

3-manifold Invariants and TQFT's

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Topological quantum field theories (TQFT's) can be motivated by physicists' approach using the path integral (partition function) in quantum field theory and statistical mechanics. In this talk, we introduce the concept of a quantization functor and discuss when such an object defines a TQFT. Then, we describe a particular quantization functor defined using invariants of 3-manifolds in [3] and how, in certain cases, this functor defines a TQFT.

1 Quantization functors

Before jumping into the formalism of quantization functors, we provide some motivation for why one would consider these objects.

1.1 Motivation and physics

We will begin with the following definition/joke (see [5]):

Definition 1.1.1. Physics is a part of mathematics devoted to the calculation of integrals of the form $\int g(x)e^{f(x)}dx$.

In the cases we are interested in, these integrals will be associated to a closed system, which we take to be a closed (smooth, oriented) 3-manifold M . Usually the integrals are performed over some very large space of *fields* on M denoted by $\Phi(M)$, but we will not focus on difficulties of defining such an object. For now, we'll just assume that these integrals work approximately like integrals in a finite dimensional setting. We use the notation $Z(M)$ to denote the integral associated to M . In addition, we will sometimes want our integrals to depend on additional parameters, i.e.

$$Z(\vec{p}, M) = \int_{\Phi(M)} g(\vec{p}, x)e^{f(\vec{p}, x)} dx.$$

We call this integral the *partition function* associated to M and the parameters \vec{p} and think of it as a function of these parameters.

The difficulty is that such an object is usually difficult to compute if $\Phi(M)$ is complicated enough. In these notes, this happens when M is topologically interesting. We may hope to compute $Z(\vec{p}, M)$ in simple cases, such as when M is the three sphere. If M is topologically complicated, we can try to cut M into simpler pieces. This idea is most easily visualized in the lower dimensional case of closed surfaces, when it is possible to cut a surface Σ into a collection of disks and cylinders (with holes). Once we have done this, we can then try to compute the partition function of each piece and then combine each of these contributions to obtain $Z(\vec{p}, M)$.

However, this procedure as described above has a problem. The partition function was only defined for a closed system (3-manifold). Once we have cut M into pieces, each piece will have a boundary and integrating over these pieces requires choosing boundary conditions for our fields on these boundaries. More precisely, we first decompose M as

$$M = M_1 \cup M_2 \cup \dots \cup M_n$$

where each intersection $M_i \cap M_j$ is either empty or a disjoint union of closed surfaces. Then, for each M_i we set a boundary condition on ∂M_i by choosing a fixed field φ on ∂M_i and perform the integral

$$Z(\vec{p}, M_i, \varphi) = \int_{\substack{x \in \Phi(M_i) \\ \text{s.t. } x|_{\partial M_i} = \varphi}} g(\vec{p}, x) e^{f(\vec{p}, x)} dx.$$

We can then combine the integrals on each of the components to compute the partition function for M .

To get a feel for this procedure, let's examine the simplest non-trivial case when we cut M into two pieces M_1 and M_2 along a surface Σ . Then the partition function for M can be written as

$$Z(\vec{p}, M) = \sum_{\varphi \in \Phi(\Sigma)} Z(\vec{p}, M_1, \varphi) Z(\vec{p}, M_2, \varphi).$$

We can think of this formula as Fubini's theorem on the space of fields $\Phi(M)$. Ignoring any issues of convergence, this is just the formula for the dot-product of two vectors:

$$Z(\vec{p}, M) = \sum_{\varphi \in \Phi(\Sigma)} v(\vec{p})_{\varphi} w(\vec{p})_{\varphi} = \vec{v}(\vec{p}) \cdot \vec{w}(\vec{p}).$$

The next simplest case is when M is cut into three pieces M_1 , M_2 , and M_3 along two surfaces Σ and Σ' . In this case, $\partial M_2 = \Sigma \cup \Sigma'$ has two components and the partition function for M can be written as

$$Z(\vec{p}, M) = \sum_{\varphi \in \Phi(\Sigma)} \sum_{\psi \in \Phi(\Sigma')} Z(\vec{p}, M_1, \varphi) Z(\vec{p}, M_2, (\varphi, \psi)) Z(\vec{p}, M_3, \psi).$$

This is the formula for the evaluation of a bilinear form on two vectors:

$$Z(\vec{p}, M) = \sum_{\varphi \in \Phi(\Sigma)} \sum_{\psi \in \Phi(\Sigma')} v(\vec{p})_{\varphi} M(\vec{p})_{\varphi\psi} w(\vec{p})_{\psi} = \vec{v}(\vec{p})^{\top} M(\vec{p}) \vec{w}(\vec{p}).$$

By thinking this way, we have translated the computation of $Z(\vec{p}, M)$ into two problems. First, we compute partition functions of topologically simpler 3-manifolds with boundary conditions. Second, we combine these calculations using "linear algebra." Formalizing this procedure leads one to define quantization functors.

1.2 Quantization functors

We begin with the definition of a quantization functor. Let \mathbf{Bord}_{23} be the category whose objects are smooth, closed, oriented surfaces and whose morphisms are isotopy classes of smooth, oriented, compact cobordisms between surfaces. We should note that the constructions in [3] take place on a "decorated" bordism category, in which surfaces and their cobordisms come with

additional data and structure.¹ For now we will ignore these subtleties, as the general theory is roughly the same without them.

Definition 1.2.1. A quantization functor is a functor

$$Z : \mathbf{Bord}_{23} \longrightarrow \mathbf{Vect}_{\mathbb{C}}$$

satisfying the following properties.

- i) $Z(\emptyset) = \mathbb{C}$.
- ii) For any object Σ in \mathbf{Bord}_{23} there is a non-degenerate, Hermitian, sesquilinear form $\langle \cdot, \cdot \rangle_{\Sigma}$ on $Z(\Sigma)$ such that if $\partial M_1 = \partial M_2 = \Sigma$,

$$\langle Z(M_1), Z(M_2) \rangle_{\Sigma} = Z(M_1 \cup_{\Sigma} -M_2).$$

A couple remarks are necessary. First, we note that $\mathbf{Vect}_{\mathbb{C}}$ may be replaced by \mathbf{Mod}_R for any commutative ring R with unit and conjugation operation $\overline{(-)} : R \rightarrow R$. Second, viewing a 3-manifold M with boundary $\partial M = \Sigma$ as a cobordism from \emptyset to Σ , we can think of $Z(M)$ as an element of $Z(\Sigma)$ by setting

$$Z(M) := Z(M)(1) \in Z(\Sigma).$$

A quantization functor Z is *cobordism generated* if the elements $Z(M)$ with $\partial M = \Sigma$ generate $Z(\Sigma)$.

We now explain how the cutting and gluing procedure given in terms integrals and partition functions in the previous section allows us to construct a cobordism generated quantization functor from a certain type of 3-manifold invariants.

1.2.2 The universal construction

Let $Z(-)$ be an (complex-valued) invariant of closed, smooth, oriented 3-manifolds which, for any such M , only depends on the diffeomorphism class of M . We say Z is *multiplicative* if, for any M_1 and M_2 ,

$$Z(M_1 \amalg M_2) = Z(M_1)Z(M_2)$$

and $Z(\emptyset) = 1$. We say that Z is *involutive* if, for any M ,

$$Z(-M) = \overline{Z(M)}.$$

We should think of the invariant $Z(M)$ as the value of the partition function on M . Given a multiplicative and involutive invariant of 3-manifolds Z , we can construct a quantization functor $Z : \mathbf{Bord}_{23} \rightarrow \mathbf{Vect}_{\mathbb{C}}$ as follows. First, consider the functor F corepresented by the object \emptyset in \mathbf{Bord}_{23}

$$F(\Sigma) := \mathbf{Hom}_{\mathbf{Bord}_{23}}(\emptyset, \Sigma)$$

which assigns to each Σ the set of cobordisms between \emptyset and Σ . To a morphism $N : \Sigma \rightarrow \Sigma'$, F assigns the map which is defined on any element $M \in F(\Sigma)$ by

$$F(N)(M) = M \cup_{\Sigma} N \in \mathbf{Hom}_{\mathbf{Bord}_{23}}(\emptyset, \Sigma')$$

¹In particular, [3] deals with the bordism category whose objects are as above with the additional requirement of a choice of p_1 structure and the additional data of a banded link, and similarly for morphisms. For details, see section 3.

We can extend F to a functor $\tilde{F} : \mathbf{Bord}_{23} \rightarrow \mathbf{Vect}_{\mathbb{C}}$ by defining $\tilde{F}(\Sigma)$ to be the vector space generated by $F(\Sigma)$ and defining \tilde{F} in the natural way on morphisms. We will define the quantization functor Z as a “quotient” of \tilde{F} . For each object Σ in \mathbf{Bord}_{23} , we define $Z(\Sigma) = \tilde{F}(\Sigma)/V(\Sigma)$. Here $V(\Sigma)$ is linearly generated by the elements of the form $a\tilde{F}(M_1) - b\tilde{F}(M_2)$ where a and b are complex numbers and

$$aZ(M_1 \cup_{\Sigma} M') = bZ(M_2 \cup_{\Sigma} M')$$

for all M' with $\partial M' = -\Sigma$ (if M_1 and M_2 have $\partial M_1 = \partial M_2 = \Sigma$ we consider $\tilde{F}(M_1)$ and $\tilde{F}(M_2)$ as elements in $\tilde{F}(\Sigma)$). For a morphism $N : \Sigma \rightarrow \Sigma'$ in \mathbf{Bord}_{23} we define $Z(N)$ as the linear map induced by $\tilde{F}(N)$. To check that this map is well defined, suppose that Σ is an object in \mathbf{Bord}_{23} and that M_1 and M_2 are 3-manifolds bounding Σ which represent the same element in $Z(\Sigma)$. Then, for any morphism $N : \Sigma \rightarrow \Sigma'$ in \mathbf{Bord}_{23} and for any M' with $\partial M' = -\Sigma'$ we see that by assumption,

$$Z(M_1 \cup_{\Sigma} N \cup_{\Sigma'} M') = Z(M_2 \cup_{\Sigma} N \cup_{\Sigma'} M')$$

so $Z(M_1 \cup_{\Sigma} N) = Z(M_2 \cup_{\Sigma} N)$ in $Z(\Sigma')$.

Proposition 1.2.3. *The functor $Z : \mathbf{Bord}_{23} \rightarrow \mathbf{Vect}_{\mathbb{C}}$ defined above is a quantization functor.*

Proof. The fact that $Z(\emptyset) = \mathbb{C}$ follows from the multiplicativity of Z since for any closed 3-manifolds $M, N \in Z(\emptyset)$

$$Z(M \amalg N) = Z(M)Z(N).$$

In particular, $Z(M \amalg N)$ is linearly related to $Z(M' \amalg N)$ for any other choice of M' . To check that Z satisfies property (ii), for any object Σ in \mathbf{Bord}_{23} we define a pairing on $Z(\Sigma) \times Z(\Sigma)$ by

$$\langle Z(M), Z(M') \rangle_{\Sigma} = Z(M \cup_{\Sigma} -M').$$

This pairing is sesquilinear since

$$Z(- \cup_{\Sigma} -M') = Z(-M')(-)$$

is a linear map and

$$Z(-(- \cup_{\Sigma} -M)) = \overline{Z(- \cup_{\Sigma} -M)} = \overline{Z(-M)}(-)$$

is conjugate linear. Similarly, Hermiticity follows from the fact that the invariant Z is involutive. \square

It is also immediate from the construction that the quantization functor Z is cobordism generated. The property of being cobordism generated allows us to do two things.

Exercise 1.2.4. Show that if $Z : \mathbf{Bord}_{23} \rightarrow \mathbf{Vect}_{\mathbb{C}}$ is a cobordism generated quantization functor, then one can define natural linear maps

- a) $i : Z(-\Sigma) \rightarrow Z(\Sigma)^*$,
- b) $m : Z(\Sigma) \otimes Z(\Sigma') \rightarrow Z(\Sigma \amalg \Sigma')$

where $Z(\Sigma)^*$ is the dual vector space of $Z(\Sigma)$. Show that the map m is injective.

Note that it is not always the case that the maps i and m are isomorphisms.

1.3 Quantization functors and TQFT's

In this section we discuss the relationship between quantization functors and TQFT's. We begin with a definition.

Definition 1.3.1. Let $Z : \mathbf{Bord}_{23} \rightarrow \mathbf{Vect}_{\mathbb{C}}$ be a cobordism generated quantization functor. We say Z is *involutive* if the map $i : Z(-\Sigma) \rightarrow Z(\Sigma)^*$ is an isomorphism for all objects Σ and that Z is *multiplicative* if the map $m : Z(\Sigma) \otimes Z(\Sigma') \rightarrow Z(\Sigma \amalg \Sigma')$ is an isomorphism for all objects Σ and Σ' .

We can now define a topological quantum field theory.

Definition 1.3.2. A topological quantum field theory (TQFT) is a cobordism generated quantization functor which is both involutive and multiplicative.

A remark is necessary here. It is possible to treat \mathbf{Bord}_{23} as a symmetric monoidal category, with the monoidal operation given by disjoint union and the unit given by the empty surface. Then, we see that a TQFT is a symmetric monoidal functor $Z : \mathbf{Bord}_{23} \rightarrow \mathbf{Vect}_{\mathbb{C}}$ such that the pairing given by

$$\langle Z(M_1), Z(M_2) \rangle_{\Sigma} = Z(M_1 \cup_{\Sigma} -M_2)$$

is a Hermitian, unimodular, sesquilinear form.

In [3] certain invariants of 3-manifolds, which are denoted by θ_p , are used (via the universal construction) to define quantization functors. The main result of the paper states when these quantization functors are actually a TQFT's. To prove this result they use the concept of a 1-extended quantization functor. Before discussing the invariants θ_p and outlining the proof of this result, we give an overview of 1-extended quantization functors and the relevant bicategories.

2 1-extended quantization functors

The category \mathbf{Bord}_{23} has a natural extension to the bicategory \mathbf{Bord}_{123} which is defined as follows. Its objects are smooth, closed, oriented 1-manifolds and to each pair of objects Γ and Γ' we associate the bordism category whose objects are (isotopy classes of) smooth, oriented, compact cobordisms from Γ to Γ' and whose morphisms are (isotopy classes of) smooth, oriented, compact cobordisms between these.² In other words, \mathbf{Bord}_{123} has objects, which are closed 1-manifolds, morphisms, which are 2-manifolds with boundary, and morphisms of morphisms, which are 3-manifolds with corners. Furthermore, all the information of \mathbf{Bord}_{23} is contained in $\mathbf{Bord}_{123}(\emptyset, \emptyset)$.

Our goal for this section is to understand what a 1-extended quantization functor

$$Z^{\text{ext}} : \mathbf{Bord}_{123} \longrightarrow \mathcal{C}$$

should be, where \mathcal{C} is a bicategory and Z^{ext} is a (weak) 2-functor. To do this, we first must find the appropriate extension of the category $\mathbf{Vect}_{\mathbb{C}}$ to a bicategory.

²Of course, to really define \mathbf{Bord}_{123} we would need to say what these are! There is some ambiguity in how to define cobordisms between manifolds with boundary, but we won't get into the details at this point. More on this later.

2.1 Algebroids and bimodules

Recall the definition of an algebroid over \mathbb{C} .³

Definition 2.1.1. An algebroid over \mathbb{C} is a \mathbb{C} -linear category. That is, a (small) category Λ whose hom-sets $\text{Hom}_\Lambda(X, Y)$ are \mathbb{C} -vector spaces.

Algebroids should be thought of as the “categorification” of algebras, in particular an algebroid with one object is an algebra. We now review how some basic constructions on algebras are generalized to algebroids.

Definition 2.1.2. Let Λ be an algebroid over \mathbb{C} . A left (right) Λ -module is a \mathbb{C} -linear functor from Λ (Λ^{op}) to $\text{Vect}_{\mathbb{C}}$. A bimodule for $\Lambda \times \Lambda'$ is a \mathbb{C} -linear functor from the product category $\Lambda \times (\Lambda')^{\text{op}}$ to $\text{Vect}_{\mathbb{C}}$. A morphism of left/right/bi-modules is a \mathbb{C} -linear natural transformation between the two functors. We denote the category of left Λ -modules by ${}_\Lambda\text{Mod}$, and similarly for right/bi-modules.

Exercise 2.1.3. Show that if Λ (and Λ') is an algebra that these definitions reduce to the usual definitions for Λ -modules and Λ -linear homomorphisms.

If Λ is an algebroid over \mathbb{C} , and M and N are left and right Λ -modules respectively, then we can define the tensor product $M \otimes_\Lambda N$ to be the quotient of the \mathbb{C} -vector space

$$\bigoplus_X M(X) \otimes_{\mathbb{C}} N(X)$$

(where the direct sum runs over objects in Λ) by the submodule generated by the relations

$$M(\alpha)(u) \otimes v \sim u \otimes N(\alpha)(v)$$

where $u \in M(Y)$, $v \in N(X)$ and $\alpha \in \text{Hom}_\Lambda(X, Y)$ for objects X, Y in Λ . Note that we have “used up” all of the Λ -module structure when forming the tensor product, so the resulting object is just a \mathbb{C} -vector space. If M is a $\Lambda \times \Lambda'$ -bimodule and N is a $\Lambda' \times \Lambda''$ -bimodule, then we can endow the tensor product $M \otimes_\Lambda N$ with the structure of a $\Lambda \times \Lambda''$ -bimodule by setting

$$(M \otimes_{\Lambda'} N)(X, Y) := \bigoplus_Z M(X, Z) \otimes_{\mathbb{C}} N(Z, Y)$$

and

$$(M \otimes_{\Lambda'} N)(\alpha, \beta) = \bigoplus_Z M(\alpha, \text{id}_Z) \otimes_{\mathbb{C}} N(\text{id}_Z, \beta)$$

for $(\alpha, \beta) \in \text{Hom}_\Lambda(X, X') \times \text{Hom}_{\Lambda''}(Y', Y)$. For convenience, we will occasionally use the following notation (see [3])

$$\begin{aligned} {}_X M_Y &:= M(X, Y) \\ {}_Y \Lambda_X &:= \text{Hom}_\Lambda(X, Y) \\ \alpha u \beta &:= M(\alpha, \beta)(u) \end{aligned}$$

where M is a $\Lambda \times \Lambda'$ -bimodule, $u \in M(X, Z)$, $\alpha \in \text{Hom}_\Lambda(X, Y)$, and $\beta \in \text{Hom}_{\Lambda'}(W, Z)$. In this notation, $u \in {}_X M_Z$, $\alpha \in {}_Y \Lambda_X$, $\beta \in {}_Z \Lambda'_W$, and $\alpha u \beta \in {}_Y M_W$. A good trick to remember the funny order in ${}_Y \Lambda_X$ is that $\Lambda_X := \text{Hom}_\Lambda(X, -)$ is a left Λ -module, so we would write ${}_Y(\Lambda_X)$ for the vector space ${}_Y \Lambda_X$.

Exercise 2.1.4. Let M be a left Λ -module and let N be the left Λ -module $N := \Lambda \otimes_\Lambda M$. Show that M and N are isomorphic as Λ -modules, that is, there is a natural isomorphism between the functors M and N .

³All of the following can also be defined when \mathbb{C} is replaced by a commutative ring with unit.

2.1.5 Morita equivalence

Morita equivalence of algebroids is defined exactly as in the case of algebras.

Definition 2.1.6. Two algebroids over \mathbb{C} , Λ and Λ' are *Morita equivalent* if there is a functor $F : {}_{\Lambda}\text{Mod} \rightarrow {}_{\Lambda'}\text{Mod}$ which is part of a \mathbb{C} -linear equivalence of categories.

Exercise 2.1.7 (for those who haven't seen Morita equivalence before). Let A and B be commutative \mathbb{C} -algebras. Prove that A and B are Morita equivalent if and only if they are isomorphic as algebras.

We now prove a technical result which will be useful in section 3. Let Λ be an algebroid over \mathbb{C} and let $\{X_i\}_{i \in I}$ be a family of objects in Λ . For each $i \in I$, choose an idempotent element in the algebra ${}_{X_i}\Lambda_{X_i}$ and call it ϵ_i . We define the algebroid Δ to have objects I and morphisms given by

$${}_j\Delta_i = \epsilon_j {}_{X_j}\Lambda_{X_i} \epsilon_i := \{\epsilon_j \circ \alpha \circ \epsilon_i \in {}_{X_j}\Lambda_{X_i} \mid \alpha \in {}_{X_j}\Lambda_{X_i}\}.$$

We now define two bimodules. Let E be the $\Delta \times \Lambda$ -bimodule defined by

$${}_iE_X = \epsilon_i {}_{X_i}\Lambda_X := \{\epsilon_i \circ \alpha \in {}_{X_i}\Lambda_X \mid \alpha \in {}_{X_i}\Lambda_X\}.$$

Similarly, let \mathcal{E} be the $\Lambda \times \Delta$ -bimodule defined by

$${}_X\mathcal{E}_i = {}_X\Lambda_{X_i} \epsilon_i := \{\alpha \circ \epsilon_i \in {}_X\Lambda_{X_i} \mid \alpha \in {}_X\Lambda_{X_i}\}.$$

Proposition 2.1.8. *In the situation described above, suppose the idempotents $\{\epsilon_i\}_{i \in I}$ generate Λ as a two-sided ideal. Then $E \otimes_{\Lambda} \mathcal{E} \cong \Delta$ and $\mathcal{E} \otimes_{\Delta} E \cong \Lambda$ as bimodules.*

Proof. By the definition of Δ , the correspondence $\epsilon_j \alpha \otimes \beta \epsilon_i \mapsto \epsilon_j \alpha \beta \epsilon_i$ defines an isomorphism ${}_jE \otimes_{\Lambda} \mathcal{E}_i \cong {}_j\Delta_i$ for every $i, j \in I$. Similarly, we can define a linear map $\varphi : {}_Y\mathcal{E} \otimes_{\Delta} E_X \rightarrow {}_Y\Lambda_X$ by $\alpha \epsilon_i \otimes \epsilon_i \beta \mapsto \alpha \epsilon_i^2 \beta = \alpha \epsilon_i \beta$. Then, by assumption, for each object Y in Λ , there are finitely many $\beta_i \in {}_Y\Lambda_{Y_i}$ and $\alpha_i \in {}_Y\Lambda_{Y_i}$ such that

$$\text{id}_Y = \sum_i \alpha_i \epsilon_i \beta_i.$$

Then, we define a linear map $f : {}_Y\Lambda_X \rightarrow {}_Y\mathcal{E} \otimes_{\Delta} E_X$ by

$$f(\alpha) = \sum_i \alpha_i \epsilon_i \otimes \epsilon_i \beta_i \alpha.$$

We see that f provides a two-sided inverse to the map φ . It is obvious that that composition $\varphi \circ f$ is the identity, and with some algebraic manipulations, the other composition is

$$f(\varphi(\alpha \epsilon_j \otimes \epsilon_j \beta)) = \sum_i \alpha_i \epsilon_i \otimes \epsilon_i \beta_i \alpha \epsilon_j \beta = \sum_i \alpha_i \epsilon_i \beta_i \alpha \epsilon_j \otimes \epsilon_j \beta = \alpha \epsilon_j \otimes \epsilon_j \beta$$

where we have used the fact that

$$\alpha_i \epsilon_i \otimes \epsilon_i \beta_i \alpha \epsilon_j \beta = \alpha_i \epsilon_i \otimes \epsilon_i \beta_i \alpha \epsilon_j^2 \beta = \alpha_i \epsilon_i^2 \beta_i \alpha \epsilon_j \otimes \epsilon_j \beta = \alpha_i \epsilon_i \beta_i \alpha \epsilon_j \otimes \epsilon_j \beta.$$

□

2.2 The bicategory of algebroids

There is a nicer way to view this result using the language of bicategories. We start by constructing the bicategory $\mathbf{Algoid}_{\mathbb{C}}$. This will be the bicategory whose objects are algebroids over \mathbb{C} . To each pair of objects Λ and Λ' , we assign the category ${}_{\Lambda}\mathbf{Mod}_{\Lambda'}$ of $\Lambda \times \Lambda'$ -bimodules.⁴ Composition of morphisms is given by the tensor product of bimodules. The morphisms in the category of bimodules are sometimes called intertwiners (think of what it means to be a natural transformation between modules). In this setting, proposition 2.1.8 can be restated as the following.

Proposition 2.2.1. *Under the hypotheses of proposition 2.1.8, the algebroids Λ and Δ are isomorphic in $\mathbf{Algoid}_{\mathbb{C}}$ with isomorphisms given by $E : \Lambda \rightarrow \Delta$ and $\mathcal{E} : \Delta \rightarrow \Lambda$.*

This raises the question of how isomorphic objects in $\mathbf{Algoid}_{\mathbb{C}}$ are related. We may hope that when restricted to algebras it corresponds to the usual notion of isomorphism of algebras. However, this is usually not the case.

Proposition 2.2.2. *To algebroids over \mathbb{C} are isomorphic if and only if they are Morita equivalent.*

Proof. Let Λ and Λ' be isomorphic algebroids over \mathbb{C} . Λ can be given the structure of a right Λ -module, i.e. $\Lambda \in \mathbf{Mod}_{\Lambda}$, and similarly $\Lambda' \in \mathbf{Mod}_{\Lambda'}$. Suppose the isomorphism from Λ to Λ' is given by the $\Lambda' \times \Lambda$ -bimodule \mathcal{I} and the inverse morphism is given by the $\Lambda \times \Lambda'$ -bimodule \mathcal{J} . Then we see that the equivalence of categories between ${}_{\Lambda}\mathbf{Mod}$ and ${}_{\Lambda'}\mathbf{Mod}$ is given by

$$\begin{aligned} {}_{\Lambda}\mathbf{Mod} \ni M &\mapsto \mathcal{I} \otimes_{\Lambda} M \in {}_{\Lambda'}\mathbf{Mod} \\ {}_{\Lambda'}\mathbf{Mod} \ni N &\mapsto \mathcal{J} \otimes_{\Lambda'} N \in {}_{\Lambda}\mathbf{Mod} \end{aligned}$$

since $\mathcal{J} \otimes_{\Lambda'} \mathcal{I} \otimes_{\Lambda} M \cong \Lambda \otimes_{\Lambda} M \cong M$ and $\mathcal{I} \otimes_{\Lambda} \mathcal{J} \otimes_{\Lambda'} N \cong \Lambda' \otimes_{\Lambda'} N \cong N$.

Conversely, we see that if ${}_{\Lambda}\mathbf{Mod}$ and ${}_{\Lambda'}\mathbf{Mod}$ are equivalent then for any algebroid Λ'' the subcategories ${}_{\Lambda}\mathbf{Mod}_{\Lambda''}$ and ${}_{\Lambda'}\mathbf{Mod}_{\Lambda''}$ are also equivalent. The result now follows from the Yoneda lemma. \square

We now finally come back to our original goal of defining 1-extended quantization functors. First, observe that \mathbb{C} is an object in $\mathbf{Algoid}_{\mathbb{C}}$, and that $\mathbf{Hom}_{\mathbf{Algoid}_{\mathbb{C}}}(\mathbb{C}, \mathbb{C})$ is just the category of vector spaces, $\mathbf{Vect}_{\mathbb{C}}$. In other words, just as we saw that \mathbf{Bord}_{123} contained all the information of \mathbf{Bord}_{23} in a particular category of morphisms, $\mathbf{Algoid}_{\mathbb{C}}$ contains all the information of $\mathbf{Vect}_{\mathbb{C}}$. We will say that a 1-extended quantization functor is a (weak) 2-functor

$$Z^{\text{ext}} : \mathbf{Bord}_{123} \longrightarrow \mathbf{Algoid}_{\mathbb{C}}.$$

2.3 The universal construction for 1-extended quantization functors

Recalling that the quantization functors we will encounter in section 3 will be defined from 3-manifold invariants using the universal construction, it is natural to ask if the universal construction can be extended to construct a 1-extended quantization functor. We will briefly describe this procedure here.

⁴If you haven't seen this definition before, it may seem a little surprising. In particular, you may have guessed that morphisms between algebroids would be defined as \mathbb{C} -linear functors on the underlying categories, since these correspond to morphisms of algebras. In this case, the following example may be helpful. For commutative \mathbb{C} -algebras A and B , we can use an algebra map $\varphi : A \rightarrow B$ to consider B as a right A -module (this is just extension of scalars), so B is a $B \times A$ -bimodule. Conversely, given a $B \times A$ -bimodule structure on B , we can define a map of algebras $\varphi : A \rightarrow B$ to be the composition $A \rightarrow B \otimes_A A \cong B$ where the first map sends $a \mapsto 1 \otimes a$. Exercise 2.1.7 is a consequence of this.

Recall that if Z is an involutive, multiplicative invariant of closed 3-manifolds then we can construct a cobordism generated quantization functor

$$Z : \mathbf{Bord}_{23} \longrightarrow \mathbf{Vect}_{\mathbb{C}}$$

by defining $Z(\Sigma)$ to be the quotient of the vector space generated by 3-manifolds which bound Σ by certain relations (see section 1.2.2). Now, we would like to define a way of assigning smooth, closed, oriented 1-manifolds to algebroids over \mathbb{C} . To such a 1-manifold Γ we assign the following algebroid, denoted $Z^{\text{ext}}(\Gamma)$. Its objects are given by (isotopy classes of) smooth, compact, oriented surfaces Σ which bound Γ . The morphisms of $Z^{\text{ext}}(\Gamma)$ are defined to be the vector spaces

$${}_{\Sigma_2} Z^{\text{ext}}(\Gamma)_{\Sigma_1} := Z(\Sigma_1 \cup_{\Gamma} -\Sigma_2).$$

Composition of morphisms is given by the following gluing procedure. Let Γ be a smooth, closed, oriented 1-manifold and let Σ_1 , Σ_2 , and Σ be smooth, oriented surfaces such that $\partial\Sigma_1 = \partial\Sigma_2 = \partial\Sigma = \Gamma$. We will define a linear map

$${}_{\Sigma_2} Z^{\text{ext}}(\Gamma)_{\Sigma} \otimes_{\mathbb{C}} {}_{\Sigma} Z^{\text{ext}}(\Gamma)_{\Sigma_1} \rightarrow {}_{\Sigma_2} Z^{\text{ext}}(\Gamma)_{\Sigma_1}$$

by constructing a cobordism between $\Sigma_A := \Sigma_1 \cup_{\Gamma} (-\Sigma) \amalg \Sigma \cup_{\Gamma} (-\Sigma_2)$ and $\Sigma_B = \Sigma_1 \cup_{\Gamma} (-\Sigma_2)$. To construct this cobordism, first Consider the 3-manifold with boundary

$$\widetilde{M} = (\Sigma_1 \times [0, 1]) \cup_{\Gamma \times [1/2, 1]} (-\Sigma_2 \times [0, 1]).$$

We should think of \widetilde{M} as gluing $\Sigma_1 \times [0, 1]$ and $-\Sigma_2 \times [0, 1]$ along the left and right sides of $\Gamma \times Y$ where Y is the letter ‘‘Y’’ obtained by gluing two intervals together along a subinterval. The boundary of \widetilde{M} is given by the boundaries of these two pieces, plus a piece corresponding to the top of the ‘‘Y’’

$$\partial\widetilde{M} = (\Sigma_1 \cup_{\Gamma \times \{0\}} (\Gamma \times [0, 1/2]) \cup_{\Gamma \times \{1/2\}} (\Gamma \times [0, 1/2]) \cup_{\Gamma \times \{0\}} (-\Sigma_2)) \amalg (\Sigma_1 \cup_{\Gamma} (-\Sigma_2)).$$

We can then glue in the 3-manifold with boundary $\Sigma \times [0, 1]$ along the top part of the ‘‘Y’’ to obtain

$$M = \widetilde{M} \cup_{\varphi} \Sigma \times [0, 1]$$

where $\varphi : \Gamma \times [0, 1] \rightarrow (\Gamma \times [0, 1/2]) \cup_{\Gamma \times \{1/2\}} (\Gamma \times [0, 1/2])$ is the attaching map defined by

$$\varphi(x, y) = \begin{cases} (x, y) & : 0 \leq y \leq 1/2 \\ (x, 1 - y) & : 1/2 \leq y \leq 1. \end{cases}$$

We can then check that ∂M has two disjoint components given by $-\Sigma_A$ and Σ_B . This gives a linear map $Z(\Sigma_A) \rightarrow Z(\Sigma_B)$. To obtain our desired map, we precompose with the natural linear map $m : Z(\Sigma_1 \cup_{\Gamma} -\Sigma) \otimes_{\mathbb{C}} Z(\Sigma \cup_{\Gamma} -\Sigma_2) \rightarrow Z(\Sigma_B)$.

This defines the algebroids $Z^{\text{ext}}(\Gamma)$ for each object Γ in \mathbf{Bord}_{123} . Next, for each pair of objects Γ and Γ' and each smooth, compact, oriented surface Σ with $\partial\Sigma = -\Gamma \amalg \Gamma'$ we define the $Z^{\text{ext}}(\Gamma') \times Z^{\text{ext}}(\Gamma)$ -bimodule $Z^{\text{ext}}(\Sigma)$ by setting

$${}_{\Sigma_2} Z^{\text{ext}}(\Sigma)_{\Sigma_1} := Z(\Sigma_1 \cup_{\Gamma} \Sigma \cup_{\Gamma'} -\Sigma_2)$$

for any object Σ_1 in $Z^{\text{ext}}(\Gamma)$ and Σ_2 in $Z^{\text{ext}}(\Gamma')$.

Exercise 2.3.1. Define the bimodule structure on $Z^{\text{ext}}(\Sigma)$. That is, define linear maps

$$\Sigma'_2 Z^{\text{ext}}(\Gamma')_{\Sigma_2} \otimes_{\mathbb{C}} \Sigma_2 Z^{\text{ext}}(\Sigma)_{\Sigma_1} \rightarrow \Sigma'_2 Z^{\text{ext}}(\Sigma)_{\Sigma_1}$$

and

$$\Sigma_2 Z^{\text{ext}}(\Sigma)_{\Sigma_1} \otimes_{\mathbb{C}} \Sigma_1 Z^{\text{ext}}(\Gamma)_{\Sigma'_1} \rightarrow \Sigma_2 Z^{\text{ext}}(\Sigma)_{\Sigma'_1}$$

where $\partial\Sigma'_2 = \Gamma'$ and $\partial\Sigma'_1 = \Gamma$. You will need to use a gluing procedure similar to the one above.

We now must define how Z^{ext} behaves on intertwiners of bimodules. First, we define a cobordism of manifolds with boundary. Let Σ_1 and Σ_2 be smooth, compact, oriented surfaces with $\partial\Sigma_1 = \partial\Sigma_2 = -\Gamma_a \amalg \Gamma_b$. A cobordism between Σ_1 and Σ_2 is the a 3-manifold M such that

$$\partial M = (-\Sigma_1 \amalg \Sigma_2) \cup_{\varphi} ((\Gamma_a \times [0, 1]) \amalg (\Gamma_b \times [0, 1]))$$

where the attaching map φ glues one copy of the boundary Γ_a to Σ_1 and the other to Σ_2 , and similarly for Γ_b in such a way that all orientations agree. Now suppose we are in the situation described here. We must define \mathbb{C} -linear maps of vector spaces

$$\Sigma_b Z^{\text{ext}}(M)_{\Sigma_a} : \Sigma_b Z^{\text{ext}}(\Sigma_1)_{\Sigma_a} \rightarrow \Sigma_b Z^{\text{ext}}(\Sigma_2)_{\Sigma_a}.$$

for any objects Σ_a in $Z^{\text{ext}}(\Gamma_a)$ and Σ_b in $Z^{\text{ext}}(\Gamma_b)$. Recalling that we have defined

$$\Sigma_b Z^{\text{ext}}(\Sigma_1)_{\Sigma_a} = Z(\Sigma_a \cup_{\Gamma_a} \Sigma_1 \cup_{\Gamma_b} -\Sigma_b)$$

and

$$\Sigma_b Z^{\text{ext}}(\Sigma_2)_{\Sigma_a} = Z(\Sigma_a \cup_{\Gamma_a} \Sigma_2 \cup_{\Gamma_b} -\Sigma_b)$$

we can define $\Sigma_b Z^{\text{ext}}(M)_{\Sigma_a} := Z(\Sigma_b M_{\Sigma_a})$ where

$$\Sigma_b M_{\Sigma_a} := (\Sigma_a \times [0, 1]) \cup_{\Gamma_a \times [0, 1]} M \cup_{\Gamma_b \times [0, 1]} (-\Sigma_b \times [0, 1])$$

so that

$$\partial(\Sigma_b M_{\Sigma_a}) = -(\Sigma_a \cup_{\Gamma_a} \Sigma_1 \cup_{\Gamma_b} -\Sigma_b) \amalg (\Sigma_a \cup_{\Gamma_a} \Sigma_2 \cup_{\Gamma_b} -\Sigma_b).$$

This defines the functor Z^{ext} , though there are a number of compatibility conditions left to be checked which we will not discuss here. Note that the algebroid $Z^{\text{ext}}(\emptyset)$ is very far from being equal to \mathbb{C} . In particular, it has many objects (each corresponding to a closed surface). This will be a key feature of the extended theory, allowing us to prove the following important result. This will allow us to reduce the problem of showing that a quantization functor is multiplicative to proving that $Z^{\text{ext}}(\emptyset)$ and \mathbb{C} are Morita equivalent. This will be discussed further in section 3.

Proposition 2.3.2. *Let Σ_1 and Σ_2 be surfaces with boundary $\partial\Sigma_1 = -\Gamma_1 \amalg \Gamma$ and $\partial\Sigma_2 = -\Gamma \amalg \Gamma_2$. Then the natural map*

$$-Z^{\text{ext}}(\Sigma_2)_{-} \otimes_{Z^{\text{ext}}(\Gamma)} -Z^{\text{ext}}(\Sigma_1)_{-} \rightarrow -Z^{\text{ext}}(\Sigma_1 \cup_{\Gamma} \Sigma_2)_{-} \quad (*)$$

is an isomorphism of $Z^{\text{ext}}(\Gamma_2) \times Z^{\text{ext}}(\Gamma_1)$ -bimodules.

Partial proof. We will show that $(*)$ is an epimorphism. In particular, on each component, we will construct a linear map

$$s : \Sigma_c Z^{\text{ext}}(\Sigma_1 \cup_{\Gamma} \Sigma_2)_{\Sigma_a} \rightarrow \Sigma_c Z^{\text{ext}}(\Sigma_2)_{\Sigma_b} \otimes_{Z^{\text{ext}}(\Gamma)} \Sigma_b Z^{\text{ext}}(\Sigma_1)_{\Sigma_a}$$

which is a section for $(*)$. Define Σ to be to surface $\Sigma_a \cup_{\Gamma_1} \Sigma_1$ with boundary Γ . Consider the cobordism between Σ and itself formed by gluing the product $\Sigma \times [0, 1]$ along the boundary component $\Gamma \times [0, 1]$. This gives an element $\epsilon \in {}_{\Sigma}Z^{\text{ext}}(\Sigma_1)_{\Sigma_a}$. Now, Consider the linear map

$${}_{\Sigma_c}Z^{\text{ext}}(\Sigma_1 \cup_{\Gamma} \Sigma_2)_{\Sigma_a} \rightarrow {}_{\Sigma_c}Z^{\text{ext}}(\Sigma_2)_{\Sigma}$$

given by the identity cobordism. We can compose this map with the map $-\otimes_{Z^{\text{ext}}(\Gamma)} \epsilon$ to obtain a map the map s . Then, recalling the definition of $(*)$ we see that the composition of s with this map is the identity cobordism. \square

Exercise 2.3.3. Draw the construction of the cobordism s from the proof above and convince yourself that s is indeed a section.

3 TQFT's from the Kauffman bracket

We will now discuss a specific class of examples of 3-manifold invariants and outline the proof that, via the universal construction, we can use them to construct multiplicative quantization functors. It is proved in [3] that these quantization functor are also involutive, hence define TQFT's. We will mostly give a “highlight reel” presentation of these results, omitting some proofs and finer details of the constructions.

3.1 The invariants θ_p

Our first step in defining invariants of 3-manifolds follows [4]. Recall from Piotr's talks that any closed (smooth, oriented) 3-manifold can be obtained from surgery on a framed link⁵ in S^3 . Using this fact, we can proceed to define invariants of closed 3-manifolds by defining invariants of framed links in S^3 (which must be stable under certain moves between links which do not change the diffeomorphism type of the resulting 3-manifold). In the following, $R = \mathbb{Z}[A, A^{-1}]$ will be the ring of Laurent polynomials with integer coefficients.

Definition 3.1.1. Let M be a closed 3-manifold. The *Jones-Kauffman module* $K(M)$ is the R -module generated by isotopy classes of framed links in M , quotiented by the following relations:

$$\langle \text{X} \rangle = A \langle \text{Y} \rangle + A^{-1} \langle \text{Z} \rangle \langle \text{Z} \rangle,$$

$$L \amalg U = -(A^2 + A^{-2})L$$

where L is any link and U is the trivial unlink.

One may complain that these are relations for link diagrams in the plane, but since every framed link in S^3 may be represented unambiguously by such a link diagram, and since these relations are locally defined, we see that the definition makes sense.

Using these relations we can show that $K(M)$ is generated by unlinks which bound some interesting topology in M . For example, $K(S^3) \cong R$ where we can define this isomorphism by setting value of the empty link to 1. Similarly, we can see that $K(D^2 \times S^1) \cong R[z]$, where the generator z corresponds to the “standard link” $L = \{0\} \times S^1 \subset D^2 \times S^1$. We should note here that L is given the trivial framing (i.e. no twists).

⁵In [4] these are referred to as “banded links.”

Definition 3.1.2. Given a framed link $L \subset S^3$, we can choose an embedding of solid tori $D^2 \times S^1$ into S^3 such that this map sends each standard link to a component of L . We then define the *meta-bracket*

$$\langle -, \dots, - \rangle_L : R[z]^{\otimes n} \cong R[z_1, \dots, z_n] \rightarrow R$$

by setting $\langle b_1, \dots, b_n \rangle$ to be the element of R corresponding to the link L .

Note that this is a linear function on $R[z_1, \dots, z_n]$. We will now use the meta-bracket to construct an invariant of 3-manifolds. Let $t : R[z] \rightarrow R[z]$ be the map induced by one positive twist (using the right hand convention). Define the bilinear map

$$\langle -, - \rangle : R[z, w] \rightarrow R$$

to be the meta-bracket associated to the Hopf link (where each component has no twists) and the linear map

$$\langle - \rangle : R[z] \rightarrow R$$

to be the meta-bracket associated to the unknot with no twist. It is a nice exercise to verify, using our definitions, that $\langle b, 1 \rangle = \langle b \rangle$ for any $b \in R[z]$.

Proposition 3.1.3. *Let B be a commutative R -algebra. Suppose we are given an element $\Omega \in R[z] \otimes B \cong B[z]$ such that*

- i) $\langle t^{\pm 1}(\Omega), t^{\pm 1}(b) \rangle = \langle t^{\pm 1}(\Omega) \rangle \langle b \rangle$ for all $b \in R[z]$,*
- ii) $\langle t^{\pm 1}(\Omega) \rangle$ is invertible in B .*

Then, for all framed links $L \subset S^3$, the number

$$\theta_{\Omega}(L) = \frac{\langle \Omega, \dots, \Omega \rangle_L}{\langle t(\Omega) \rangle^{b_+(L)} \langle t^{-1}(\Omega) \rangle^{b_-(L)}} \in B,$$

where $b_{\pm}(L)$ are the numbers of positive and negative eigenvalues of the linking matrix⁶ of L , is an invariant of the manifold M_L obtained from L via surgery.

The proof follows from checking that $\theta_{\Omega}(L)$ does not change under Kirby moves. This proposition appears in [4], where they use this, along with the following theorem, to construct invariants of 3-manifolds.

Theorem 3.1.4. *Let $p \geq 3$ and $B_p = R[p^{-1}]/\phi_{2p}(A)$ where $\phi_d(x)$ is the d -th cyclotomic polynomial. Then there are elements $\Omega_p \in B_p$ which satisfy conditions (i) and (ii) of proposition 3.1.3, hence the numbers $\theta_p(L) = \theta_{\Omega_p}(L)$ are invariants of the manifold M_L .*

We would like to use the universal construction to obtain a quantization functor from these invariants. However, in order to do so we need to check a few things. First of all, we would need to define a conjugation map on the rings B_p . Furthermore, we would need to check that the invariants θ_p are involutive and multiplicative. This last requirement is easy to arrange, since the invariants were only defined on connected 3-manifolds (these are the only ones we can obtain via surgery on a framed link in S^3), so we can just define an invariant on all 3-manifolds $I_p(M)$ which is generated multiplicatively by $\theta_p(M_i)$ on each component M_i of M .

⁶The linking matrix lk_{ij} of L is defined to have (i, j) -th entry equal to the linking number of the i -th and j -th components of L . While lk_{ij} depends on an orientation of L , the numbers $b_{\pm}(L)$ do not.

To define the conjugation operation on B_p , we observe that A is a primitive $2p$ -th root of unity in this ring, so we may define the conjugation action on B_p by sending $A \mapsto A^{-1}$. Using the formula for θ_p , we can then check that these invariants are involutive since under an orientation reversal, the skein relation undergoes an interchange of A and A^{-1} .

The invariants of 3-manifolds defined in [3] are derived from the invariants θ_p , but are slightly more complicated. They introduce a B_p algebra \tilde{B}_p which contains an additional variable κ (it is connected to A via the relation $\kappa^6 = A^{-6-(p+1)p/2}$). They then define invariants of 3-manifolds $J_p(M)$ which additionally depend on a p_1 structure⁷ on M in terms of the invariants θ_p :

$$J_p(M) = f(A, A^{-1}, \kappa)I_p(M)$$

where $f(A, A^{-1}, \kappa)$ is polynomial in A , A^{-1} , and κ which depends on the choice of p_1 structure on M .

Finally, we note that this definition can be extended to an invariant of 3-manifolds with an embedded framed link, which we denote as a pair (M, K) . We first define

$$\theta_p(M_L, K) := \frac{\langle \Omega, \dots, \Omega, z, \dots, z \rangle_{L \cup K}}{\langle t(\Omega) \rangle^{b_+(L)} \langle t^{-1}(\Omega) \rangle^{b_-(L)}}.$$

Then, we can define $I_p(M, K)$ and $J_p(M, K)$ as above in terms of the invariants $\theta_p(M_L, K)$.

In summary, we have defined invariants of 3-manifolds with p_1 structure and an embedded link, which we will refer to as decorated 3-manifolds. Using an analogue of the universal construction, we can define the quantization functor Z_p and the 1-extended quantization functor Z_p^{ext} which are generated by the invariants J_p . Note that in this case, the category \mathbf{Bord}_{23} and the bicategory \mathbf{Bord}_{123} will be modified to include surfaces which contain a number of embedded intervals and cobordisms between such surfaces which contain an embedded link which must intersect the boundary surface in these intervals. The codomain of these functors has also changed, from vector spaces of algebroids over \mathbb{C} to modules or algebroids over \tilde{B}_p . We will now review the argument in [3] used to show that, for certain values of p , Z_p is a TQFT.

3.2 The Temperley-Lieb algebroid and idempotents

Recall from proposition 2.1.8 that if we can find a set of generating idempotents $\{\epsilon_i\}_{i \in I}$ in an algebroid Λ , then that algebroid is Morita equivalent to an algebroid Δ whose objects are elements of the index set I . Also recall from 2.3.2 that in order to show the quantization functor Z_p is multiplicative, it is enough to show that $Z_p^{\text{ext}}(\emptyset)$ is morita equivalent to the trivial algebroid over \tilde{B}_p . Our goal now is to find a generating set of idempotents for the algebroid $Z_p^{\text{ext}}(\emptyset)$. After doing so, the strategy in [3] is to show that this set has one element and the algebra Δ is just \tilde{B}_p .

The method for identifying these idempotent elements employed in [3] is to find idempotents in another algebroid and then showing how map these elements into $Z_p^{\text{ext}}(\emptyset)$.

Definition 3.2.1. The n -th Temperley-Lieb algebra TL_n over a commutative ring R is the

⁷We will not discuss this in detail here. p_1 structures can be thought of as homotopy classes of trivializations of the “double tangent bundle” $E = TM \oplus TM$. The quantum field theory defined by Witten in [6] constructs invariants of 3-manifolds which also depend on a choice of p_1 structure. For more details, see [6] and [2].

algebra generated by elements U_1, \dots, U_{n-1} subject to the relations

$$\begin{aligned} U_i U_{i+1} U_i &= U_i, \\ U_i U_{i-1} U_i &= U_i, \\ U_i U_j &= U_j U_i \text{ for } |i - j| > 1, \\ U_i^2 &= \delta U_i \text{ for some fixed } \delta \in R. \end{aligned}$$

We can think of elements of TL_n as sets of non-crossing lines in a square between sets of n points placed on the top and bottom sides. In this picture, the generators U_i are described by $n - 2$ vertical lines between points labeled by $1, \dots, i - 1, i + 2, \dots, n$ and a cap (cup) joining the i -th and $(i + 1)$ -st points on the bottom (top) side. The multiplication operation in this description is given by vertical composition. The relations $U_i U_{i+1} U_i = U_i$ and $U_i^2 = \delta U_i$ can be displayed as the following pictures (taken from [1], which contains other helpful pictures).

$$\begin{aligned} U_1 U_2 U_1 &= U_1 & U_1^2 &= \delta U_1 \end{aligned}$$

There is an operation tr on TL_n called *trace* which counts the number of “identity” portions of an element (in the pictorial version, it counts the number of vertical lines). Alternatively, for any element of TL_n , attach the top and bottom points in the corresponding picture with curves. This produces a link in the plane, and the trace counts the number of components of this link (see [1]).

There is another description of TL_n which connects it to the preceding discussion. Consider (n, n) tangles in $D^2 \times [0, 1]$. Such objects are given by braids between n points, and the set of braids is given by elements of the braid group \mathcal{B}_n . If we then consider the Jones-Kauffman skein module generated by such braids, we see that it is exactly the Temperley-Lieb algebra TL_n over the algebra R where we have chosen $\delta = -(A^2 + A^{-2})$. We have associated the element B_i in the braid group with the element $AU_i + A^{-1}$ in TL_n using the skein relations. We can use this description of TL_n to define its categorification.

Definition 3.2.2. The Temperley-Lieb algebraoid TL over an R -algebra B has objects $n \in \mathbb{N}$ and morphisms ${}_m TL_n$ given by $K(D^2 \times [0, 1], (a_m, a_n)) \otimes_R B$, the Jones-Kauffman skein module (with coefficients in B) on framed links in $D^2 \times S^1$ which intersect the “top” boundary in m intervals and intersect the “bottom” boundary in n intervals.

Let S_n^2 denote the 2-sphere with n embedded intervals. In the case when $B = \tilde{B}_p$, we can define a linear map

$${}_n TL_n \rightarrow S_n^2 Z_p^{\text{ext}}(\emptyset)_{S_n^2}.$$

To see how this is done, first observe that an element $u \in {}_n TL_n$ is given by the class of a link $K \subset D^2 \times [0, 1]$ which intersects each boundary $D^2 \times \{0\}$, $D^2 \times \{1\}$ in n intervals. We can embed such a link into $S^2 \times [0, 1]$ along the embedding of $D^2 \times [0, 1]$ in $S_n^2 \times [0, 1]$ (where we can perform an isotopy on the boundary spheres to ensure that the link K is mapped to a link which intersects each boundary S_n^2 in the given intervals).

We now quote a result from [3] about existence of idempotents in TL .

Proposition 3.2.3. *Let TL be the Temperley-Lieb algebroid over \tilde{B}_p . If $p \geq 4$ is even, then there exist idempotent elements $f_n \in {}_n TL_n$ for $0 \leq n \leq (p-2)/2$ such that, for each $x \in {}_n TL_n$,*

$$f_n x = x f_n = \text{tr}(x) f_n.$$

The trace identity given in this proposition is enough to uniquely specify these idempotent elements, and there is a recursive formula for the f_n due to Wenzl, which is described in [3].

3.3 Morita equivalence of $Z_p^{\text{ext}}(\emptyset)$ and \tilde{B}_p and other results

We now state a subset of the main results from [3] which show that $Z_p^{\text{ext}}(\emptyset)$ and \tilde{B}_p are Morita equivalent, hence the functor Z defines a TQFT. Denote the image of the idempotent $f_n \in {}_n TL_n$ in ${}_n S_n^2 Z_p^{\text{ext}}(\emptyset) {}_n S_n^2$ by f'_n .

Theorem 3.3.1. *If $p \geq 4$ is even, $Z_p^{\text{ext}}(\emptyset)$ is generated by the idempotent elements f'_n for $0 \leq n \leq (p-2)/2$.*

Proposition 3.3.2. *If p is even, then the elements $f'_n = 0$ for $n > 0$ and f'_0 is in the ideal generated by $1_\emptyset \in {}_\emptyset Z_p^{\text{ext}}(\emptyset) {}_\emptyset$.*

These two results are provided (along with more general statements) in Theorem 3.4 in [3].

Corollary 3.3.3. *If $p \geq 4$ is even, then $Z_p^{\text{ext}}(\emptyset)$ is generated by the single idempotent 1_\emptyset .*

Applying proposition 2.1.8 we have the following corollary.

Corollary 3.3.4. *If $p \geq 4$ is even, then $Z_p^{\text{ext}}(\emptyset)$ is Morita equivalent to the algebra $\Delta = {}_\emptyset Z_p^{\text{ext}}(\emptyset) {}_\emptyset$.*

Since ${}_\emptyset Z_p^{\text{ext}}(\emptyset) {}_\emptyset = Z_p(\emptyset) = \tilde{B}_p$, we see that, under the above hypotheses, $Z_p^{\text{ext}}(\emptyset)$ is Morita equivalent to \tilde{B}_p . The following is then a corollary of proposition 2.3.2.

Corollary 3.3.5. *If $p \geq 4$ is even, then the quantization functor Z_p is multiplicative.*

The following also appears in [3] as a consequence of Theorems 1.14 and 1.15.

Theorem 3.3.6. *For any p , the quantization functor Z_p is involutive.*

Finally, we may quote a portion of the Main Theorem 1.4 in [3].

Theorem 3.3.7. *If $p \geq 4$ is even, then the quantization functor Z_p is a TQFT.*

We should now investigate why (if?) this TQFT is equivalent to the one defined by witten in [6].

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