MATH 448: RESHETIKHIN–TURAEV INVARIANTS
WEDNESDAY, JANUARY 6, 2016

THEO JOHNSON-FREYD

Today continues the story of the Potts model, based on Jones' notes[1], Baxter's book[2], and the original Temperley–Lieb paper[3].

1. Potts model and Temperley–Lieb algebra

Last time I introduced the *Potts model*, which is the statistical mechanical model on a graph G in which each vertex carries one of N colors (the "spins"), and each edge has a *Bolztmann weight*

$$w(\sigma, \sigma') = 1 + v\delta(\sigma, \sigma')$$

where σ, σ' are the values of the state at the two ends of the edge. What we're interested in understanding is the *partition function*

$$Z_G(v, N) = \sum_{\text{states:vertices}\to\text{spins}} \prod_{\text{edges}} w.$$

As a warm-up, let's consider the one-dimensional case, and be more general, allowing arbitrary Boltzmann weights:

Example 1.1. There's only one type of one-dimensional lattice, and it's the same whether you decide that the sites are the edges of the vertices (since you can always move to the Poincaré dual lattice). Suppose that there are N possible spins, and that the only interactions are on adjacent vertices. Then the Boltzmann weights are a $N \times N$ matrix W with nonnegative entries. Think of W as an endomorphism of $\mathcal{V} = \mathbb{C}^N$ and let's take periodic boundary conditions. Then if there are m sites,

$$Z(T) = \mathrm{tr}_{\mathcal{V}} W^m$$

As $m \to \infty$, this is going to be dominated by the largest eigenvalue of W:

$$\text{free energy per site} = \text{largest eigenvalue of } W$$

So the problem is solved if you know how to relate the largest eigenvalue for some matrix with the largest eigenvalue for the matrix formed by raising every entry of your original matrix to the power $1/T$. Anyway, the point is that it's "just" a problem of finding the spectrum of an operator (and, in fact, the largest eigenvalue). $\diamond$

Returning to the Potts model, let's consider the problem on an $m \times n$ lattice (with free boundary conditions). Set $\mathcal{V} = \mathbb{C}^N$.

The strategy will be to try to turn this into the one-dimensional spectral problem like in Example 0.1. So let's consider each (length n) row to be a single "atom," which can take any of N^n

[1]Vaughan F. R. Jones. In and around the origin of quantum groups. http://arxiv.org/abs/math/0309199

[2] Baxter, Rodney J. Exactly solved models in statistical mechanics. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London, 1982. xii+486 pp. ISBN: 0-12-083180-5. https://physics.anu.edu.au/theophys/_files/Exactly.pdf

[3]Temperley, H. N. V.; Lieb, E. H. Relations between the "percolation" and "colouring" problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the "percolation" problem. Proc. Roy. Soc. London Ser. A 322 (1971), no. 1549, 251–280. http://dx.doi.org/10.1098/rspa.1971.0067

spins. So its vector space is $\mathcal{V}^{\otimes n}$. A typical basis vector of $\mathcal{V}^{\otimes n}$ is a list of n spins $\vec{\sigma} = (\sigma_1, \dots, \sigma_n)$. What's the energy of m rows? Actually, we just want the exponentiated energy.

Each row itself contributes an exponentiated energy of

$$E_{\text{horizontal}}(\sigma) = \sum_{i=1}^{n-1} K\delta(\sigma_i, \sigma_{i+1})$$

and two adjacent rows also contribute an energy of

$$E_{\text{vertical}}(\sigma, \sigma') = \sum_{i=1}^{n} K\delta(\sigma_i, \sigma'_i)$$

Set A to be the $N^n \times N^n$ diagonal matrix whose σth entry is $\exp E_{\text{horizontal}}(\sigma)$, and B to be the $N^n \times N^n$ whose (σ, σ')th entry is $\exp E_{\text{vertical}}(\sigma, \sigma')$.

Exercise 1. *Show that the partition function is*

$$Z = \xi_{N^n}^t (AB)^{m-1} A \xi_{N^n}$$

where ξ_{N^n} is the length-N^n (column) vector which is all 1*s, and $\xi_{N^n}^t$ is its transpose.*

So we're interested in spectral problems related, for example, to AB.

Let g be the $N \times N$ matrix which is all $N^{-1/2}$s, i.e. $g = N^{-1/2}\xi_N \xi_N^t$ where ξ_N is the length-N all-1s vector. Bear with me for the funny normalizations — there's a reason I want them. For $i = 1, \dots, n$, set

$$U_{2i-1} = \mathbf{1}_N \otimes \cdots \otimes \mathbf{1}_N \otimes g \otimes \mathbf{1}_N \otimes \cdots \otimes \mathbf{1}_N$$

where $\mathbf{1}_N$ is the unit $N \times N$ matrix and the g, thought of as a map $\mathcal{V} \to \mathcal{V}$, is in the ith spot. So U_{2i-1} acts on $\mathcal{V}^{\otimes n}$. Then

$$B = \prod_{i=1}^{n} (v + N^{1/2} U_{2i-1}).$$

This follows from the case when $n = 1$, in which case I'm claiming that

$$\exp\big(K\delta(\sigma, \sigma')\big) = (e^K - 1)\delta(\sigma, \sigma') + N^{1/2} N^{-1/2}$$

Let f be the $N^2 \times N^2$ diagonal matrix whose (σ, σ')th diagonal entry (where now the ordered pair $(\varsigma, \varsigma') \in \{\text{spins}\}^{\times 2}$ parameterizes the column of the matrix, and not the matrix entry, since I already said it was diagonal) is $N^{1/2}\delta(\varsigma, \varsigma')$. For $i = 1, \dots, n-1$, set

$$U_{2i} = \mathbf{1}_N \otimes \cdots \otimes \mathbf{1}_N \otimes f \otimes \mathbf{1}_N \otimes \cdots \otimes \mathbf{1}_N$$

where the f, thought of as a map $\mathcal{V}^{\otimes 2} \to \mathcal{V}^{\otimes 2}$ is in the ith and $i+t$th spots. So U_{2i} acts on $\mathcal{V}^{\otimes n}$. Then

$$A = \prod_{i=1}^{n-1} (1 + N^{-1/2} v U_{2i}).$$

This follows from the case when $n = 2$, which I leave to you.

Now, the point of the funny normalizations is that

$$U_i U_{i\pm 1} U_i = U_i.$$

We also have

$$U_i^2 = N^{1/2} U_i$$

and

$$U_i U_j = U_j U_i, \quad |i - j| \geq 2$$

where in these relations $i, j = 1, \dots, 2n-1$ (except U_0 and U_{2n} are undefined...).

Definition 1.2. The $2n$th *Temperley–Lieb algebra* TL_{2n} is the unital associative algebra generated by $U_1, \dots, U_{2n-1}$ with the above relations. ◇

The point is that the Potts model furnishes a representation of Temperley–Lieb algebra.

Exercise 2. *In TL_{2n}, set*

$$R = N^{-n/2} U_1 U_3 \dots U_{2n-1}.$$

Prove that there is a unique linear functional τ on TL_{2n-1} such that

$$RXR = \tau(X) R$$

for all $X \in TL_{2n-1}$. For example, $\tau(1) = 1$.

Exercise 3. *In the Potts representation, $R = N^{-n} \xi_{N^n} \xi_{N^n}^t$, $\tau(X) = N^{-n} \xi_{N^n}^t X \xi_{N^n}$, and $Z = N^n \tau(ABA \dots BA)$.*

Warning 1.3. The functional τ can be hard to evaluate. ◇