MATH 448: RESHETIKHIN–TURAEV INVARIANTS
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My goal today is to describe the (Iwahori–)Hecke algebra in some detail, following papers of Jones[1]. We're working towards the HOMFLYPT polynomial.

1. The Hecke algebra

Last time we found ourselves interested in the *Braid group* $\mathcal{B}_n$, with presentation

$$\langle T_1, \dots, T_{n-1} | T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \rangle$$

The relation is called the *braid relation*. Among its quotients is the *symmetric group*, which which you force $T_i^2 = 1$.

Exercise 1. *All the generators T_i are in the same conjugacy class. Therefore the one-dimensional representations of $\mathcal{B}_n$ are parameterized by a single number $\lambda \in \mathbb{C}^\times$, with $T_i \mapsto \lambda$.*

More generally, in any *representation all the T_is have the same spectrum.*

Rather than looking just at two-dimensional representations, the next thing after one-dimensional representations are those in which the spectrum of the T_is is just two numbers $a, b \neq 0$ (not zero since the T_is are invertible).

Remark 1.1. This really is the next thing because, like any good physicist or functional analyst, I am assuming that "representation" includes the word "unitary." The unitary representation theory of *anything* is completely reducible, meaning that every subobject is a direct summand, because you can always take the orthogonal complement of any subrep. Anyway, if s_i acts with a single eigenvalue, it's acting as a scalar, because unitary things are always diagonalizable (because of complete reducibility), and so the representation is a direct sum of one-dimensional reps. $\diamond$

Remember that the *group algebra* of a (discrete) group G is the ring $\mathbb{C}G$ with basis G and multiplication extending that of G by linearity. I am going to completely confuse groups with their algebras. The reason is that they have the same representation theory — $\text{Rep}(G) = \text{Mod}(\mathbb{C}G)$ — and the way you study *anything* in mathematics is through its representations.

Then note that the braid relation is homogeneous, and so $T_i \mapsto \lambda T_i$ is an automorphism of $\mathcal{B}_n$ for any $\lambda \in \mathbb{C}^\times$. So up to rescaling, I can set one of the eigenvalues to whatever I want. I will set one of them to -1 and the other to $q \in \mathbb{C}^\times$. Then the equation imposing that a diagonalizable (unitarity!) matrix T has eigenvalues $-1, q$ is:

$$T_i^2 = (q-1)T_i + q \qquad \text{(Hecke)}$$

Definition 1.2. The *Hecke algebra* $\mathcal{H}_{n,q}$ is the quotient of (the group algebra of) the Braid group $\mathcal{B}_n$ by the relation (Hecke). $\diamond$

The reason for the normalization is so that $\mathcal{H}_{n,q=1}$ is the symmetric group, or rather its group algebra.

Proposition 1.3 (PBW-type theorem for Hecke algebras)**.** $\dim \mathcal{H}_{n,q} = n!$.

[1]Vaughan F. R. Jones. In and around the origin of quantum groups. http://arxiv.org/abs/math/0309199

— Hecke Algebra Representations of Braid Groups and Link Polynomials. Annals of Mathematics, Second Series, Vol. 126, No. 2 (Sep., 1987), pp. 335–388. http://www.jstor.org/stable/1971403

I will prove the claim except for perhaps finitely many values of q, but actually the claim holds for all q, as established in Wenzl's thesis[2].

Proof. Note first that $T_i^{-1} = q^{-1}(T_i - q + 1)$, so we can work just with polynomials in the T_is.

Think about the symmetric group. You can draw a permutation by connecting each element of $\{1, \dots, n\}$ with where it goes by a straight line. Choose a way to resolve triple (and higher) crossings: you find a *minimal length word* for each element of the symmetric group. Of course, the elements of the symmetric group are the basis for its group algebra.

How do you multiply (in the symmetric group) in this basis? You stack your pictures and try to straighten out the kinks. The point is that when you do this: (1) stacking the pictures adds the lengths of the presenting words; (2) you spend a lot of time applying the braid relations, which do not change length; (3) occasionally you apply a relation $s_i^2 = 1$, which lowers the length. In particular length never increases, and you can always get to a minimal-length word (satisfying whatever choice you made about triple- etc.- points).

So now take your words in the s_is as your basis for $\mathcal{H}_{n,q}$. You can multiply just as before, except now occasionally instead of using $s_i^2 = 1$ you use the Hecke relation — it still reduces the length.

So that shows that your words are a spanning set for $\mathcal{H}_{n,q}$. It could happen, I guess, that for some qs, they are linearly dependent. Certainly this cannot happen when $q \approx 1$ (since it doesn't happen when $q = 1$), and the point is that "$q \approx 1$" actually means a *Zariski* open set containing 1. Why Zariski open? Because it's an algebraic problem. Non-empty Zariski-open sets are cofinite. □

Remark 1.4. Similar arguments show that for $q \approx 1$, $\mathcal{H}_{n,q}$ is semisimple with representations indexed by the same Young diagrams as for the symmetric group. In fact, by Wenzl's thesis, everything works except when q is a root of unity (of order at most something depending on n). ◇

2. Connection to YBE

Now here's how to find solutions to (YBE):

Step 1: Use $T_i^2 = (q-1)T_i + q$ to find $R_i = \alpha T_i$ such that $R_i + R_i^{-1} = \kappa$ for some $\kappa \in \mathbb{C}$.
Step 2: Define $R_i(x) = \kappa^{-1}(xR_i + x^{-1}R_i^{-1})$.

Then if you have a representation of $\mathcal{H}_{n,q}$ on $\mathcal{V}^{\otimes n}$ where $T_i = \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes T \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}$ with the T acting in the $i, i+1$th spots, you get the desired R-matrices.

Connecting back, we have such a representation:

Exercise 2. *Recall the* Temperley–Lieb algebra $\mathcal{TL}_{n,\delta}$ *generated by* $U_1, \dots, U_{n-1}$ *with* $U_i^2 = \delta U_i$ *and* $U_i U_{i\pm 1} U_i = U_i$. *Find the (essential unique) homomorphism* $\mathcal{H}_{n,q} \to \mathcal{TL}_{n,\delta}$ *of the form* $T_i \mapsto \alpha U_i + \beta$, *where I mean you need to find* α, β, δ *as functions of* q.

Recall from last time that, with $\delta = -t - t^{-1}$, *we had a favorite representation of* $\mathcal{TL}_n$ *on* $(\mathbb{C}^2)^{\otimes n}$ *in which* U_i *acted by*

$$U_i \mapsto \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \begin{pmatrix} 0 & & & \\ & -t^{-1} & 1 & \\ & 1 & -t & \\ & & & 0 \end{pmatrix} \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}$$

Pull this representation back to $\mathcal{H}_{n,q}$ *and use it to find a solution to YBE. Check that the solution is the one* $R_{x,t}$ *from last time.*

[2] H. Wenzl, Representations of Hecke algebras and subfactors, Thesis, Univ. of Pennsylvania (1985).

3. A useful lemma

Set $\mathcal{H}_n = \mathcal{H}_{n,q}$. There are obvious inclusions $\mathcal{H}_n \hookrightarrow \mathcal{H}_{n+1}$ given by $T_i \mapsto T_i$. Set $\mathcal{H} = \bigcup_n \mathcal{H}_n$ along these inclusions. This is the algebra we're after. Why? Because last week we were interested in a certain linear functional on $\mathcal{TL}$, and this week we'll study the pullback of that functional to $\mathcal{H}$. A linear functional on $\mathcal{H}$ is just a system of functionals, all compatible, on the $\mathcal{H}_n$s.

To build such a system, we need to understand how $\mathcal{H}_n$ looks from the point of view of $\mathcal{H}_{n-1}$. Note that there is a linear map

$$\mathcal{H}_n \oplus \mathcal{H}_n \otimes_{\mathcal{H}_{n-1}} \mathcal{H}_n \to \mathcal{H}_{n+1}$$

given by

$$x \oplus y \otimes y' \mapsto x + yT_n y'.$$

This is well-defined because in the tensor product, you quotient by $yz \otimes y' = y \otimes zy'$ if $z \in \mathcal{H}_{n-1}$, and this doesn't bother the map because $[z, T_n] = 0$ if $z \in \mathcal{H}_{n-1}$.

Note that this map is obviously a map of $\mathcal{H}_n$-bimodules.

Lemma 3.1. *The map $\mathcal{H}_n \oplus \mathcal{H}_n \otimes_{\mathcal{H}_{n-1}} \mathcal{H}_n \to \mathcal{H}_{n+1}$ given by $x \oplus y \otimes y' \mapsto x + yT_n y'$ is an isomorphism of $\mathcal{H}_n$-bimodules.*

What I'm saying is that every $w \in \mathcal{H}_{n+1}$ has a presentation as $w = x + yT_n y'$ for $x, y \in \mathcal{H}_n$, unique up to $yzT_n y' = yT_n zy'$ for $z \in \mathcal{H}_{n-1}$.

Proof. First, count dimensions: the RHS is $(n+1)!$, whereas the LHS is $n! + n!^2/(n-1)! = n! + n \cdot n! = (n+1)n! = (n+1)!$.

Thus it suffices to prove surjectivity, i.e. check that any w can be written $x + yT_n y'$, without checking uniqueness — uniqueness follows from the dimension count.

A generic element of $\mathcal{H}_{n+1}$ is a (sum of) word(s) in the T_is. If there are no or one T_n, we're done. Suppose that you have more than one T_n. Then somewhere in the word you have $T_n w T_n$ for $w \in \mathcal{H}_n$. It suffices to rewrite this with only one T_n; then, repeating if necessary, we can keep reducing the number of T_ns until we're ≤ 1.

By induction, w (is a sum of words each of which) has at most one T_{n-1}. If $w \in \mathcal{H}_{n-1}$, i.e. if it does not have any T_{n-1}s, then $T_n w T_n = T_n^2 w = (q-1)T_n w + qw$ by the Hecke relation, and we've reduced the number of T_ns. If w has a T_{n-1} in it, then $w = w' T_{n-1} w''$ (or rather is a sum of such things) and $T_n w T_n = w' T_n T_{n-1} T_n w'' = w' T_{n-1} T_n T_{n-1} w''$ by the braid relation, and we've reduced the number of T_ns. $\square$