MATH 448: RESHETIKHIN–TURAEV INVARIANTS
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My goal today is to explain the equivalence between the Potts and Ice models found be Temperley and Lieb[1] and also to introduce the equivalent State Sum model of Kauffman[2]. The lecture is also based on Jones' notes[3] and Baxter's book[4].

1. An ice-type model

Last time we studied the *Potts model* in statistical mechanics. It was defined on any graph, although we focused on the case of a lattice. It was a *nearest neighbor spin* model in the sense that the *vertices* carry "spins" and the Boltzmann weights are assigned to edges. The most important example of a Potts model is the Ising model of ferromagnetism.

For comparison, a *vertex* model is also defined on an arbitrary graph, except that now the *edges* carry spin, and the *vertices* get the Boltzmann weights. The most important vertex models are *ice type*. See, when ice crystals form, there's some array of oxygen atoms, and each oxygen is bonded strongly to two hydrogens. But each of those hydrogens is also weakly bonded to another oxygen, and this is what holds the ice together. More specifically, a weak bond connects a hydrogen to some neighbor oxygen *on a side of that oxygen which does not have a hydrogen.* You can model this as follows. Describe the locations of the oxygens by the vertices of a graph, and there's one hydrogen on each edge, with the ends of the edge indicating the oxygen that it's strongly bonded to and the oxygen it's weakly bonded to. Then (1) each edge is pointed in one direction or the other, to say which end has the strong bond and which the weak; and (2) every vertex has exactly two incoming edges.

Let's be even more specific. Let's say that we're on a square lattice. Then there are six possible configurations at each vertex, and so we're doing what's called a *six vertex model.*

Now, water molecules prefer to have their oxygens at some specific angle. So there will be different energies / Boltzmann weights to the configuration depending on which of the six vertices we're in. On a square lattice with no exterior forces there would be two Boltzmann, depending on whether the covalent bonds are at a right or 180-degree angle. But you can change this by applying some external electric field or by putting some shear force on the ice, so that the angles are no longer right.

The case I want to consider is where the Boltzmann weights are as follows:

draw them

[1]Temperley, H. N. V.; Lieb, E. H. Relations between the "percolation" and "colouring" problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the "percolation" problem. Proc. Roy. Soc. London Ser. A 322 (1971), no. 1549, 251–280. http://dx.doi.org/10.1098/rspa.1971.0067

[2]Kauffman, Louis H. State models and the Jones polynomial. Topology 26 (1987), no. 3, 395–407. http://www.sciencedirect.com/science/article/pii/0040938387900097

[3]Vaughan F. R. Jones. In and around the origin of quantum groups. http://arxiv.org/abs/math/0309199

[4]Baxter, Rodney J. Exactly solved models in statistical mechanics. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London, 1982. xii+486 pp. ISBN: 0-12-083180-5. https://physics.anu.edu.au/theophys/_files/Exactly.pdf

Put another way, let's define a 4×4 matrix h, where $4 = \{\uparrow\uparrow, \uparrow\downarrow, \downarrow\uparrow, \downarrow\downarrow\}$, via

$$h = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q^{-1} & 1 & 0 \\ 0 & 1 & q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then the Boltzmann weights are $w = 1 + \alpha h$. More generally, perhaps we have different types of vertices, and the αs can depend on the vertex.

Last time I introduced the *Temperley–Lieb algebra* TL_n, which was the associative algebra with generators $U_1, \dots, U_{n-1}$ and relations

$$U_i^2 = \delta U_i$$
$$U_i U_{i\pm 1} U_i = U_i$$
$$U_i U_j = U_j U_i, \quad |i - j| \geq 2$$

We called $\delta = N^{1/2}$ last time, and n was even. Today let's set $\delta = q^{-1} + q$. We discovered that the Potts model was a representation of this algebra, but in fact everything about it was entirely determined by this algebra alone, without its representation.

Today we can make the same discovery about ice. Namely:

Exercise 1. *Check that*

$$U_i = \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes h \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}$$

satisfies the Temperley–Lieb relations, when we're acting on $(\mathbb{C}^2)^{\otimes n}$ *and* $h : (\mathbb{C}^2)^{\otimes 2} \to (\mathbb{C}^2)^{\otimes 2}$ *is in the* i *and* $(i+1)$*th spots.*

Last time, up to some constant factors that are easy to compute, we were interested in the partition function

$$Z = \xi^t ABABABAB\xi$$

where

$$A = \prod_{j=1,3,5,\dots} (1 + \alpha U_j), \quad B = \prod_{j=2,4,6,\dots} (1 + \beta U_j)$$

More generally, we could have allowed α and β to depend on which edge it was in the graph that was contributing that factor — we got these products by multiplying over the edges, after all.

But in our new ice-type model, this is nothing but the partition function of a "diagonal" grid:

picture

2. Arbitrary graphs and Kaufmann's model

I said this in terms of square lattices, but really you could work on an arbitrary graph — both models are defined on arbitrary graphs (up to some way of indicating at the 4-valent vertices which are which). The equivalence will work for a *planar* graph.

Let G be the underlying graph of a Potts model. Let me define a new planar, 4-valent graph G' as follows: the vertices of G' are the edges of G; the edges of G' correspond to cyclic adjacencies in G.

picture

Note that then the faces of G' are the vertices of G union the faces of G. Checkboard-color the faces of G' black or white depending on whether they came from vertices of faces of G. Then you can clearly recover G by taking the vertices to be the black faces of G' and the edges to be the corner-to-corner connections. But every planar 4-valent graph has a unique checkerboard color (**Exercise!**) and so every G' determines a unique G.

Now the point is that G' is the graph for our six-vertex model. If you and I disagree about how to orient a vertex, then just multiply through by some power of v.

Let's work in the ice model on G', which I always consider checkerboard-colored. We want the partition function

$$Z_G = \sum_{\text{states}} \prod_{\text{vertices}} (1 + \alpha h)$$

where α is allowed to depend on the choice of vertex, and h is the matrix above, read in terms of the four states. (This matrix is symmetric, so it doesn't matter how you orient a vertex.) Consider expanding out the sum $\prod_{\text{vertices}}(1+\alpha h)$. You get $2^{\#\text{vertices}}$ many terms. What does each term look like?

We described the edges in the six-vertex model as indicating which end of a bond a hydrogen atom was hanging out at, but let's instead read it as describing a direction of "current" along the edge. Then the allowed configurations of current in the six-vertex model all have "conservation of current". But the splitting $w = 1 + \alpha h$ actually breaks this up more. See, the matrix 1 only allows current on one "channel":

vertical lines = *four options*

and the matrix h only allows current on another channel:

horizontal lines.

These might be the "black channel" and the "white channel", because they correspond to the two different ways to connect diagonal faces at a vertex.

(In terms of the Potts graph G, the sums correspond to keeping or removing edges from the graph G, based on whether you allow the colors at the two ends to be arbitrary, or force them to be the same with a factor of v. So we're counting ways to remove some edges from G and then color the resulting graph so that every connected component is the same.)

So each summand in Z_G is a smoothing of the 4-valent vertices. Given such a smoothing, what are the allowed states? Current has to circulate around each resulting circle in the smoothing, so you have $2^{\#\text{circles}}$ of states, depending on whether current circulates clockwise or counterclockwise.

Let's fix some smoothing and some choice, for each circle, of whether current will flow clockwise or counterclockwise. What's the contribution to the partition function of that state? The black channels don't contribute anything except to force conservation of current — all of the interesting part is on the white channels. The white channels each contribute a factor of α, so let's record that separately. Finally, the white channels have a sort of "magnetic force" if the currents are flowing opposite directions to each other, which is q^{-1} or q depending on which side the currents are on.

Here's a different way to say count these factors. If a current turns 180 degrees counterclockwise, pick up a factor of $q^{1/2}$; if clockwise, pick up $q^{-1/2}$.

Then the factors of q are just counting total winding numbers of a circle: you get a total factor of q for each counterclockwise circle, and a total factor of q^{-1} for each clockwise circle. Thus summing over orientations we have $\delta^{\#\text{circles}}$, where $\delta = q + q^{-1}$ from before. All together, we see that

$$Z_G^{\text{Potts}} = Z_{G'}^{\text{Ice}} = \sum_{\text{smoothings}} \alpha^{\#\text{white smoothings}} \delta^{\#\text{circles}}$$

or, if α is allowed to depend on the vertex in question, then $\alpha^{\#\text{white smoothings}}$ is replaced by a product over white-smoothed edges of those αs.

This is the *Kaufmann state sum* model, where the spins are the two possible smoothings of each vertex, so that a state is a smoothing.

3. Temperley–Lieb algebra redux

Return to the Temperley–Lieb algebra

$$U_i^2 = \delta U_i$$
$$U_i U_{i\pm 1} U_i = U_i$$
$$U_i U_j = U_j U_i, \quad |i - j| \geq 2$$

This algebra has a natural representation in terms of smoothings, which some Morse theory checks is faithful.

Exercise 2. *Use this description to show that* $\dim \mathcal{TL}_n$ *is the nth Catalan number.*