)\times(k+1)$ matrix

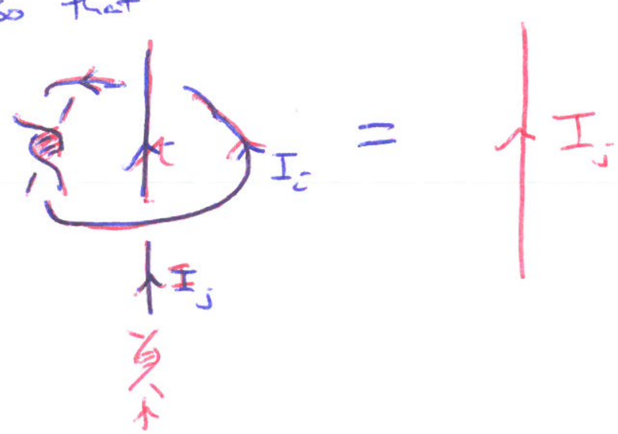
$S_{ij} =$ 

is invertible.

<u>Note</u>: Since I_j is simple, 



<u>Exercise</u>: Since S_{ij} is invertible, there are numbers $\alpha_i \in \mathbb{C}$ so that

$\sum_i \alpha_i$ 

(*)

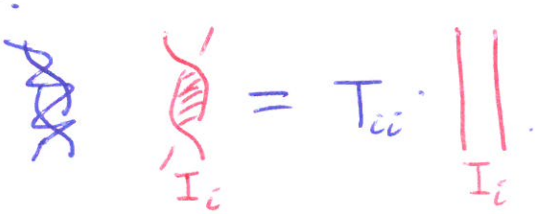
(should have been red)

(I might have switched which is  and which .)

Anyway, consider the "object" $J = \bigoplus_i \alpha_i \cdot I_i$, which isn't really an object (because α_i usually $\notin \mathbb{N}$) but which may label (closed) links. Given a framed link, label every component by J (which, note, is self-dual, so orientations don't matter), and then (*) says that the value is invariant under the Kirby moves — well, up to a framing correction, but really, see, S^3 has a standard framing, and surgery naturally produces framed 3-folds.... If $\mathcal{C}$ is Hermitian, then the invariant is.

(I'm not sure how accurate it is, but I like to think of J as the <u>regular representation</u> of the "finite quantum group" whose representation theory is $\mathcal{C}$. See, if G is a finite group, then $\mathbb{C}\cdot G$ as a G-module is $\mathbb{C}\cdot G = \bigoplus_i \dim(I_i^*) \cdot I_i$. We <u>don't</u> have $\alpha_i = \dim I_i$, but ....)

<u>Remark</u>: Set T_{ij} to be the diagonal $(k+1)\times(k+1)$ matrix s.t.



Then $\{S, T\}$ define a projective rep of $SL(2,\mathbb{Z})$.
This is why it's called <u>modular</u>.

II. Review of BHMV

Sean explained another step of the construction, due to Blanchet, Habegger, Masbaum, and Vogel:

$$\left\{ \begin{matrix} \text{invariants of} \\ \text{3-manifolds} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{~~Lax monoidal functors~~} \\ \text{“quantization functors”} \end{matrix} \right\}$$

$\{$invariants of 3-manifolds$\}$ = multiplicative functions "Bord_3" $\to \mathbb{C}$ ~~xxxxxxxx~~

$\{$"quantization functors"$\}$ = lax monoidal functors $\mathrm{Bord}_{23} \to \mathrm{Vect}_{\mathbb{C}}$

the "Universal Construction"

$$\longrightarrow \left\{ \begin{matrix} \text{“extended} \\ \text{QFs”} \end{matrix} \right\} \longrightarrow \dots$$

$\{$"extended QFs"$\}$ = lax monoidal lax bifunctors $\mathrm{Bord}_{123} \to \mathrm{Algoid}_{\mathbb{C}}$

For many reasons, we would like to end up with TQFTs, which are when all those "lax"es become strong.

("Lax" means "equipped with a map $F(X) \otimes_{F(N)} F(Y) \to F(X \cup_N Y)$". "Strong" means "this map is an iso".)

Sean moreover explained a strategy involving Morita equivalence for showing that RT invariants do give TQFTs.

Let me remind the very basic idea. By construction, ~~End Bord~~ $\mathrm{End}(\emptyset)$ in Bord_{23} is "Bord_3" = monoid of 3-folds. Let's linearize freely — I won't write a different word for it, so that ~~$\mathrm{End}_{\mathrm{Bord}}(\emptyset)$~~

$$\mathrm{End}_{\mathrm{Bord}_{23}}(\emptyset) = \begin{matrix} \mathbb{C}\text{-alg w/ basis} \\ \{\text{3-manifolds}\} \end{matrix}.$$

An invariant ~~is~~ of 3-folds is then an ~~too~~ alg homomorphism
$$\mathrm{End}_{\mathrm{Bord}_{23}}(\emptyset) \xrightarrow{\langle\rangle} \mathbb{C}.$$

You can "base change" along this homomorphism; I will abbreviate $\mathcal{B} := \mathrm{Bord}_{23}$: it ~~~~ is a category over $\mathcal{B}(\emptyset,\emptyset) := \mathrm{End}_{\mathcal{B}}(\emptyset)$; consider instead

$$\mathcal{B} \underset{\mathcal{B}(\emptyset,\emptyset)}{\otimes} \mathbb{C} = \mathcal{C}.$$

You get a ($\mathbb{C}$-linear) sym $\otimes$ cat w/ $\mathrm{End}(\mathbb{1}) = \mathbb{C}$.

Now, as you probably know, $\mathcal{B}$ is rigid in the sense that every object has a dual. ~~~~ It follows that so is $\mathcal{C} = \mathcal{B} \underset{\mathcal{B}(\emptyset,\emptyset)}{\otimes} \mathbb{C}$. Using this, given objects $X, Y \in \mathcal{C}$, you can define a "trace" pairing

$$\langle , \rangle : \mathcal{C}(X,Y) \otimes \mathcal{C}(Y,X) \longrightarrow \mathbb{C} = \mathrm{End}_{\mathcal{C}}(\mathbb{1}).$$

In general, this pairing is degenerate.

<u>Definition</u>: A morphism is <u>negligible</u> if it is in the kernel of this pairing.

<u>Exercise</u>: Prove that {negligible morphisms} is a "symmetric monoidal categorical ideal" in the sense that if f is negligible, so are $g \circ f$, $f \circ g$, $f \otimes g$, ...

So you can define a quotient $\mathcal{C}/\{\text{negligibles}\}$ which is again a sym $\otimes$ cat w/ ~~the same objects~~ ~~as~~ (strong) sym $\otimes$ functors

$$\mathcal{B} \longrightarrow \mathcal{B} \underset{\mathcal{B}(\emptyset)}{\otimes} \mathbb{C} = \mathcal{C} \longrightarrow \mathcal{B} \underset{\mathcal{B}(\emptyset)}{\otimes} \mathbb{C} / \{\text{negligibles}\}$$

Finally, the "quantization functor" is the composition of all this with

$$\hom_{\mathcal{C}/N}(\mathbb{1}^{\text{"}\emptyset\text{"}}, -).$$

<u>Exercise</u>:

$\hom(\mathbb{1}, -)$ is always lax monoidal.

So that explains the universal construction. (The "extended" version is a beat-for-beat repeat.)

<u>Example</u>: If you start with an MTC, then the TQFT you get sends

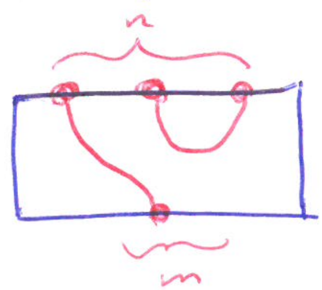
$$T^2 \longmapsto \mathbb{C}\text{-span of simple objects}.$$

and $S_{ij}, T_{ij} \in SL(2,\mathbb{Z}) = MCG(T^2)$ act via S_{ij}, T_{ij}. The action is projective because of the framing anomaly.

III. Actual start of this talk

Missing from this story is any construction of interesting MTCs. I mentioned a week or two ago an uninteresting — for our purposes — MTC called $Z(\mathrm{Rep}_{\mathbb{C}}(\mathbb{Z}/2))$, which is an interesting braiding on $\mathrm{Rep}_{\mathbb{C}}(\mathbb{Z}/2 \times \mathbb{Z}/2)$. It is very important physically, but boring for us because the invariants it determines just count sizes of certain $\mathbb{Z}/2$ cohomology groups.

To build interesting MTCs, I need to go back weeks to my earlier lectures and the Temperley-Lieb algebras. What I didn't say then was that these algebras naturally form an algebroid, i.e. a $\mathbb{C}$-linear category. Let me state that version. Pick $q \in \mathbb{C} \setminus \{0\}$. The algebroid TL has objects = linear sets ~~ordered finite sets~~ $= \mathbb{N}$. A morphism is a planar cobordism; or rather a linear combination thereof:

$$\mathrm{hom}(m,n) = \mathbb{C}\text{-span}\Big($$  $$\Big)$$

Composition is by stacking, where

$$\bigcirc = -[2] = -\frac{q^2 - q^{-2}}{q - q^{-1}} = -(q + q^{-1}).$$

This is where "q" appears.

<u>Remark</u>: By Morse theory, $TL(m,m)$ is generated by elements

 $= U_i$, subject to

$U_i^2 =$  $= -(q+q^{-1})$  $= -(q+q^{-1})U_i$

and $U_i U_{i\pm1} U_i =$  $=$  $= U_i$.

These were the relations from weeks ago.

We can make TL <u>hermitian</u> if $q \in U(1) \subseteq \mathbb{C}$ since then we can declare $U_i^* = U_i$.

You can make TL monoidal by <u>horizontal</u> stacking and ribbon by

 $= q^{1/2}$)($+ q^{-1/2}$ .

~~Now, rather than just looking at TL, let's look at its (f.d., say) representation theory.~~

~~The Quantum Schur-Weyl duality~~

Another version of this category is constructed as follows. There is a Hopf algebra called $U_q\,sl(2)$. Recall that $sl(2)$

is generated by E, F, and H with

$$[E,F] = H, \qquad [H,E] = 2E, \quad [H,F] = -2F.$$

(To get $su(2)$, you just need to ~~make~~ declare the right a Hermitian structure.) The interesting rep theory is the f.d. part, on which H acts with integer eigenvalues, so let's work with $K = \cancel{\exp(\hbar H/2)} \cdot q^{H/2}$

for some value q. Then the idea is to deform just the first relation to

$$\cancel{[E,F] = \frac{\sinh(\hbar H/2)}{\sinh(\hbar/2)} = \frac{q^{H/2} - q^{-H/2}}{q^{1/2} - q^{-1/2}}}$$

$$[E,F] = \frac{q^{H} - q^{-H}}{q - q^{-1}} = \frac{K^2 - K^{-2}}{q - q^{-1}} = \text{"quantum } H\text{"}.$$

Note that as $q \to 1$, this (formally) goes to H.

It's Hopf w/

$$\Delta(E) = E \otimes K + K^{-1} \otimes E, \quad \text{etc.} \ldots$$

The representation theory of this deformed Hopf algebra is essentially the same as that of $U_q\,sl(2)$.

For example, there's a 2-dim "Vector" rep, and in general the n-dim rep which is just like the usual one except you use *quantum* numbers:

$$K = \begin{pmatrix} q^{n/2} & & & \\ & q^{\frac{n}{2}-1} & & \\ & & \ddots & \\ & & & q^{-\frac{n}{2}} \end{pmatrix}, \qquad \cancel{E=}$$

$$E = \begin{pmatrix} 0 & [n-1] & & \\ & 0 & \ddots & \\ & & \ddots & [1] \\ & & & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & & & \\ [1] & 0 & & \\ & \ddots & \ddots & \\ & & [n-1] & 0 \end{pmatrix}$$

where $[n] = \dfrac{q^n - q^{-n}}{q - q^{-1}}$ = "quantum n".

~~This category is bra~~

The major difference is that this representation theory is braided, not symmetric. You've already seen the braiding in my earlier talks. Let $V = \mathbb{C}^2$ be the vector rep. Then

 $=$ six-vertex matrix $\begin{pmatrix} 1 & & & \\ & * & * & \\ & * & * & \\ & & & 1 \end{pmatrix}$

(I don't want to write it down precisely for fear of having formulae not match.)

As you know from $sl(2)$ there are maps $V \otimes V \to \mathbb{C}$ and $\mathbb{C} \to V \otimes V$ of $sl(2)$-reps. So the punchline is that TL is a picture of the part of $\mathrm{Rep}(U_q sl(2))$ ~~generated~~ whose objects are $V^{\otimes n}$s.

How do you get other modules? As you know, in usual $sl(2)$, you can find the n-dim module as a direct summand of $V^{\otimes(n-1)}$, and all f.d. modules are direct sums of these basic ones. So let's extend TL allowing direct sums, direct summands, ...

A slightly larger thing you could do is to work with all (f.d., say) TL-modules. The first choice ~~~~ (direct sums + summands) gives the subcat of (f.g.) projective modules. Just like in the case of algebras "all modules" is abelian, ~~by~~ whereas "projective modules" might not be!

Thm (Quantum Schur-Weyl duality): $U_q sl(2)$ and TL are "finitely Morita equivalent" in the sense that their f.d. rep thy are the same.

V Back to main story

What we learned from comparing w/ $sl(2)$ is that it's good to look for idempotents. This splits the rep thy into ~~indecomposable projectives~~ blocks by finding the indecomposable projectives for the algebra.

0. $\mathrm{End}(\emptyset) = \mathbb{C}$. No idempotents.
1. $\mathrm{End}(\cdot) = \mathbb{C}$. " "
2. $\mathrm{End}(\cdot\!-\!\cdot) = \mathbb{C}[U] / U^2 = -[2]U$.

So if $[2] = q + q^{-1} \neq 0$, then there is an idempotent:

$$\frac{-1}{[2]} U = \frac{-1}{[2]} \begin{smallmatrix}\cup\\ \cap\end{smallmatrix},$$

and another one:

$$1 - \frac{-1}{[2]} U = \| + \frac{1}{[2]} \begin{smallmatrix}\cup\\ \cap\end{smallmatrix} =: \boxed{JW_2}.$$

The letters "JW" stand for "Jones + Wenzl", who first found it.

~~The (Jones + Wenzl): Suppose that none of the quantum numbers $[1], \ldots, [n]$ vanish. Then $TL(n,n)$ is semisimple. There is a unique idempotent~~ ~~$\boxed{JW_n}$~~ n

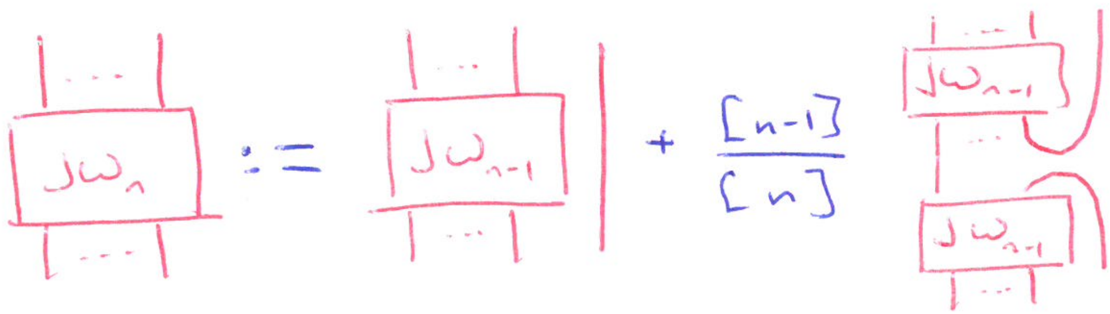
Note that there is an algebra homomorphism

$$TL(n,n) \longrightarrow \mathbb{C}, \qquad U_i \longmapsto 0.$$

Exercise: $TL(n,n)$ has at most one idempotent $JW_n \in TL(n,n)$ such that this homomorphism is $a \mapsto (JW_n) a (JW_n)$, i.e. projection onto this summand.

Theorem (Jones-Wenzl): If all the quantum numbers $[1], \dots, [n]$ are non-zero, i.e. if q is not a low root of unity, then JW_n exists.

Pf:



works. $\square$.

Corollary: If q is not a root of unity, TL is semisimple.

Pf: The JWs provide enough projections; then do a dimension count.

Ex: When $q=1$, JW_n = antisymmetrization, and in terms of $U\mathfrak{sl}(2)$ projects ~~onto~~ $V^{\otimes n} \twoheadrightarrow \Lambda^n V$.

Because TL is rigid, we can take traces, etc. Note that trace(idempotent) = "dimension" of the corresponding direct summand.

~~Fact~~ Exercise: $\mathrm{Tr}(JW_n) = [n+1]\cdot(-1)^n$.

Exercise: Let $V^{(n)} = |_n$ denote the direct summand corresponding to JW_n. Find an isomorphism

$$V^{(n)} \otimes V = V^{(n+1)} \oplus V^{(n-1)}$$

(assuming ~~q~~ $[1], \ldots, [n+1]$ are non-zero).

Now, recall that if $[2]=0$, then $TL(2,2) = \mathbb{C}[u]/u^2$ is not semisimple. In fact, if $[n]=0$, then $TL(n,n)$ is not semisimple for essentially the same reason.

Suppose that $[n]=0$ but $[1], \ldots, [n-1] \neq 0$. (How can $[n]=0$? $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$, so this happens when q is a $2n$th root of unity.) Then we have simple objects $V^{(0)} = \mathbb{1}$, $V^{(1)} = V$, ..., $V^{(n-1)}$, and plenty else, with dimensions

$$[1],\ [2],\ \ldots,\ [n]=0.$$

In particular, $id_{V^{(n-1)}}$ is negligible in the sense that $tr(f)=0$ for any $f \in hom(V^{(n-1)}, V^{(n-1)})$.

Just as before, we can form a new category by quotienting by the negligible morphisms.

Exercise: If $id_X \in N$ is an ideal in a category $\mathcal{C}$, then ~~$X \in \mathcal{C}$~~ $X \cong \emptyset$ in $\mathcal{C}/N$.

zero object

Thm: TL/negligible in this case is semisimple with simple objects

$$V^{(0)} = \mathbb{1},\ V^{(1)} = V,\ \ldots,\ V^{(n-2)}.$$

The fusion rules (i.e. multiplication up to iso) are

$$V^{(i)} \otimes V \cong \begin{cases} V^{(i+1)} \oplus V^{(i-1)}, & i = 1, \ldots, n-3 \\ V^{(1)}, & i = 0 \\ V^{(n-3)}, & i = n-2. \end{cases}$$

This is called, incidentally, the Verlinde ring.

Now, TL / negligible is still a ribbon category.

<u>Theorem</u>: If q is a primitive $(2n)$th root of 1, It is modular.

<u>Idea of proof</u>: 

is not too hard to compute

because the JW relations force lots of smoothings of the 2-i-j crossings to contribute $\emptyset$ to the sum.

<u>Schur-Weyl</u> <u>dual</u>: This is, of course, equivalent to taking $U_q sl(2)$ at $q^{2n}=1$ and quotienting out by something to make it semisimple. By what? By the relations that hold in $V^{(0)}, \ldots, V^{(n-2)}$, which are

$$E^n = 0, \quad F^n = 0, \quad K^{4n} = 1.$$

↖ "because" $K = q^{H/2}$, so $K^{4n} = (q^{2n})^H$.

<u>Definition</u>: The number $K = n-2$ of nontrivial irreps is called the <u>level</u>. The MTC constructed above is called

<u>quantum $SU(2)$ at level K</u>.

This is the "K" in Witten's paper.

Example: When $[3]=0$, we have $q^6=1$ and $[2]^2=1$. The S-matrix is

$$\begin{pmatrix} 1 & [2] \\ [2] & q^3 \end{pmatrix}$$

which has determinant q^3-1. This shows that for n odd, we really do want q to be a primitive $(2n)$th root. (When n even, we already know that $q^n \neq 1$, because by assumption $[\frac{n}{2}] \neq 0$.)

But! TL has a subcategory of "even weight" reps, i.e. (direct summands of) tensor powers of $V^{\otimes 2}$. This subcategory is defined in terms of q^2, not q, ~~Thus~~ and so can't care whether $q^n = \pm 1$.

Under Schur-Weyl duality, this subcategory corresponds to $SO(3)$.

Defn: If n is odd and q is a primitive nth root, this subcat is modular and defines quantum $SO(3)$ at level k.

This is why Sean talked about even and odd cases.