MATH 448: RESHETIKHIN–TURAEV INVARIANTS
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With some luck, today we will describe the Jones/Ocneanu[1] presentation of the HOMFLYPT polynomial.

1. Whence "Hecke"?

Last time I described what I called the *Hecke algebra*, which is the algebra $\mathcal{H} = \bigcup \mathcal{H}_n$ with generators $T_1, T_2, \dots$ and relations

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \tag{Braid}$$

$$T_i^2 = (q-1)T_i + q \tag{Hecke}$$

There was a question last time of why this is called "Hecke" and what was the relation with the "Hecke operatorss" on modular forms.

Hecke was interested in the following general question. Suppose G is a group and $H \subseteq G$ is a subgroup. I will assume them finite — you can do it with much more general groups, but you have to think a bit about functional analysis. We talked already about the *group algebra* $\mathbb{C}G$. It is isomorphic to the *convolution algebra* of G, which is the algebra structure on $\mathcal{O}(G) = \mathbb{C}^G$ with multiplication $(\phi \star \psi)(g) = \int_{g' \in G} \phi(gg')\psi((g')^{-1})$, where $\phi, \psi : G \to \mathbb{C}$. You can describe this as follows. There are three maps $G \times G \to G$, given by projection onto the two factors and multiplication. So you can "push forward" a function on $G \times G$ along the multiplication map, and that's convolution. (Technically you should think of this as a "pull push".)

What Hecke did was to study convolution not just on G but on spaces of *double cosets* $X = H\backslash G/H$. Then $\mathcal{O}(H\backslash G/H) \subseteq \mathcal{O}(G)$ as the subset of functions satisfying $\phi(g) = \phi(hgh')$ for $h, h' \in H$. Consider the space $X^{(2)} = H\backslash G \times_H G/H$, where $G \times_H G$ means the set of pairs (g, g') modulo $(gh, g') = (g, hg')$ for $h \in H$. This has three maps to $X = H\backslash G/H$, again given by the two projection maps and the multiplication.

$$X \times X \leftarrow X^{(2)} \to X$$

Pulling back along the projection maps $\mathcal{O}(X) \otimes \mathcal{O}(X) = \mathcal{O}(X \times X) \to \mathcal{O}(X^{(2)})$ and then pushing forward along the multiplication map $\mathcal{O}(X^{(2)}) \to \mathcal{O}(X)$ defines a "convolution" operation $m_*(p^* \otimes q^*)$ on $\mathcal{O}(X)$, which you can check is an associative algebra.

$\mathcal{H}_{G,H} = \mathcal{O}(H\backslash G/H)$ with this "convolution" multiplication is the *Hecke algebra* for $H \subseteq G$. Why would you want to do this? A similar "convolution" defines an action of $\mathcal{H}_{G,H}$ on $\mathcal{O}(G/H)$, where G/H is the usual coset space (e.g. $G/H = S^n$ if $G = \mathrm{SO}(n)$ and $H = \mathrm{SO}(n-1)$), which commutes with the natural G-action. In some (e.g. finite) situations, $\mathcal{H}_{G,H} = \mathrm{End}_G(\mathcal{O}(G/H))$.

Now what's the connection with our other Hecke algebras?

Exercise 1. *The "Hecke algebra" for elliptic curves (i.e. the algebra of Hecke operators) in characteristic p ends up being this construction with $G = \mathrm{GL}_2(\mathbb{Q}_p)$ and $H = \mathrm{GL}_2(\mathbb{Z}_p)$. It is abelian.*

The "Hecke algebra" we're interested in is in some sense even deeper:

[1]Vaughan F. R. Jones. Hecke Algebra Representations of Braid Groups and Link Polynomials. Annals of Mathematics, Second Series, Vol. 126, No. 2 (Sep., 1987), pp. 335–388. http://www.jstor.org/stable/1971403

Exercise 2. *Let $G = \mathrm{GL}_n(\mathbb{F}_q)$ and $H =$ the* Borel *subgroup of (all) upper-triangular matrices (arbitrary diagonal). Then the corresponding Hecke algebra is the algebra $\mathcal{H}_{n,q}$ presented above, where q now is the order of the finite field we're working over.*

If you know about classical and reductive groups over finite fields, you can of course play this game for other things replacing GL_n. And the limit $\mathcal{H} = \bigcup \mathcal{H}_n$ is like "GL_∞".

Exercise 3. *If G is not finite but Lie, you have to think more about the pushforward. Let's work with* constructible *functions as the meaning of "$\mathcal{O}$", and then pushforward is integration against the Euler characteristic. Set $G = \mathrm{GL}_n(\mathbb{C})$ and $H =$ the Borel of upper-triangular matrices. Then the Hecke algebra is the $q \mapsto 1$ limit, i.e. the symmetric group.*

In both exercises, an important part of the proof is that the set of double cosets $B\backslash G/B$ is parameterized by the Weyl group, for any reductive group G.

Incidentally, the "elliptic curve" case is roughly the "$\widehat{\mathrm{SL}_2}$" version of this, where $\widehat{\mathrm{SL}_2}$ is the first *affine* Kac–Moody group.

My impression of the history is that Jones was the one to first really notice that the braid group representations arising in (quantum or) statistical mechanics were the same as the ones that the Hecke algebraists had been studying in order to understand finite groups of Lie type.

2. Ocneanu's trace

Ok, I turn now to my main story. Remember that last time we proved:

Lemma 2.1. *The map $\mathcal{H}_n \oplus \mathcal{H}_n \otimes_{\mathcal{H}_{n-1}} \mathcal{H}_n \to \mathcal{H}_{n+1}$ given by $x \oplus y \otimes y' \mapsto x + yT_ny'$ is an isomorphism of $\mathcal{H}_n$-bimodules.* □

Theorem 2.2 (Ocneanu)**.** *For each $\zeta \in \mathbb{C}$, there is a unique linear functional* tr *on $\mathcal{H}$ satisfying*

(1) *Cyclicity:* $\mathrm{tr}(ab) = \mathrm{tr}(ba)$
(2) $\mathrm{tr}(1) = 1$
(3) $\mathrm{tr}(wT_n) = \zeta\,\mathrm{tr}(w)$ *for $w \in \mathcal{H}_n$.*

Proof. By Lemma 2.1, there is a unique not-necessarily-cyclic linear functional on $\mathcal{H}$ defined inductively by $\mathrm{tr}(1) = 1$ and $\mathrm{tr}(xT_ny) = \zeta\,\mathrm{tr}(xy)$ for $x, y \in \mathcal{H}_n$. Clearly if a tr as in the theorem exists, then it satisfies these formulas. So it suffices to show that this definition is in fact cyclic.

So I take $a, b \in \mathcal{H}_{n+1}$ and I want to show $\mathrm{tr}(ab) = \mathrm{tr}(ba)$. By Lemma 2.1, I can write a, b each as a sum of terms in $\mathcal{H}_n$ and terms of the form $a'T_na''$ for $a', a'' \in \mathcal{H}_n$. If both $a, b \in \mathcal{H}_n$ I'm already done. If one of them is, then I have $\mathrm{tr}(ab'T_nb'') \overset{?}{=} \mathrm{tr}(b'T_nb''a)$ for $a, b', b'' \in \mathcal{H}_n$, but this follows by the definition and the cyclicity in $\mathcal{H}_n$. So the only case to consider is

$$\mathrm{tr}(a'T_na''b'T_nb'') \overset{?}{=} \mathrm{tr}(b'T_nb''a'T_na'')$$

for $a', a'', b', b'' \in \mathcal{H}_n$.

Now,

$$\mathrm{tr}(a'T_na''b'T_nb'') = \mathrm{tr}(T_na''b'T_nb''a')$$

by the case already established, since $a' \mapsto a \in \mathcal{H}_n$ and $T_na''b'T_nb'' \mapsto b \in \mathcal{H}_{n+1}$, and also

$$\mathrm{tr}(a''b'T_nb''a'T_n) = \mathrm{tr}(b'T_nb''a'T_na'')$$

for the same reason. So the only case to handle is

$$\mathrm{tr}(T_nxT_ny) \overset{?}{=} \mathrm{tr}(xT_nyT_n)$$

for $x, y \in \mathcal{H}_n$.

Appeal to Lemma 2.1 yet again. Expand $x, y \in \mathcal{H}_n$ as a summand in $\mathcal{H}_{n-1}$ and summands of the form $x' T_{n-1} x''$. For the two summands in $\mathcal{H}_{n-1}$, the problem is trivial. We've reduced the problem to two cases:

$$\operatorname{tr}(T_n x' T_{n-1} x'' T_n y) \stackrel{?}{=} \operatorname{tr}(x' T_{n-1} x'' T_n y T_n), \quad x', x'', y \in \mathcal{H}_{n-1} \tag{$\dagger$}$$

$$\operatorname{tr}(T_n x' T_{n-1} x'' T_n y' T_{n-1} y'') \stackrel{?}{=} \operatorname{tr}(x' T_{n-1} x'' T_n y' T_{n-1} y'' T_n), \quad x', x'', y', x'' \in \mathcal{H}_{n-1} \tag{$\ddagger$}$$

In (†), commuting T_n past x', x'', y, the LHS simplifies to

$$\begin{aligned}\operatorname{tr}(T_n x' T_{n-1} x'' T_n y) = \operatorname{tr}(x' T_n T_{n-1} T_n x'' y) = \operatorname{tr}(x' T_{n-1} T_n T_{n-1} x'' y) = \zeta \operatorname{tr}(x' T_{n-1}^2 x'' y) \\ = \zeta(q-1) \operatorname{tr}(x' T_{n-1} x'' y) + \zeta q \operatorname{tr}(x' x'' y)\end{aligned}$$

whereas the RHS simplifies to

$$\begin{aligned}&\operatorname{tr}(x' T_{n-1} x'' T_n y T_n) = \operatorname{tr}(x' T_{n-1} x'' T_n^2 y) \\ &\quad = (q-1) \operatorname{tr}(x' T_{n-1} x'' T_n y) + q \operatorname{tr}(x' T_{n-1} x'' y) = \zeta(q-1) \operatorname{tr}(x' T_{n-1} x'' y) + \zeta q \operatorname{tr}(x' x'' y)\end{aligned}$$

where at the line breaks I applied the Hecke relation.

In (‡), similar application of the braid relation and the definition of tr gives

$$\begin{aligned}\operatorname{tr}(T_n x' T_{n-1} x'' T_n y' T_{n-1} y'') &= \zeta \operatorname{tr}(x' T_{n-1}^2 x'' y' T_{n-1} y''), \\ \operatorname{tr}(x' T_{n-1} x'' T_n y' T_{n-1} y'' T_n) &= \zeta \operatorname{tr}(x' T_{n-1} x'' y' T_{n-1}^2 y'')\end{aligned}$$

which agree by the Hecke relation and the definition of tr on terms of the form $x''' T_{n-1} y'''$, $x''', y''' \in \mathcal{H}_{n-1}$. □

3. Connection to knots

The goal is to get invariants of knots. Ocneanu's trace is a conjugation-invariant of braids. The relation between knots and braids is given by the following classical theorem of Alexander and Markov[2]:

Theorem 3.1 (Alexander)**.** *Every oriented link is the closure of a braid.*

(Markov) *Suppose that $b \in \mathcal{B}_n$ and $b' \in \mathcal{B}_{n'}$ are braids whose closures are equivalent as oriented links. Then b and b' are related by some sequence of "moves" of two types. Namely, for each $b \in \mathcal{B}_n$, the allowed moves are*

"Type 1" or "conjugation": *$b \leftrightsquigarrow gbg^{-1}$ for some $g \in \mathcal{B}_n$.*
"Type 2" or "stabilization": *$b \leftrightsquigarrow bt_n^{\pm 1} \in \mathcal{B}_{n+1}$.* □

I've called the generators of $\mathcal{B}_n$ "t_i" with lower case t, because I now do want to distinguish them from the generators of $\mathcal{H}_n$.

The proof of the "Alexander" part is a good exercise (Hint: Maypole dance). The "Markov" part is proved similarly to the Reidemeister theorem: a Markov-1 move is like a Reidemeister-2 or Reidemeister-3 move, and a Markov-2 move is like a Reidemeister-1 move. If you understand how to prove the Reidemeister theorem, then you should try Markov's theorem as an exercise.

It's a solved problem to decide (via algorithm) if two braids are related by conjugation. It's a hard open problem to decide if two braids are related by Markov moves, because Type 2 moves can make the calculation very large.

Suppose you want to construct an invariant of links. Theorem 3.1 implies it's enough to define a functional on the braid group(s) invariant under Type 1 and Type 2 moves. A *trace* on an algebra is a linear functional such that $\operatorname{tr}(ab) = \operatorname{tr}(ba)$, or equivalently (if the algebra is spanned by invertibles) a conjugation-invariant linear functional — so Type 1 invariance is just being a trace on the braid group.

[2]A. A. Markov, Uber die freie Aequivalenz geschlossener Zopfe, Mat. Sb. 1 (1935), 73–78

Now note the similarity between the Type 2 moves and item (3) from Theorem 2.2. To make Ocneanu's trace into a Type 2 invariant, we just need to use not the canonical map $\pi : t_i \mapsto T_i$ from the braid group to the Hecke algebra, but some rescaling thereof $\pi_\lambda : t_i \mapsto \lambda T_i$ such that $\operatorname{tr}(w\lambda T_i) = \operatorname{tr}(w(\lambda T_i)^{-1})$. Then the two Type 2 moves act the same, and our link invariant will take $b \in B_n$ to $(\zeta\lambda)^{-n+1} \operatorname{tr}(\pi_\lambda(b))$. (In functional analysis, π is the usual name for a representation. The "+1" in the exponent is so that $1 \in B_1$ maps via our invariant to 1.) Indeed, given $b \in B_n$,

$$(\zeta\lambda)^{-n+1} \operatorname{tr}(\pi_\lambda(b)) \stackrel{?}{=} (\zeta\lambda)^{-(n+1)+1} \operatorname{tr}(\pi_\lambda(bt_n)) = (\zeta\lambda)^{-(n+1)+1} \operatorname{tr}(\pi_\lambda(b)\lambda T_n) = (\zeta\lambda)^{-(n+1)+1+1} \operatorname{tr}(\pi_\lambda(b)) \quad \checkmark$$

What is this λ? $\operatorname{tr}(w\lambda T_i) = \lambda\zeta \operatorname{tr}(w)$, whereas

$$\operatorname{tr}(w\lambda^{-1}T_i^{-1}) = \lambda^{-1} \operatorname{tr}(w(q^{-1}T_i + q^{-1} - 1)) = \lambda^{-1}(q^{-1}\zeta + q^{-1} - 1) \operatorname{tr}(w)$$

and so

$$\lambda = \sqrt{q^{-1} + (q^{-1} - 1)\zeta^{-1}}$$

Given a link L, let $P(L) = (\zeta\lambda)^{-n+1} \operatorname{tr}(\pi_\lambda(b))$ denote its invariant defined as above, where b is any braid whose closure is L. $P(L)$ is called the *HOMFLYPT invariant* of L.