TMF talk

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1 Genera

A major goal of algebraic topology has been the classification of manifolds. To classify manifolds, we define manifold invariants, of which the Reshetikhin-Turaev invariants are one example. I want to focus today on a different kind of classification for a different kind of manifold. Let's take manifolds with some sort of extra structure on their tangent bundles, and classify them up to cobordism that preserves that structure. The set of cobordism classes forms a ring, where

$$[M] + [M'] = [M \sqcup M']$$

$$[M] \cdot [M'] = [M \times M'].$$

One approach to manifold asks: what are these rings? Many of them have been calculated by Milnor, Quillen, and others. In the unoriented cobordism ring, MO_*, every manifold is 2-torsion (the cylinder $M \times [0, 1]$ is a cobordism between $M \sqcup M$ and $\emptyset$), and we have

$$MO_* = \mathbb{F}_2[\{x_i : i \neq 2^j - 1\}].$$

The complex cobordism ring MU_* is torsion-free and satisfies

$$MU_* \otimes \mathbb{Q} = \mathbb{Q}[\{\mathbb{C}P^n\}].$$

The only torsion in the oriented cobordism ring MSO_* is 2-torsion. The rest looks like

$$MSO_* \otimes \mathbb{Q} = \mathbb{Q}[\{\mathbb{C}P^{2n}\}].$$

These theorems were mostly proved by finding cobordism invariants of manifolds. Often, such invariants can be found in the manifold's characteristic numbers. Recall that a vector bundle $E \to B$ has characteristic classes:

- If E is complex, there are Chern classes $c_i(E) \in H^{2i}(B; \mathbb{Z})$.
- If E is real, there are Stiefel-Whitney classes $w_i(E) \in H^i(B; \mathbb{F}_2)$.
- If E is real and oriented, there are Pontryagin classes $p_i(E) \in H^{4i}(B; \mathbb{Z})$.

For a complex manifold M^{2n}, one can then define Chern numbers as follows. Let $I = (i_s)$ be a sequence of natural numbers with $\sum s i_s = n$; then the corresponding Chern number is

$$c_I[M] = \left\langle \prod c_s(TM)^{i_s}, [M] \right\rangle.$$

Thom proved that

Theorem 1.1. *Two complex manifolds are complex cobordant if and only if they have the same Chern numbers.*

Similar theorems hold for real manifolds and the Stiefel-Whitney numbers, and oriented real manifolds and the Pontryagin and Stiefel-Whitney numbers.

Now let's turn to the general problem of finding all the cobordism invariants. Here's one not on the list so far: if M is an oriented $4n$-manifold, then the cup product on $H^{2n}(M;\mathbb{Z})$ is a symmetric bilinear form, and one can prove that its signature $\sigma(M)$ is an invariant of oriented cobordism. By the above, $\sigma(M)$ is some linear combination of Pontryagin numbers – but what? The answer, discovered by Hirzebruch, is a little unusual at first. Consider the power series

$$f(z) = \frac{\sqrt{z}}{\tanh\sqrt{z}} = 1 + \frac{z}{3} - \frac{z^2}{45} + \cdots.$$

The infinite product

$$f(z_1)f(z_2)\cdots$$

is invariant under permutations of the z_i, so it can be written as a power series in the elementary symmetric polynomials:

$$f(z_1)f(z_2)\cdots = L(p_1, p_2, \cdots) = 1 + \frac{p_1}{3} + \frac{7p_2 - p_1^2}{45} + \cdots$$

where p_i is the ith elementary symmetric polynomial in the z_i. Write L_n for the degree n component of this expression. Given an oriented $4n$-manifold M, we reinterpret L_n as an expression in terms of the Pontryagin classes of TM, and define $L_n[M]$ to be the corresponding Pontryagin number.

Theorem 1.2 (Hirzebruch)**.** *We have $L_n[M] = \sigma(M)$.*

For example, take $M = \mathbb{C}P^2$, which clearly has signature 1. Since TM is complex, we can calculate its Pontryagin classes from its Chern classes:

$$c(TM) = (1+x)^3 \quad \text{where } H^*M = \mathbb{Z}[x]/(x^3),\ x = c_1(\mathcal{O}(-1)).$$
$$p_1(TM) = c_1(TM)^2 - 2c_2(TM) = (3x)^2 - 2 \cdot 3x^2 = 3x^2.$$

Thus $p_1[M]/3 = 1 = \sigma(M)$.

One very nontrivial consequence of this theorem is integrality results for the Pontryagin numbers: since the signature is an integer, p_1 must always be divisible by 3, $7p_2 - p_1^2$ by 45, and so on.

Notice also that this sequence (L_n) is multiplicative: writing $L = \sum L_n$, if we have

$$\sum p_i = \left(\sum p_i'\right)\left(\sum p_i''\right),$$

then $L(p_i) = L(p_i')L(p_i'')$. Of course, this formula holds for the Pontryagin classes of a sum of two bundles – for example, the Pontryagin classes of $T(M \times M') \cong TM \oplus TM'$. As a result, we have defined a ring homomorphism $L : MSO_* \to \mathbb{Q}$.

Definition 1.3. A **genus** (plural **genera**) is a ring homomorphism from a cobordism ring to some other ring.

In other words, a genus is a cobordism invariant for manifolds with whatever structure, that sends disjoint unions to sums and products to products. The above considerations have shown us how to construct genera from any power series with constant term 1, at least starting from cobordism rings generated by some kind of elementary symmetric polynomials, that is, MO_*, MU_*, and $MSO_* \otimes \mathbb{Q}$. A few other examples:

- Taking $f(z) = \frac{z}{1-e^{-z}}$ and letting the Chern classes be the elementary symmetric polynomials in z_i gives the **Todd genus** $\mathrm{Td} : MU_* \to \mathbb{Z}$.
- Taking $f(z) = \frac{z/2}{\sinh(z/2)}$ and letting the Pontryagin classes be elementary symmetric functions gives the $\hat{A}$ **genus** $\hat{A} : MSO_* \to \mathbb{Q}$. This has the property that it sends spin manifolds to integers.
- Taking
$$f(z) = \frac{z/2}{\sinh(z/2)} \prod_{n \geq 1} \frac{(1-q^n)^2}{(1-q^n e^z)(1-q^n e^{-z})}$$
gives the **Witten genus** $W : MSO_* \to \mathbb{Q}[[q]]$. This takes values in $\mathbb{Z}[[q]]$ on spin manifolds. Furthermore, if M is a spin manifold with $(p_1/2)(TM) = 0$, then $W[M]$ is a modular form.

2 Spin manifolds and cobordism theories

Let's pause to say what is meant by spin manifold. There's a natural double cover $\mathrm{Spin}(n) \to SO(n)$ and a Spin manifold is just an oriented manifold with a lift of its structure group to $\mathrm{Spin}(n)$.

Homotopy theory gives us a natural way of describing and generalizing this situation. First, recall that a real bundle is orientable if and only if its first Stiefel-Whitney class vanishes. This means that there's a fiber sequence

$$BSO \to BO \xrightarrow{w_1} K(\mathbb{F}_2, 1);$$

a (stable) bundle is classified by a map to BO, and it lifts to BSO if and only if the composition to $K(\mathbb{F}_2, 1)$, which classifies w_1 of the bundle, is zero. (I'm working with stable bundles both because the homotopy theory's easier and because any flavor of cobordism cares about the stable tangent bundle of a manifold, not its actual tangent bundle.)

The cohomology of BO is a polynomial ring on the w_i, and taking this fiber has killed w_1. So BSO is simply connected, and the next lowest cohomology class is $w_2 \in H^2(BSO; \mathbb{F}_2)$. As it turns out, an oriented bundle has a spin structure if and only if its w_2 vanishes. Thus, there's another fiber sequence

$$B\mathrm{Spin} \to BSO \xrightarrow{w_2} K(\mathbb{F}_2, 2).$$

The insinuation is that we can define classifying spaces for more highly structured manifolds by just taking successive connective covers of BO. The lowest cohomology class in $H^* B\mathrm{Spin}$ is an integral class $p_1/2 \in K(\mathbb{Z}, 4)$. (Note that this isn't in the image of H^*BSO – but two times it is.) We take another fiber

$$B\mathrm{String} \to B\mathrm{Spin} \xrightarrow{p_1/2} K(\mathbb{Z}, 4).$$

$B\mathrm{String}$ is now the 7-connected cover of BO – its lowest cohomology class is in H^8. Despite the name, $B\mathrm{String}$ is no longer the classifying space of a group. However, we can still use it as a sort of generalized structure group – define a **string manifold** to be a spin manifold with $p_1/2 = 0$.

There's a similar diagram for complex structures, a tower of fibrations $BU\langle 6\rangle \to BSU \to BU$.

Now, just as we defined the cobordism rings MU_*, MSO_*, and MO_*, there are cobordism rings $M\mathrm{Spin}_*$, $M\mathrm{String}_*$, and so on. Homotopy theory lets us do more. Each of these generalized classifying spaces is associated to an object called a **Thom spectrum**, which is a highly structured (E_∞) ring spectrum, and thus defines a multiplicative cohomology theory with power operations. There are maps of E_∞ ring spectra

$$M\mathrm{String} \to M\mathrm{Spin} \to MSO \to MO.$$

What do these cohomology theories do? One point of view is that $MG^{-n}(X)$ is the cobordism classes of families of G-manifolds over X. You have to do a little work to set this up right, but if X is a manifold, it's strictly true: a class in $MG^{-n}(X)$ is a codimension $-n$ G-manifold over X, mod G-cobordism over X.

We can now make a fancier version of our definition.

Definition 2.1. A **genus** is a map of E_∞ ring spectra from a Thom spectrum to something else.

And now to review the above examples:

- The Todd genus is naturally a map $MU \to K$, into complex K-theory.
- Restricting to Spin manifolds, the $\hat{A}$-genus is naturally a map $M\mathrm{Spin} \to KO$ into real K-theory. This version of the genus is actually *better* than the power series version above. The power series version was $MSO_* \to \mathbb{Q}$ that happened to send $M\mathrm{Spin}_* \to \mathbb{Z}$. Here, the torsion in KO_* actually captures some of the torsion in $M\mathrm{Spin}_*$, that would be annihilated by the power series version.
- Restricting to String manifolds, we can write the Witten genus as $M\mathrm{String} \to KO[[q]]$. However, this definition factors through the $\hat{A}$ genus, and doesn't explain this modularity property.

3 Complex oriented cohomology theories and formal groups

Definition 3.1. A cohomology theory E is **complex oriented** if there's a class $x \in E^2(\mathbb{C}P^\infty)$ that maps to 1 under the maps

$$E^2(\mathbb{C}P^\infty) \to E^2(\mathbb{C}P^1) \cong E^0(S^0).$$

Such a class is to be thought of as a universal first Chern class for complex line bundles, valued in E. Using the splitting principle, one can go on to define a theory of Chern classes for all complex vector bundles, just as they're defined for ordinary cohomology. Complex orientability also forces

$$E^*(\mathbb{C}P^\infty) \cong (\pi_* E)[[x]].$$

Since $\mathbb{C}P^\infty$ is a topological abelian group (multiplication representing the tensor product of complex line bundles), we get a cocommutative cogroup structure on $\pi_* E[[x]]$ – that is, a map of $(\pi_* E)$-algebras

$$\begin{aligned} F : & E^*(\mathbb{C}P^\infty) \to & E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong E^*(\mathbb{C}P^\infty) \otimes_{\pi_* E} E^*(\mathbb{C}P^\infty) \\ F : & (\pi_* E)[[x]] \to & (\pi_* E)[[x, y]] \end{aligned}$$

This can be identified with a power series $F \in (\pi_* E)[[x, y]]$ satisfying the conditions

$$F(0, x) = F(x, 0) = x$$

$$F(x, y) = F(y, x)$$

$$F(F(x, y), z) = F(x, F(y, z))$$

$$F(i(x), x) = 0 \text{ for some power series } i(x).$$

Such a structure is called a (commutative, one-dimensional) **formal group law**. It is equivalent to a group structure on the formal affine line over $\pi_* E$. As will be important in a second, you can also get such formal group laws from algebraic geometry: you get one by completing any smooth, commutative, one-dimensional group scheme over $\pi_* E$ at the group identity.

Here are a few basic examples:

1. Ordinary cohomology $H\mathbb{Z}$ has a theory of Chern classes already, which corresponds to a complex orientation. If V and W are complex line bundles, we have

   $$c_1(V \otimes W) = c_1(V) + c_1(W).$$

   Thus we get the formal group law

   $$F(x, y) = x + y,$$

   called the **additive formal group law**. This is the completion of the scheme $\mathbb{A}^1_{\mathbb{Z}}$, with the additive group structure, at 0. The same holds for ordinary cohomology with coefficients in any ring.

2. Complex K-theory is also complex orientable. Here $\pi_* K = \mathbb{Z}[\beta^{\pm 1}]$ with $\beta \in \pi_2 K$ the Bott periodicity class. The formal group law is

   $$F(x, y) = x + y + \beta xy,$$

   called a **multiplicative formal group law**. If we re-coordinatize by putting $x' = -\beta^{-1} x$, then it has the form

   $$1 - F(x', y') = (1 - x')(1 - y')$$

   which is the formal completion of the scheme $\mathbb{G}_m$ at 1.

3. Finally, complex cobordism MU has the universal complex orientation: there's a first Chern class $c_1 \in MU^2(\mathbb{C}P^\infty)$, and the complex orientations of a ring spectrum E are exactly the images of this c_1 under the maps of ring spectra $MU \to E$. It is a surprising and deep theorem, due to Quillen, that the formal group law F_{MU} over $\pi_* MU \cong \mathbb{Z}[a_1, a_2, \ldots]$ is also the universal formal group law. That is, formal group laws over a ring R are precisely the base changes of F_{MU} under ring homomorphisms $\pi_* MU \to R$.

4 Elliptic spectra

So there's a suggestion of a deep connection between formal group laws and stable homotopy theory. The next step in its exploration is finding cohomology theories that carry other formal group laws, particularly ones coming from algebraic geometry.

Definition 4.1. An **elliptic cohomology theory** is the data of: an even periodic, complex orientable cohomology theory E, an elliptic curve $C \to E_0$, and an isomorphism $\widehat{C} \to \mathbb{G}_E$ between the formal completion of C at the identity and the formal group of E.

I haven't proved that such things exist, but they do in wide generality: any elliptic curve $C \to \operatorname{Spec} R$ such that the corresponding map from $\operatorname{Spec} R$ to the moduli of elliptic curves is *flat* is part of an elliptic cohomology theory.

Theorem 4.2. *There is a natural map of multiplicative cohomology theories $M\mathrm{String}_* \to E_*$ for every elliptic cohomology theory E.*

Now, the Witten genus, as described above, is the genus associated to a particular elliptic cohomology theory E. Namely, we take the Tate curve over $\mathbb{Z}[[q]]$ and form an elliptic spectrum $K[[q]]$. The Witten genus is naturally a map $M\mathrm{String}_* \to K[[q]]_*$. And just as we saw that the $\hat{A}$ genus was better understood by lifting to a map to KO, it would be nice to lift the Witten genus further – to build a genus from $M\mathrm{String}$ to some sort of universal elliptic cohomology theory.

5 TMF

Unfortunately, no such thing exists. What we think instead is: elliptic curves are sections of a sheaf over the moduli stack of elliptic curves. An elliptic cohomology theory, likewise, could be the sections of a sheaf of elliptic cohomology theories. So we can take the global sections of this sheaf – which is no longer an elliptic cohomology just because the stack itself has nontrivial cohomology, which, for example, destroys even periodicity.

This doesn't quite work either. Part of the problem is that multiplicative cohomology theories themselves are not a nice enough category, and don't have a good sheaf theory. We can fix this by further rigidifying into the category of elliptic E_∞ ring spectra. Briefly, an E_∞ ring spectrum has enough structure to take homotopy limits, and we can define a sheaf of E_∞ spectra on some site to be a presheaf of E_∞ spectra that sends covers to homotopy limits. Finally, we can construct a sheaf of E_∞ ring spectra on the moduli of elliptic curves, and realize the various elliptic genera as maps of E_∞ ring spectra into sections of this sheaf. The spectrum TMF is the global sections of this sheaf, and the Witten genus in its full glory is a map $M\mathrm{String} \to TMF$.

This program has been carried out by work of Goerss, Hopkins, and Lurie. The Lurie version runs as follows. A scheme is a functor from rings to sets satisfying a sheaf condition, that's covered by representable objects. Likewise, a derived stack is a functor from E_∞ rings to spaces, satisfying this homotopy sheaf condition, and that's covered by representable objects. In particular, we can define a derived moduli stack of derived elliptic curves, as the stack representing the functor that sends an E_∞ ring R to the space of derived elliptic curves over R. One then has to prove:

1. this functor is representable;
2. it restricts to elliptic spectra on sufficiently small open affine sets.

The second part is where the relationship between elliptic curves and formal groups really takes off. Localize everything at p; an elliptic curve C has a ***p*-divisible group** $C[p^\infty]$, defined as the colimit of the subschemes of p^n-torsion points of C. $C[p^n]$ is a finite flat group scheme of rank p^{2n}; in characteristic p, it can take one of two forms.

1. If C is ordinary, then $C[p^n]$ has p^n different connected components, each of which is a formal group scheme of order p^n. $C[p^\infty]$ is some Galois twist of an extension
$$0 \to \mathbb{G}_m \to C[p^\infty] \to \underline{\mathbb{Q}_p/\mathbb{Z}_p} \to 0.$$

2. If C is supersingular, then $C[p^n]$ has a single connected component, and $C[p^\infty]$ corresponds naturally to a height 2 formal group.

Theorem 5.1 (Serre-Tate)**.** *Deformations of a p-local elliptic curve C are the same as deformations of the p-divisible group of C.*

And we can describe this deformation space completely. If C is supersingular, then it's the deformation space of a formal group. A theorem of Goerss, Hopkins, and Miller says that this deformation space can *canonically* be lifted to a derived affine formal scheme, the formal spectrum of an E_∞ ring spectrum called **Morava E-theory**, E_2. If C is ordinary, then its p-divisible group is an extension of a formal group $\mathbb{G}_m$ by an étale group. The étale part deforms uniquely, so the deformation space of $C[p^\infty]$ is an extension of the deformation space of the formal part by the space of extensions. Again, the deformation space of the formal part corresponds to an E_∞ ring spectrum E_1, and the space of extensions gives us a function spectrum of the form $\mathrm{Hom}(\mathbb{C}P^\infty, E_1)$.

The upshot is that we can describe the local structure of the derived moduli stack in terms of these two kinds of E-theories. And proving that the moduli of derived elliptic curves is built from elliptic spectra reduces to looking at its local structure. As a result, the construction of TMF generalizes to give derived versions of *any* nice enough stack over the moduli of p-divisible groups. In particular, Behrens and Lawson have constructed spectra of topological automorphic forms associated to moduli of higher-dimensional abelian varieties with extra symmetry. These spectra are still pretty poorly understood, and people are very interested both in understanding them better and in developing other variants of this whole game. I'll conclude by mentioning a few very open questions.

1. Are there higher analogues of the Witten genus? That is, can we fiber off more of the cohomology of BO and get a map from the corresponding Thom spectrum to some other 'chromatic' spectrum – maybe a topological automorphic forms spectrum?

2. Instead of seeing elliptic curves as one-dimensional abelian varieties, we could think of them as one-dimensional Calabi-Yau varieties. Calabi-Yau varieties of dimension n, like K3 surfaces, have formal groups associated to their n-dimensional étale cohomology. Can we understand higher chromatic phenomena through moduli of Calabi-Yau varieties? One immediate obstacle is that the deformations of a K3 surface are not controlled by the deformations of a p-divisible group. However, they bear a close relation to something that looks like the Dieudonné crystal of a p-divisible group.

3. Is there a moduli of derived p-divisible groups, such that the sheaves giving TMF, TAF, and so on are pulled back from a sheaf over this derived stack? If so, what are the corresponding global sections?