MATH 448: RESHETIKHIN–TURAEV INVARIANTS MONDAY, JANUARY 4, 2016

THEO JOHNSON-FREYD

Welcome to "Math 448: Topology and Geometry." This year the course will focus on Reshetikhin–Turaev Invariants and related quantum topology. We should touch on hyperbolic geometry, statistical mechanics, representation theory, functional analysis, category theory. . . .

But first, we have a few administrivia to dispense with:

(1) There is a course website at http://math.northwestern.edu/~theojf/RTinvariants/. I'll try to keep it updated as the course progresses. I will also put together an email list, so please add your name to the sheet of paper circulating. (If you are enrolled in the class, I already have your email address. Please enroll! It makes many things easier.)

(2) The bulk of the course will consist of you folks lecturing. I'll lecture the first four days, which leaves, if I counted correctly, 25 classes. So each of you will give about three lectures.

Also, for each lecture you give, please prepare typed lecture notes. They don't have to be flawless, but it's a good place to include full citations, etc. I'll post the notes on the website. At the end of the quarter, I'm hoping we'll edit together the different lecture notes into some sort of document expositing this part of quantum topology.

If you are "young," then I expect your lectures will cover some classic paper on the subject. If you are "old," then I expect you to explain some part of your thesis work, setting your story within the context of the class.

(3) We need to move meeting times, because already there are conflicts with this slot. Rather than figuring that out now, I've created a two Doodle Polls:

MWF options: The main one is at http://doodle.com/poll/3sf353vcxukqtq4y. There are three options for the poll: "Yes, this time is great." "(Yes), I can make that time, but it's not good." and "No, I cannot make that time."

T and Th options: Most of you have TA duties on Tuesdays and Thursdays, so grad classes aren't normally scheduled then. But it might be nice to move one of our sessions to a T or Th to free up Friday or Monday for travel. So I have a second poll at http://doodle.com/poll/75f9ixbnqu2knqkr to try to find a T or Th slot.

I will also email these addresses out to the email list.

Please complete the poll as early as possible, and no later than Tomorrow (Tuesday, January 5) at 3pm. That should give enough time to find a classroom and send out an announcement for Wednesday.

0. Preamble

Ok, now onto mathematics. This is a class, as I said, on Reshetikhin–Turaev Invariants[1] and more generally on the quantum topology of 3-manifolds. The actual "history" of this story begins with Witten's foundational paper[2] in which he outlines the whole subject. Most of the mathematical

[1] Reshetikhin, N.; Turaev, V. G. Invariants of 3-manifolds via link polynomials and quantum groups. Invent. Math. 103 (1991), no. 3, 547–597. http://link.springer.com/article/10.1007%2FBF01239527

[2] Witten, Edward. Quantum field theory and the Jones polynomial. Comm. Math. Phys. 121 (1989), no. 3, 351–399. http://projecteuclid.org/euclid.cmp/1104178138

details of Witten's paper were filled in by Reshetikhin–Turaev[3], Axelrod–Della Pietra–Witten[4], Jeffrey–Weitsman[5], Axelrod–Singer[6], and Blanchet–Habegger–Masbaum–Vogel[7]. Given this early history, it's fairly standard for discussions of the subject to begin with Witten's paper.

Exercise 1 (Homework for next two weeks)**.** *Read Witten's article:*

> *Witten, Edward. Quantum field theory and the Jones polynomial. Comm. Math. Phys. 121 (1989), no. 3, 351–399. http://projecteuclid.org/euclid.cmp/1104178138.*

But that really isn't where the story begins: there is a rich "prehistory" that I think sometimes gets lost, which is focused on statistical mechanics. (Witten does a good job of citing a lot of this prehistory.) That's where I'd like to begin. My story today will follow Jones' notes[8] and Baxter's book[9].

1. Generalities about statistical mechanics

A *statistical mechanics model* consists of a regular lattice (the size of which is supposed to be a variable in the model, and we normally want to take the size to ∞ and study the leading behavior). On certain "sites" in the model (say, the vertices, or the edges, or…) we attach "spins" ranging over some finite set. A *state* then is a function $\{\text{sites}\} \to \{\text{spins}\}$, so there are $\#\{\text{spins}\}^{\#\{\text{sites}\}}$ number of states. To each state we associate an *energy* (possibly $+\infty$, which means that that state is disallowed). Usually the energy is given as a sum of "local energies": in addition to the sites, let's say that the *interaction sites* are some other part of the lattice (say, the edges if the sites are vertices) and then

$$\text{energy} = \sum_{\text{interaction sites}} \text{local energy}$$

where "local energy" at a given interaction site is some function of the value of the state just on the sites "neighboring" the interaction site.

[3]Reshetikhin, N.; Turaev, V. G. Ibid.

—. Ribbon graphs and their invariants derived from quantum groups. Comm. Math. Phys. 127 (1990), no. 1, 1–26. http://projecteuclid.org/euclid.cmp/1104180037

[4]Axelrod, Scott; Della Pietra, Steve; Witten, Edward. Geometric quantization of Chern-Simons gauge theory. J. Differential Geom. 33 (1991), no. 3, 787–902. http://projecteuclid.org/euclid.jdg/1214446565

[5]Jeffrey, Lisa C.; Weitsman, Jonathan. Bohr-Sommerfeld orbits in the moduli space of flat connections and the Verlinde dimension formula. Comm. Math. Phys. 150 (1992), no. 3, 593–630. http://projecteuclid.org/euclid.cmp/1104251961.

—. Half density quantization of the moduli space of flat connections and Witten's semiclassical manifold invariants. Topology 32 (1993), no. 3, 509–529. http://www.sciencedirect.com/science/article/pii/004093839390003E.

—. Geometric quantization and Witten's semiclassical manifold invariants. Low-dimensional topology and quantum field theory (Cambridge, 1992), 317–322, NATO Adv. Sci. Inst. Ser. B Phys., 315, Plenum, New York, 1993.

[6]Axelrod, Scott; Singer, I. M. Chern-Simons perturbation theory. Proceedings of the XXth International Conference on Differential Geometric Methods in Theoretical Physics, Vol. 1, 2 (New York, 1991), 3–45, World Sci. Publ., River Edge, NJ, 1992.

—. Chern-Simons perturbation theory. II. J. Differential Geom. 39 (1994), no. 1, 173–213. https://projecteuclid.org/euclid.jdg/1214454681.

[7]Blanchet, C.; Habegger, N.; Masbaum, G.; Vogel, P. Topological quantum field theories derived from the Kauffman bracket. Topology 34 (1995), no. 4, 883-927. http://www.sciencedirect.com/science/article/pii/0040938394000514

[8]Vaughan F. R. Jones. In and around the origin of quantum groups. http://arxiv.org/abs/math/0309199

[9] Baxter, Rodney J. Exactly solved models in statistical mechanics. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London, 1982. xii+486 pp. ISBN: 0-12-083180-5. https://physics.anu.edu.au/theophys/_files/Exactly.pdf

What you actually want to compute is the *partition function*, defined as

$$Z(T) := \sum_{\text{states}} \exp\left(-\frac{1}{T}\text{energy(state)}\right) = \sum_{\text{states}} \prod_{\text{interaction sites}} w$$

where T is a parameter of the theory called the *temperature* (and I am using units in which the Boltzmann constant k_B equals 1), and the values of $w = \exp(-(\text{local energy})/T) \in [0, \infty)$ are the *Boltzmann weights*. Sometimes there are other parameters in the theory, in which case Z is a function of those parameters as well. Note that Z depends on the size of the lattice, and also on the choice of boundary conditions. As the lattice gets big, probably Z is dominated by contributions from the far interior of the lattice, and so hopefully the boundary conditions don't matter much. The *free energy per site* is

$$\lim_{\text{size of lattice}\to\infty} \frac{1}{\#\{\text{sites}\}} \log Z.$$

To *solve* a model basically means to give an explicit formula for the free energy as a function of the temperature (and other parameters).

Oh, one other comment is that of course you can shift all energies by some fixed amount without changing anything of physical importance.

2. Potts model

Our story really begins with the *Potts model*, which is a generalization of the *Ising model* of feromagnetism that you might have heard of. Take any graph (in a moment, a two-dimensional rectangular lattice) G, and consider coloring the vertices (the "sites") with N colors (the "spins"). Let's suppose that there's an interaction along each edge, with energy

$$\text{local energy of an edge} = \begin{cases} 0, & \text{different colors at the two ends} \\ -J, & \text{same colors} \end{cases}$$

so that, setting $K = -J/T$ and $v = e^K - 1$, we have Boltzmann weights

$$w(\sigma, \sigma') = 1 + v\delta(\sigma, \sigma')$$

where σ, σ' are the values of the state at the two ends of the edge.

Exercise 2. *Investigate relations between the Potts model and coloring problems, e.g. the four color theorem. First, show that the chromatic polynomial of G (i.e. the number of colorings of G with N colors) is the partition function when $v = -1$. In general, show that the partition function, as a function of N and v, makes sense even when N is not an integer (in particular, the chromatic polynomial is a polynomial in N).*

In 1971, Temperley and Lieb[10] made major progress on solving the Potts model for a rectangular lattice.

[10] Temperley, H. N. V.; Lieb, E. H. Relations between the "percolation" and "colouring" problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the "percolation" problem. Proc. Roy. Soc. London Ser. A 322 (1971), no. 1549, 251–280. http://dx.doi.org/10.1098/rspa.1971.0067