MATH 448: RESHETIKHIN–TURAEV INVARIANTS
FRIDAY, JANUARY 15, 2016

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Today we will describe the Jones/Ocneanu[1] presentation of the HOMFLYPT polynomial.

1. The HOMFLYPT polynomial

In late September and early October 1984, the Bulletin of the AMS received research announcements from Freyd–Yetter, Hoste, Lickorish–Millet, and Ocneanu. These were published all as the same article[2], because the work was done independently, each of the articles would have been published if it were the only one, and they didn't want to publish all four articles. All four works construct the same invariant of knots, all four motivated by the Jones and Alexander(–Conway) polynomials. The article gives in a few paragraphs four separate constructions — I will basically follow Ocneanu's, because it's the closest to Jones' construction (and to statistical mechanics). Also, Przytycki and Traczyk independently found the same invariant via yet another derivation; it wasn't published until 1988[3]. The invariant is often called "HOMFLY" or "HOMFLYPT" because these are the most pronounceable acronyms of the various names. I know one person who likes to add "Unknown" to the list — it really is the obvious generalization of Jones & Alexander, and so probably was discovered by other people as well — and acronymize as "LYMPH TOFU."

Here's how Ocneanu did the construction. Let $\mathcal{B}_n$ denote the braid group on n strands Recall from last time the following classical theorem of Alexander and Markov[4]:

Theorem 1.1 (Alexander)**.** *Every oriented link is the closure of a braid.*

(Markov) *Suppose that $b \in \mathcal{B}_n$ and $b' \in \mathcal{B}_{n'}$ are braids whose closures are equivalent as oriented links. Then b and b' are related by some sequence of "moves" of two types. Namely, for each $b \in \mathcal{B}_n$, the allowed moves are*

- **"Type 1" or "conjugation":** $b \leftrightsquigarrow gbg^{-1}$ *for some* $g \in \mathcal{B}_n$.
- **"Type 2" or "stabilization":** $b \leftrightsquigarrow bt_n^{\pm 1} \in \mathcal{B}_{n+1}$. □

Here t_n denotes the crossing in $\mathcal{B}_{n+1}$ between the nth and $(n+1)$th strands, and we have the standard embedding $\mathcal{B}_n \hookrightarrow \mathcal{B}_{n+1}$ on the first n strands. $\mathcal{B}_n$ is generated by crossings $t_1^{\pm 1}, \dots, t_{n-1}^{\pm 1}$ modulo the braid relation $t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$.

Recall also the *Hecke algebra* $\mathcal{H}_n$. It is generated by $T_1, \dots, T_{n-1}$ module the braid relation $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ and the Hecke relation $T_i^2 = (q-1)T_i + q$. Equivalently, $T_i^{-1} = q^{-1}T_i + (q^{-1} - 1)$. There are standard embeddings $\mathcal{H}_n \hookrightarrow \mathcal{H}_{n+1}$ on the obvious generators. Last time we stated the following theorem of Ocneanu:

Theorem 1.2 (Ocneanu)**.** *For each $\zeta \in \mathbb{C}$, there is a unique linear functional $\mathrm{tr}_\zeta : \mathcal{H}_n \to \mathbb{C}$ such that:*

- **"Normalization":** $\mathrm{tr}_\zeta(1) = 1$

[1] Vaughan F. R. Jones. Hecke Algebra Representations of Braid Groups and Link Polynomials. Annals of Mathematics, Second Series, Vol. 126, No. 2 (Sep., 1987), pp. 335–388. http://www.jstor.org/stable/1971403

[2] Freyd, P., Yetter, D., Hoste, J., Lickorish, W.B.R., Millett, K.. Ocneanu, A.: A new polynomial invariant of knots and links. Bull. AMS 12, 239 (1985). https://projecteuclid.org/download/pdf_1/euclid.bams/1183552531

[3] Przytycki, J.H., Traczyk, P.: Invariants of links ofconway type. Kobe J. Math.,4,115 (1988)

[4] A. A. Markov, Uber die freie Aequivalenz geschlossener Zopfe, Mat. Sb. 1 (1935), 73–78

"Cyclicity" or "conjugation": $\mathrm{tr}_\zeta(ab) = \mathrm{tr}_\zeta(ba)$
"Markov" of "stabilization": $\mathrm{tr}_\zeta(wT_n) = \zeta\,\mathrm{tr}_\zeta(w)$ *if* $w \in \mathcal{H}_n$.

Note then that $\mathrm{tr}_\zeta(wT_n^{-1}) = \mathrm{tr}_\zeta(w(q^{-1}T_i + (q^{-1}-1))) = (q^{-1}\zeta + (q^{-1}-1))\,\mathrm{tr}_\zeta(w)$.

The idea is to match these with the Markov theorem. Consider the homomorphism $\pi_\lambda : \mathcal{B}_n \to \mathcal{H}_n$ sending $t_i \mapsto \lambda T_i$. Then define, for $b \in \mathcal{B}_n$,

$$I_{q,\zeta}(b) = \mu^n\,\mathrm{tr}_\zeta(\pi_\lambda(b)).$$

Note that $I(bt_n) = \mu\lambda\zeta I(b)$ and $I(bt_n^{-1}) = \mu\lambda^{-1}(q^{-1}\zeta + (q^{-1}-1))I(b)$. Then the point is to tune μ and λ so that

$$\mu\lambda\zeta = 1 = \mu\lambda^{-1}(q^{-1}\zeta + (q^{-1}-1)).$$

If you can — and you obviously can, at least for generic q, ζ — then I will be invariant under both types of Markov moves (the first because tr is cyclic, the second by the tuning). Then I is an invariant of oriented links.

Definition 1.3. This I is the *unnormalized HOMFLYPT polynomial*. It's called "unnormalized" because $I(\text{unknot}) = \mu \neq 1$. The *normalized* HOMFLYPT polnyomial is $P = \mu^{-1}I$. ◇

Now, recall that in the Hecke algebra

$$q^{-1/2}T_i - q^{1/2}T_i^{-1} = q^{1/2} - q^{-1/2}$$

In terms of the representation $\pi_\lambda : \lambda^{-1}t_i \mapsto T_i$, we have

$$q^{-1/2}\lambda^{-1}t_i - q^{1/2}\lambda t_i^{-1} = q^{1/2} - q^{-1/2}$$

Let's set $\alpha = q^{-1/2}\lambda = (1 + (1-q)\zeta^{-1})^{-1/2}$, and set $z = q^{1/2} - q^{-1/2}$. It should be clear that the change of variables $(q, \zeta) \to (\alpha, z)$ is generically locally invertible.

Suppose now that $L = L_+$ is any oriented link, and take any positive crossing in it (a crossing in an oriented link is *positive* if it satisfies the right-hand rule). Let L_- be the link formed by reversing that crossing, and L_0 the link formed by smoothing the crossing (which has one more or one fewer component than has $L_\pm$.) If you understand the proof of the Alexander part of Theorem 1.1, you can find a presentation of $L = L_+$ as a braid such that the given crossing is a t_i for some i, by which I mean that L_- and L_0 are presented by replacing that t_i by t_i^{-1} or 1. Then we see that the invariants of L_+, L_-, and L_0 are related by the *skein relation*:

$$\alpha P(L_+) - \alpha^{-1}P(L_-) = zP(L_0)$$

Exercise 1.
(1) *Prove that $P(-)$ is determined by the skein relation, together with the values of $P(U^{\sqcup n})$, where $U^{\sqcup n}$ is an unlinked disjoint union of n unknots. Actually, it is enough to specify $P(U)$.*
(2) *Prove that $P(U^{\sqcup n}) = ((\alpha^{-1} - \alpha)z^{-1})^{n-1}$.*
(3) *Conclude that for every link L, $P(L)$ is a Laurent polynomial in α, z.*
(4) *Prove that $I(L \sqcup L') = I(L)I(L')$.*
(5) *Prove that $P(L\#L') = P(L)P(L')$, where $\#$ denotes any connect sum (i.e. connect sum any component of L to any component of L').*

Remark 1.4. The HOMFLYPT polynomial (for any normalization) is the universal polynomial satisfying a skein relation, in the sense that you could consider a homogeneous three-variable polynomial defined by

$$xP(L_+) + yP(L_-) = zP(L_0)$$

for arbitrary x, y, z, and we've chosen to work in coordinates on $\mathbb{P}^2$ in which $xy = 1$. ◇

2. Specializations

Definition 2.1. The *Alexander polynomial* is the knot invariant with skein relation $\Delta(L_+) - \Delta(L_-) = (q^{1/2} - q^{-1/2})V(L_0)$. So this corresponds to the specialization $\alpha = 1$, where we retain $z = q^{1/2} - q^{-1/2}$ as above. ◇

This is Conway's skein relation for the Alexander polynomial. Alexander defined his polynomial in terms of the homotopy theory of the knot complement.

Remark 2.2. If you specialize $\alpha = q^{-1/2}$, then the invariant $P \equiv 1$ satisfies the skein relation. ◇

Definition 2.3. The *Jones polynomial* is the knot invariant with skein relation $q^{-1}V(L_+) - qV(L_-) = (q^{1/2} - q^{-1/2})V(L_0)$. So this corresponds to the specialization $\alpha = q^{-1}$, where we retain $z = q^{1/2} - q^{-1/2}$ as above. ◇

Most people recognize that "V" stands for "Vaughan." Vaughan used the letter "V" in his first paper on the subject[5] without

Exercise 2. *For what values of α (or λ, or ζ, ...) does Ocneanu's trace on $\mathcal{H}$ descend to the Temperley–Lieb algebra $\mathcal{TL} = \bigcup \mathcal{TL}_n$?*[6]

Hint: *Prove that $\mathcal{TL}$ is the quotient of $\mathcal{H}$ by the relation*

$$T_iT_{i+1}T_i + T_iT_{i+1} + T_{i+1}T_i + T_i + T_{i+1} + 1 = 0.$$

Take the trace of both sides.

Remark 2.4. The representations of $\mathcal{H}$ are parameterized by Young diagrams, just like those of the symmetric group. The representations that satisfy the above relation, i.e. those that descend to $\mathcal{TL}$, are the ones with at most two rows. ◇

Exercise 3. *Recall that, back when we were studying the Potts model, we wanted the linear functional $\rho : \mathcal{TL} \to \mathbb{C}$ satisfying $PxP = \rho(x)P$, where P was the product of odd U_is, up to some normalization. Show that $\rho(x) \propto \operatorname{tr}(xP)$, and work out the normalizations.*

Theorem 2.5 (Jones)**.** *If L has an odd number of components, $V(L)$ is a Laurent polynomial in q. If L has an even number of components, $V(L)$ is $q^{1/2}$ times a Laurent polynomial in q.*

Exercise 4. *Calculate the Jones polynomial of the trefoil and of its mirror image. Conclude that the trefoil is not* amphichiral, *meaning not equivalent to its mirror.*

Jones used his polynomial to prove a number of results about braid representations of knots: obstructing the existence of a braid presentation in a small number of strands, obstructing amphichirality, ...

Definition 2.6. The specialization $\alpha = q^{-N/2}$ of the HOMFLYPT polynomial is the the *quantum* $\mathrm{SL}(N)$*-invariant in the vector representation.* ◇

3. Generalizations

The braid group is an *operad*, because you can *cable* strands by replacing each strand by a bundle of n strands, and putting some fixed braid along it. If you have any braid invariant, you can first cable the strands in your braid and then apply the invariant.

[5] Vaughan F. R. Jones. A polynomial invariant for knots via von Neumann algebras. Bull. AMS. Volume 12, Number 1, January 1985. https://projecteuclid.org/euclid.bams/1183552338.

[6] I think I have an arithmetic mistake somewhere in these notes, since the specialization $P \rightsquigarrow V$ would, if I did it right, correspond to $\zeta = \infty$. Jones' formulas, with $U_i^2 = \delta U_i$ and $\delta = q^{1/2} + q^{-1/2}$, are $T_i \mapsto q^{1/2}U_i - 1$ and the trace on $\mathcal{TL}$ satisfies $\operatorname{tr}(wU_i) = \delta^{-1}\operatorname{tr}(w)$, and I might be off by a sign somewhere.

Let's apply HOMFLYPT polynomial. Because of the skein relation, we can understand a lot about cabling just by understanding what happens when you cable the one-strand braid $1 \in \mathcal{B}_1$. Then the choice of cabling is clearly an element of $\mathcal{B}_n$, but actually maps to $\mathcal{H}_n$. And because we'll close up the braid, we only want conjugation classes in $\mathcal{H}_n$, and there are as many of these as there are Young diagrams with n boxes. This is just like in the symmetric group, and is dual to the statement that the irreps of $\mathcal{H}_n$ are indexed by Young diagrams. It turns out that there is a God-given assignment $\{\text{Young diagrams}\} \to \{\text{conjugacy classes in } \mathcal{H}_n\}$ that deforms the usual assignment from symmetric groups. Moreover, just like in the case of symmetric groups, you can pick out a projection $p_\Upsilon \in \mathcal{H}_n$, i.e. $p^2 = p$, for each Young diagram Υ, and $pq = 0$ if p and q correspond to different Young diagrams. This is the Peter–Weyl decomposition: $\mathcal{H}_n = \bigoplus_{\text{irreps }\Upsilon} I_\Upsilon \otimes I_\Upsilon^*$ as a $\mathcal{H}_n$-bimodule, and you can set p_Υ to be the identity element in $I_\Upsilon \otimes I_\Upsilon^*$. So these are a complete set of orthogonal minimal projections (if q is any projection and $p = p_\Upsilon$, then $qp = p$ or $qp = 0$), and $\sum_\Upsilon p_\Upsilon = 1$.

Anyway, the upshot of this is that for any n-strand cabling by some $b \in \mathcal{B}_n$ of any link L, the cabling by b is the same as the cabling by $\sum_\Upsilon b_\Upsilon p_\Upsilon$, where $b_\Upsilon \in \mathbb{C}$, and so it's enough to understand the cablings by the p_Υs.

Definition 3.1. The link invariant "cable your link by p_Υ and then apply the HOMFLYPT polynomial" is the *colored HOMFLYPT polynomial* P_Υ, where the Young diagram Υ is the *color*. ◇

Usually people use other names. For example, the Young diagram that's all one row is the *trivial* representation of the symmetric group $\mathbb{S}_n$, or the *nth symmetric power* representation of $\mathrm{GL}(N)$. The Young diagram with just one column is the *sign* or *antisymmetric power*. You one-box Young diagram is the *vector* representation of $\mathrm{GL}(N)$.

Exercise 5. *Find the complete set of minimal projections in $\mathcal{H}_2$. There should be two of them corresponding to the two Young diagrams with 2 boxes.*

How about $\mathcal{H}_3$?

Exercise 6 (Hard)**.** *Show that in the Jones polynomial, it's enough to consider Young diagrams with at most one row. (Or at most one column, depending on your conventions.) This corresponds to the representations of* $\mathrm{SL}(2)$*. The corresponding projections in $\mathcal{TL}$ are the* Jones–Wenzl idempotents*.*

Exercise 7 (Harder)**.** *Show that for the quantum* $\mathrm{SL}(N)$*-invariant, you only need Young diagrams with at most $N-1$ rows.*

These *colored* $\mathrm{SL}(N)$*-invariants* of links are given statistical mechanical interpretation in[7].

[7] V.F.R. Jones. On knot invariants related to some statistical mechanical models. Pacific Journal of Mathematics. Vol. 137, No. 2, 1989.