MATH 448: RESHETIKHIN–TURAEV INVARIANTS
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Today I follow mostly various Jones' notes[1].

1. Commuting transfer matrices

Let me say a few more words about vertex models. To repeat the set-up, the "sites" of the model are the edges of a lattice, and there is a "spin" at each site, ranging over a finite set $\{1, \ldots, N\}$. The "interactions" occur at the vertices, and the main thing are the *Boltzmann weights* which up to some choice of conventions are given by $N^2 \times N^2$ matrices

$$R_x(a, b|c, d),$$

where x is some parameter, often called "spectral." We're interested in multiplying these matrices (technically, contracting these tensors) in some complicated way.

Let's think again about a usual rectangular lattice, say $m \times n$, and let's impose periodic boundary conditions. Again let's try to make this into a one-dimensional problem, so I will treat each row as an individual atom. Suppose a row has n vertices on it. Then for any choice $\vec{\sigma}, \vec{\sigma}'$ of spins on the edges that are not participating in the necklace, the contribution to the partition function for the row is

$$T_x(\sigma, \sigma') = \sum_{\vec{a} \in \{\text{spins}\}^n} \prod_i R_x(a_i, \sigma_i | a_{i+1}, \sigma'_i) = \operatorname{tr}_{\mathbb{C}^N} \prod_i R_x(-, \sigma_i | -, \sigma'_i)$$

where for fixed b, d, you should think of $R_x(-, b|-, d)$ as an $N \times N$ matrix.

Now think of T_x as a Boltzmann weight for a one-dimensional lattice model with spins $\sigma \in N^n$. Again we want $\operatorname{tr} T_x^m$.

The problem is that as $n \to \infty$, the spectrum of T_x can get very complicated, so it's not just "figure out the largest eigenvalue."

Baxter's idea was the following. Maybe T_x and T_y commute for different values of the parameter x, y! Then we could simultaneously diagonalize them, and perhaps that would give more information. And if this happens, perhaps you'd get *infinitely many* commuting matrices, which should be plenty of information to determine the diagonalization uniquely. This is the origin of "integrability" — having lots of commuting matrices.

How to assure that T_x and T_y commute? Well, suppose that for some $z(x, y)$, the matrices R_x, R_y, and R_z satisfy

$$(R_x \otimes \mathbf{1})(\mathbf{1} \otimes R_z)(R_y \otimes \mathbf{1}) = (\mathbf{1} \otimes R_y)(R_z \otimes \mathbf{1})(\mathbf{1} \otimes R_x). \tag{YBE}$$

Suppose moreover that R_z is invertible. Then

diagrammatic argument

verifies that T_x and T_y commute.

The equation (YBE) is called the *Yang–Baxter equation.* It's actually many coupled equations — many more than unknowns. Nevertheless, solutions can be found, using the machinery of *quantum*

[1] Vaughan F. R. Jones. In and around the origin of quantum groups. http://arxiv.org/abs/math/0309199

groups. For example, fixing t,

$$R_{x,t} = \frac{1}{xt - x^{-1}t^{-1}} \begin{pmatrix} xt^{-1} - x^{-1}t & & & \\ & x^{-1}(t^{-1} - t) & x - x^{-1} & \\ & x - x^{-1} & x(q^{-1} - q) & \\ & & & xq^{-1} - x^{-1}t \end{pmatrix}$$

Exercise 1. (1) *Check YBE, where $z(x,y) = xy$.*
(2) *What is $R_{x,t=1}$?*
(3) *What is $R_{x=1,t}$?*
(4) *What is $(R_{x,t})^{-1}$?*

Note that this is just the 6-vertex model from last time with $v = x - x^{-1}$ and $t = -q$, up to a global normalization that doesn't matter; it's just that in the old variables, the function $z(x,y)$ is more complicated. In particular, it follows that the 6-vertex model from last time in fact satisfies YBE, which provides further control over its solution.

2. The Hecke algebra

Definition 2.1. Without the spectral parameters, the equation

$$(R \otimes \mathbf{1})(\mathbf{1} \otimes R)(R \otimes \mathbf{1}) = (\mathbf{1} \otimes R)(R \otimes \mathbf{1})(\mathbf{1} \otimes R),$$

where $R : \mathcal{V}^{\otimes 2} \to \mathcal{V}^{\otimes 2}$, is called the *braid relation.* $\diamond$

Exercise 2. *Compute* $\lim_{x\to 0} R_{x,t}$ *and* $\lim_{x\to\infty} R_{x,t}$ *and check that they satisfy the braid relation.*

Of course, solutions to the braid relation lead to representations of the *braid group* $\mathcal{B}_n$, which is generated by $T_1, \dots, T_{n-1}$ subject to

$$\text{(braid)} \qquad \begin{cases} T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \\ T_i T_j = T_j T_i, \quad |j - i| \geq 2 \end{cases}$$