Conformal blocks

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In this talk I will give an idea of how the main result of [1] is proved. The relation of this with the class is that one can mathematically prove a result which was predicted by the physics of 2d conformal field theories, which we know are related to 3d Chern-Simons theories that in turn are related to knot invariants. Here is the set up.

Let X be a smooth projective curve over k an algebraically closed field of characteristic 0, fix $x \in X(k)$ some rational point of X and t_x an uniformizer at x. Let $\mathscr{O}$ be the completion of the local ring at x, $\mathscr{K}$ its fraction field and $F_X = \Gamma(X\backslash\{x\}, \mathscr{O}_X)$, the ring of functions on the open complement. Let $\mathrm{Bun}^{\mathrm{ss}}_{\mathrm{SL}_r}$ be the moduli space of isomorphism classes of semisimple stable vector bundles $\mathscr{F}$ of rank r on X, with a trivialization of their top wedge power, i.e. $\eta : \bigwedge^r(\mathscr{F}) \simeq \mathscr{O}_X$. From its construction this moduli space carries a tautological rank r vector bundle $\mathscr{E}$, let $\mathscr{L} \equiv \bigwedge^{\mathrm{top}}\left(R\Gamma(\mathrm{Bun}^{\mathrm{ss}}_{\mathrm{SL}_r}, \mathscr{E})\right)$[1] this is called the *determinant line bundle*.

The main result will relate the global sections of $\mathscr{L}$ to a vector space obtained from the representation theory of $\hat{\mathrm{SL}}_r$, the Kac-Moody group associated to $\mathrm{SL}_r(\mathscr{K})$. More precisely,

Theorem 1. *For every integer $n \geq 1$, one has a canonical isomorphism of $\hat{SL}_r$-modules:*

$$\Gamma(Bun^{\mathrm{ss}}_{SL_r}, \mathscr{L}^n) \simeq (V_n^\vee)^{SL_r(F_X)}.$$

Here V_n is the basic representation of $\hat{\mathrm{SL}}_r$ of level n, and the superscript $SL_r(F_X)$ means we are taking invariants with respect to this subgroup. Recall this representation is uniquely defined by the following properties.

a. it is irreducible;

b. $1 \in k$, the center of $\hat{\mathfrak{sl}}_r$, acts by multiplication by n;

c. there is a (*vacuum*) vector $v \in V_n$, such that $\mathfrak{sl}_r(\mathscr{O}) \cdot v = 0$, and v generates V_n.

[1] Here $\bigwedge^{\mathrm{top}}(K)$ for K a perfect complex is just the tensor product of its top wedge product (resp. its dual) on even (resp. odd) parity degree.

The above vector spaces are the so-called space of *confomal blocks*, they are sometimes denoted by $\mathscr{B}_n(X)$.

Example. For $X = \mathbb{P}^1_{\mathbb{C}}$ one obtains the following. One has $\mathscr{B}_n(\mathbb{P}^1) \simeq (V_n)^{\mathfrak{sl}_r(k[t^{-1}])}$. Here $V_n \simeq k \oplus U^+(\mathfrak{sl}_r(k[t^{-1}]))$, where $U^+(\mathfrak{sl}_r(k[t^{-1}]))$ is the augmentation ideal of the enveloping algebra. Thus, $\mathscr{B}_n \simeq V_n/U^+(\mathfrak{sl}_r(k[t^{-1}]))V_n \simeq k$.

Remark. Let's briefly just mention one interesting consequence of this result. Suppose one wants to compute the dimensions of the spaces $\mathscr{B}_n(X)$. Then, the fact that they are calculated as the cohomology of a *factorizable* line bundle $\mathscr{L}^n$, implies that they form a rational Conformal Field Theory, which, in particular implies that $\dim(\mathscr{B}_n(X))$ as X varies over the moduli space of Riemann surfaces and $n \in \mathbb{Z}_{\geq 1}$ is a fusion rule[2]. Thus, it determines a fusion ring F over the integers. The following result is normally attributed to Verlinde. For any X of genus g, $\bar{x} \in \mathrm{Sym}^k(X)$ and $\bar{n} \in \mathbb{Z}^k_{\geq 1}$ the dimension of $\dim(\otimes_{i=0}^k \mathscr{B}_{\bar{n}_i})(X)$ is $\sum_{\chi \in F^*} \chi(\bar{n}_1) \cdots \chi(\bar{n}_k)\chi(\omega)^{g-1}$, where $F^* = \mathrm{Hom}_{\mathbf{Alg}_{\mathbb{C}}}(F \otimes_{\mathbb{Z}} \mathbb{C}, \mathbb{C})$ and $\chi(\omega) = \sum_{j \in \mathbb{Z}_{\geq 1}} |\chi(j)|^2$.

The idea of the proof of the theorem is to relate the global moduli space to a local description in terms of the affine Grassmannian. This gives a $SL_r(F_X)$-equivariant cover of $\mathrm{Bun}^{\mathrm{ss}}_{\mathrm{SL}_r}$. Then one can describe a $SL_r(F_X)$-equivariant line bundle on the affine Grassmannian in two ways: (i) abstractly using the theory of Tate objects; and (ii) more concretely in terms of the group $\hat{SL}_r$. Both descriptions are used to study this line bundle and eventually compute its global sections. This, modulo some considerations of infinite-dimensionality and differences between moduli space (and moduli stacks of bundles), gives the result.

The relation of this story to my research is the description (i) of the line bundle in terms of Tate objects can possibly be generalized to higher dimensions. I am working to make that precise.

1 Affine Grassmannian

We will work with a finer object than the moduli space of SL_r-bundles over X, namely the algebraic stack, whose underlying space of isomorphism objects[3] recovers the moduli space that we want. We shall introduce some notation. Let R be a commutative k-algebra, for $x \in X(R)$, let Γ_x be the graph of x in $R \times_k X$, $\mathbb{D}_x = \mathrm{Spec}(\hat{\mathscr{O}}_{X,x})$ and let $\mathbb{D}^\times_x = \mathrm{Spec}(\hat{\mathscr{K}}_{X,x})$. Let $\mathscr{F}_0$ denote the trivial SL_r-bundle over any base (we abuse notation for legibility). Consider $SL_r(\mathscr{K})$ the functor of points, which is a group prestack, that associates to any $R \in \mathbf{Alg}_{\Bbbk}$ the set $SL_r(R((t)))$, i.e. the set of $r \times r$ matrices with coefficients in Laurent polynomials in R with determinant 1.

[2]We refer the reader to [2] for a nice and short introduction to this topic.

[3]Recall that an algebraic stack is, in particular, a functor from $\mathbf{Alg}_k$ to groupoids, so one can take the composition to sets by looking at isomorphism classes of objects in these groupoids.

Lemma 1. *The functor of points $SL_r(\mathscr{K})$ is equivalent to the one that associates to any R triples $(\mathscr{F}, \eta, \epsilon)$, where $\mathscr{F}$ is an SL_r-bundle over X_R, $\eta : \mathscr{F}|_{X_R \backslash \Gamma_x} \xrightarrow{\simeq} \mathscr{F}_0$ and $\epsilon : \mathscr{F}|_{\mathbb{D}_x} \xrightarrow{\simeq} \mathscr{F}_0$.*

Proof. This is a descent statement. We start by noticing that the data of a pair $(\mathscr{F}, \eta)$ is equivalent to that of $(\mathscr{F}', \eta')$ where $\mathscr{F}'$ is a bundle over $\mathbb{D}_x$ and η' a trivialization on $\mathbb{D}_x^\times$ (cf. Lemma 2.). However, this can be described as a quotient space $\mathrm{SL}_r(\mathscr{K})/\mathrm{SL}_r(\mathscr{O})$. So one sees easily that the stabilizer of ϵ in $\mathrm{SL}_k(\mathscr{K})$ is $\mathrm{SL}_r(\mathscr{O})$, so one gets that the extra data of ϵ determines a point $\gamma \in SL_k(\mathscr{K})$. $\square$

One now introduces the so-called affine Grassmannian.

Definition 1. Let $\mathscr{G}\mathrm{r}_{SL_r}$ be the functor of points which associates to any $R \in \mathbf{Alg}_k$ the set of pairs $(\mathscr{F}, \eta)$, where $\mathscr{F}$ is an SL_r-bundle on $\mathbb{D}_x$ and $\eta : \mathscr{F}|_{\mathbb{D}_x^\times} \xrightarrow{\simeq} \mathscr{F}_0$.

Lemma 2. *The functor of points $\mathscr{G}r_{SL_r}$ is equivalent to the functor of points which associates to any R the set of pairs $(\mathscr{F}', \eta')$ where $\mathscr{F}'$ is a SL_r-bundle on X_R and $\eta' : \mathscr{F}'|_{X_R \backslash \Gamma_x} \xrightarrow{\simeq} \mathscr{F}_0$.*

Proof. This is essentially the main result of [3]. A more abstract way of seeing it is as follows. Any SL_r-bundle $\mathscr{F}$ over a scheme X is a equivalent to a symmetric monoidal functor

$$\mathscr{F} : \mathbf{Rep}_{\mathrm{SL}_r} \to \mathrm{Vect}_X,$$

where Vect_X is the category of rank r vector bundles on X. By [4] one has an equivalence of ∞-categories:

$$\mathrm{QCoh}(X) \simeq \mathrm{QCoh}(X \backslash \Gamma_x) \times_{\mathrm{QCoh}(\mathbb{D}_x^\times)} \mathrm{QCoh}(\mathbb{D}_x).$$

In particular, one has an equivalence between the underlying homotopy categories, i.e. the triangulated categories of complexes of sheaves with quasi-coherent cohomology up to quasi-isomorphism. Inside it $\mathrm{Vect}(X)$ sits as complexes of degree 0, so one restricts to an equivalence

$$\mathrm{Vect}(X) \simeq \mathrm{Vect}(X \backslash \Gamma_x) \times_{\mathrm{Vect}(\mathbb{D}_x^\times)} \mathrm{Vect}(\mathbb{D}_x).$$

Since $\mathrm{Fun}(\mathbf{Rep}_{\mathrm{SL}_r}, -)$ commutes with limits of categories we are done.

Alternatively, here is a more direct argument. The statement is local on the curve X, so it is enough to check that SL_r-bundles on $\mathbb{A}_k^1 = \mathrm{Spec}(k[t])$ with a trivialization on $\mathbb{A}_X^1 \backslash \{0\} \simeq \mathrm{Spec}(k(t))$ is equivalent to SL_r-bundles on $\mathbb{D}_0 \simeq \mathrm{Spec}(k[[t]])$ with a trivialization on $\mathbb{D}_0^\times \simeq \mathrm{Spec}(k((t)))$. One now notice that the following fact: *The category of SL_r-bundles on $\mathbb{D}_0$ is equivalent to the category of SL_r-bundles on Spec$(k_{(t)}[t])$ whose restriction to Spec$($Frac$(k_{(t)}[t]))$ descends to Spec$(k(t))$.* This is essentially fpqc-descent for G-bundles, which is equivalent to étale descent for G smooth. The trivialization over $k((t))$ clearly gives a descent to $k(t)$ so one has a bundle defined on the whole affine line minus the origin. Then by étale descent with respect to the cover $\mathbb{A}^1 \backslash \{0\}$ and $\mathbb{D}_0$ one can construct a SL_r-bundle over $\mathbb{A}^1$ with the appropriate trivialization[4]. $\square$

Definition 2. The *moduli stack of SL_r-bundles* over X is given by the following functor of points, to any $R \in \mathbf{Alg}_{\Bbbk}$ one associates the groupoid of $\mathscr{F}$ a SL_r-bundle over the curve X_R.

One has a map $\pi : \mathscr{G}\mathrm{r}_{\mathrm{SL}_r} \to \mathscr{B}\mathrm{un}_{\mathrm{SL}_r}$ which forgets the trivialization. In other words one can define $\mathscr{B}\mathrm{un}_{\mathrm{SL}_r}$ as the quotient stack of $\mathscr{G}\mathrm{r}_{\mathrm{SL}_r}$ by the action of $SL_r(F_X)$ on the left.

Here is another way to view the affine Grassmannian. Consider $k((t))$ as a Tate object of $k{-}mod^{f.g.,p.}$, i.e. an admissible ind-pro-object. Let $V_k = k((t))^{\oplus r}$ and consider the functor of points $\mathrm{Gr}_{V_k}(R) = \{L \subset V_k \otimes_k R \,|L \text{ is a lattice}\}$, here a lattice L is a pro-subobject of V_k such that V_k/L is an ind-object. For instance, $\mathscr{O} \subset \mathscr{K}$ is a lattice.

Roughly, to identify with the previous description one notices that $SL_r(\mathscr{K})$ acts transitively on the space of lattices, by changing basis. Let's pick one lattice, say $\mathscr{O}$. Its stabilizer is the subgroup $SL_r(\mathscr{O})$. Thus one obtains that at the level of k-points $\mathscr{G}\mathrm{r}_{SL_r}(k) \simeq \mathrm{SL}_r(k((t)))/SL_r(k[[t]])$. A little more care gives the equivalence of R-points.

Here is a list of facts about the objects above.

Proposition 1. *(i) $\mathscr{G}r_{SL_r}$ is an ind-finite-projective scheme;*

(ii) $SL_r(\mathscr{K})$ is an ind-group-scheme;

(iii) $SL_r(\mathscr{O})$ is an ind-finite-group-scheme;

(iv) $SL_r(F_X)$ is an integral (i.e. reduced and irreducible) scheme.

(v) $\mathscr{G}r_{SL_r}$ is an integral ind-scheme[5].

We will not need the above facts, except for (iv). The proof of which is done by reducing to the case of $\mathbb{P}^1$, where one can concretely realize it as a colimit of finite schemes $\Gamma^{(N)}$ where one filter by the order of pole of the rational function at 0. Then checking that each $\Gamma^{(N)}$ is integral, normal and a local (actually globally as well) complete intersection.

2 Line bundles on $\mathscr{B}\mathrm{un}_{\mathbf{SL}_r}$ and $\mathscr{G}\mathrm{r}_{\mathbf{SL}_r}$.

Consider the description of the affine Grassmannian as the moduli space of lattices. Given the choice of a lattice $L \subset V_R$ one can construct a line bundle

[4]The essential input in this equivalence is that $H^1_{\text{ét}}(\mathrm{Spec}(k(t)), G) = H^1_{\text{ét}}(\mathrm{Spec}(k'), G) = 0$, for any k'/k a finite extension. So here this follows from Tsen's and Springer's theorems for arbitrary smooth connected G since k is algebraically closed of characteristic 0. If one wants to consider k a finite field, then G needs in addition to be simply-connected and semisimple, due to theorems of Harder and Lang.

[5]This is defined as the colimit of schemes which are integral (cf. Lemma. 6..3. [1] for equivalent conditions).

over Gr_V by the following assignment for any R-point $x : R \to \mathrm{Gr}_{\mathscr{K}}$ one defines

$$\mathscr{L}^L_{L'} \equiv \Lambda^{\mathrm{top}}(L'/N) \otimes \Lambda^{\mathrm{top}}(L/N),$$

where $N \subset L, L'$ is a common sublattice of both L and L'[6].

The check that this actually defines a line bundle, i.e. satisfies the appropriate descent conditions, can be made in different ways. For instance, one can pick a stratification $\mathrm{Gr}_V^{(n)}$ of Gr_V, for $n \geq 1$, and check that it is well-defined on a subscheme of $\mathrm{Gr}_V^{(n)}$ of codimension at least 2. By Hartog's lemma on gets a line bundle on each $\mathrm{Gr}_V^{(n)}$, which gives a compatible family on the colimit, hence a line bundle on Gr_V.

Alternatively, one can also consider the following simplicial prestack

$$(\mathrm{Gr}_V)_n = \{L_0 \subset \cdots \subset L_n \subset V\},$$

where $\{L_i\}_{0\leq i\leq n}$ are lattices. This forms a hypercover of $\mathrm{Gr}_V \simeq (\mathrm{Gr}_V)_0$ and the assignment above is compatible with the simplicial maps, so defines a line bundle on this hypercover. Hence one gets a line bundle on Gr_V by descent.

Definition 3. Let $\hat{\mathrm{SL}}_r$ be the group of automorphisms of the total space of $\mathscr{L}^{\mathscr{O}^{\oplus r}}$ over Gr_V.

Since $\mathrm{SL}_r(\mathscr{K})$ acts transitively on Gr_V, one sees that $\hat{\mathrm{SL}}_r$ is a $k^\times$-central extension. It's actually independent of the choice of the choice of lattice $\mathscr{O}^{\oplus r}$. [7]

Here is a more concrete description of $\mathscr{L}^{\mathscr{O}^{\oplus r}}$, or rather $\hat{\mathrm{SL}}_r$, which will be useful to relate to the usual universal Kac-Moody extension. For $W \in \mathrm{Pro}(k-\mathrm{mod}^{f.g.,p.})$ consider $\mathrm{End}^f(W)$ the subalgebra of $\mathrm{End}(W)$ consisting of elements with finite rank. And let $\mathscr{T}(W)^0$ be the group of invertible elements of $\mathrm{End}(W)/\mathrm{End}^f(W)$, equivalently the image of $\mathrm{Aut}(W)$ under the natural projection. Let $(1+\mathrm{End}^f(W))_1$ be the set[8]

$$\Big\{1+u \mid u \in \mathrm{End}^f(W), \text{ s.t. } \det(u)=1\Big\}.$$

One has the extension

$$1 \to k^\times \to \mathrm{Aut}(W)/(1+\mathrm{End}^f(W))_1 \to \mathscr{T}^0(W) \to 1. \tag{1}$$

We can apply the above to $W = V/L$ for $L \subset V$ any lattice[9]. Let $\gamma \in \mathrm{SL}_r(\mathscr{K})$ and take the lattice $\mathscr{O}^{\oplus r} \subset V$, then one can write

$$\gamma = \begin{pmatrix} a(\gamma) & b(\gamma) \\ c(\gamma) & d(\gamma) \end{pmatrix},$$

[6] The general theory of [5] guarantees that these always exist.

[7] To see that one can consider the gerbe of determinantal theories on V and define the central extension as the automorphism group of the total space of this gerbe.

[8] Since u has finite rank, $\det(u)$ is well-defined.

[9] Such W's are called *co-lattices*.

where we split $\mathscr{K}^{\oplus r} \simeq \mathscr{O}^{\oplus r} \oplus (\mathscr{K}/\mathscr{O})^{\oplus r}$, using some splitting of $\mathscr{K} \to \mathscr{K}/\mathscr{O}$. Let $\bar{a}(\gamma)$ denote the projection of $a(\gamma)$ to $\mathrm{End}(W^{\oplus r})/\mathrm{End}^f(W^{\oplus r})$. We define the extension $\hat{\mathrm{SL}_r}'$ to be the pullback of (1) through $\bar{a}$. Concretely, it is the set of isomorphism classes of pairs $(\gamma, u) \in \mathrm{SL}_r \times \mathrm{Aut}(W^{\oplus r})$, such that $\bar{a}(\gamma) = u$, with respect to the relation $(\gamma, u) \sim (\gamma, v)$ if $u = wv$ for some $w \in (1 + \mathrm{End}^f(W))_1$. The Lie group structure is given by

$$[(\gamma, u), (\gamma', u')] = ([\gamma, \gamma'], [u, u']).$$

Lemma 3. *$\hat{SL_r} \simeq \hat{SL_r}'$ as $\mathbb{G}m$-central extensions of $SL_r(\mathscr{K})$.*

Proof. Idea: SL_r can be concretely described as follows. Given $(g, \lambda), (g', \lambda) \in \mathscr{L}^{\mathscr{O}^{\oplus r}}$ their commutator is

$$[(g, \lambda), (g', \lambda')] = \left([g, g'], \eta^{-1}_{g \circ g'(\mathscr{O}^{\oplus r})} \circ \mu_{\mathrm{Pic}_k}(\eta_{g(\mathscr{O}^{\oplus r})} \circ \eta_{g'(\mathscr{O}^{\oplus r})})(1)\right),$$

where $\eta_{g(\mathscr{O}^{\oplus r})}$ is the trivialization of $\mathscr{L}^{\mathscr{O}^{\oplus r}}$ induced by the lattice $g(\mathscr{O}^{\oplus r})$, and $\mu_{\mathrm{Pic}_k} : \mathrm{Pic}_k \otimes \mathrm{Pic}_k \to \mathrm{Pic}_k$ is the canonical tensor product of line bundles over k. If one traces through the definition

$$\eta^{-1}_{g \circ g'(\mathscr{O}^{\oplus r})} \circ \mu_{\mathrm{Pic}_k}(\eta_{g(\mathscr{O}^{\oplus r})} \circ \eta_{g'(\mathscr{O}^{\oplus r})})(1) = \det([a(\gamma), a(\gamma')]a^{-1}([\gamma, \gamma'])).$$

So the map $\psi : \hat{\mathrm{SL}_r}' \to \mathrm{SL}_r$ given by

$$\psi(\gamma, u) = (\gamma, \det(ua^{-1}(\gamma)))$$

gives an isomorphism between the two extensions[10]. $\square$

Let's consider the corresponding construction at the level of Lie algebras. We claim that this is equivalent to the extension $\hat{\mathfrak{sl}_r}' = \mathfrak{sl}_r \oplus k$ with the only non-trivial Lie pairing given by

$$[(g \otimes f, 0), (g' \otimes f', 0)] = ([g, g'] \otimes ff', \mathrm{Res}_{t=0}(fdf')),$$

where we identify $\mathscr{K} \simeq k((t))$, so $\gamma \in \mathfrak{sl}_r \otimes_k k((t))$ is written as $\gamma = g \otimes f$, for $g \in \mathfrak{sl}_r$ and $f \in k((t))$.

Lemma 4. *The extensions $\hat{\mathfrak{sl}_r}' \simeq \hat{\mathfrak{sl}_r}$.*

Proof. The map $\psi : \hat{\mathfrak{sl}_r} \to \hat{\mathfrak{sl}_r}'$ is given by $\psi(\gamma, u) = (\gamma, \mathrm{tr}(u - a(\gamma)))$, notice that by definition, since $u - a(\gamma) \in \mathrm{End}^f(W)$, this map is well-defined. That it is a bijection is easy to see. The center of $\hat{\mathfrak{sl}_r}$ is described by $(0, u)$, s.t. $\mathrm{tr}(u) = 0$, which is identified by ψ with the image of k inside $\hat{\mathfrak{sl}_r}'$. All that is left is to check that ψ is a map of Lie algebras. For that we can suppose that $(\gamma, u) \in \hat{\mathfrak{sl}_r}$

[10] This is essentially the same verification as Lemma 4. below, we omit the details.

is of the form $(\gamma, a(\gamma))$, i.e. the only non-zero commutators of $\hat{\mathfrak{sl}_r}'$ are between elements of the form $(g \otimes f, 0)$ for $g \in \mathfrak{sl}_r$ and $f \in k((t))$. So one wants that

$$\psi([\gamma, a(\gamma)), (\gamma', a(\gamma'))]) = [\psi((\gamma, a(\gamma))), \psi((\gamma', a(\gamma')))].$$

This becomes the formula

$$\mathrm{tr}([a(\gamma), a(\gamma')] - a([\gamma, \gamma'])) = \mathrm{Res}(\gamma d\gamma').$$

Now this is exactly Tate's residue formula. For simplicity we suppose $r = 1$. For any two elements γ (resp. γ') which belong to $t^{-N}\mathscr{O}$ (resp. $t^N\mathscr{O}$) one has a filtration V_p of V, by $V_p \equiv t^{-p}\mathscr{O} \cap V$, and $[a(\gamma), a(\gamma')] - a([\gamma, \gamma'])$ lowers this degree by 1, so its trace is 0. Thus, one can assume that γ (and γ') are actually polynomials in t, moreover by additivity it is enough to consider the case when $\gamma = t^n$ and $\gamma = t^m$. We check that

$$\mathrm{tr}\left([a(t^n), a(t^m)] - a([t^n, t^m])\right) = 1,$$

if $n + m = 0$, and 0 otherwise. This implies the formula. $\square$

If one considers the pullback of $\hat{\mathrm{SL}}_r$ to $\mathrm{SL}_r(\mathscr{O})$, it splits canonically via $\gamma \mapsto (\gamma, a(\gamma))$. Let $\chi_0 : \mathrm{SL}_r(\mathscr{O}) \to \mathbb{G}\mathrm{m}$ be the map $\chi_0(\gamma, u) = \det(u^{-1}a(\gamma))$, this is equivalent to the projection to $\mathbb{G}\mathrm{m}$ of the second factor. One obtains a line bundle $\mathscr{L}_{\chi_0}$ over $\mathscr{G}\mathrm{r}_{\mathrm{SL}_r}$ by considering the associated local system, which by Lemma 3. is isomorphic to $\mathscr{L}^{\mathscr{O}^{\oplus r}}$. One has a section $\tau_{\mathscr{O}^{\oplus r}}$ of this line bundle given by

$$\tau_{\mathscr{O}^{\oplus r}}(\gamma, u) = \det(ua(\gamma)^{-1}),$$

for any $(\gamma, u) \in \mathrm{SL}_r(\mathscr{K})$. One easily checks that $\tau_{\mathscr{O}^{\oplus r}}((\gamma, u)(\delta, v)) = \chi_0((\delta, v))\tau_{\mathscr{O}}(\gamma, u)$ for $(\delta, v) \in \mathrm{SL}_r(\mathscr{O})$, so this section descends to a section of $\mathscr{L}^{\mathscr{O}}$ over $\mathscr{G}\mathrm{r}_{\mathrm{SL}_r}$.

3 Computation of the global sections of $\mathscr{L}$

The following proposition is the key to the proof of the main theorem.

Proposition 2. $\pi^*\mathscr{L} \simeq \mathscr{L}_{\chi_0}$.

Proof. One needs to compute the determinant line bundle $\mathscr{L}$. For that let's consider $\mathscr{F} \in \mathscr{B}\mathrm{un}_{\mathrm{SL}_r}(R)$, one has an exact sequence

$$0 \to \Gamma(X_R, \mathscr{F}) \to F_X^{\oplus r} \otimes_k R \xrightarrow{\bar{\gamma}} W_R \to H^1(X_R, \mathscr{F}) \to 0,$$

where $\gamma \in \mathrm{SL}_r(\mathscr{K})(R)$ is any lift such that the image on $\mathscr{B}\mathrm{un}_{\mathrm{SL}_r}(R)$ is $\mathscr{F}$, and $\bar{\gamma}$ is the composition of the inclusion $F_X^{\oplus r} \otimes_k R \hookrightarrow R((t))^{\oplus r}$, applying $\gamma^{-1} : R((t))^{\oplus r} \to R((t))^{\oplus r}$, then projecting to W_R. This exact sequence comes from tensoring the exact sequence of sheaves[11]

$$0 \to \mathscr{F} \to j_*(\mathscr{O}^{\oplus r} \to \imath((\mathscr{K}/\mathscr{O})^{\oplus r}) \to 0$$

[11] Here, $j : X\backslash\{x\} \to X$ and $\imath$ is the inclusion of x into X.

by R and taking cohomology. The idea is to show that the choice of $\mathscr{O}^{\oplus r}$ a lattice of $\mathscr{K}^{\oplus r}$ allows one to calculate $\bar{\gamma}$. Namely, let γ_0 be any element of $\mathrm{SL}_r(\mathscr{O}^{\oplus r})(R)$ such that $\mathscr{F}_{\gamma_0}$, the associated SL_r-bundle over X_R, has trivial cohomology. Then one has an identification of $F_X^{\oplus r} \simeq (\mathscr{K}^{\oplus r}/\mathscr{O}^{\oplus r})$[12]. Thus, using this idenfitication one gets the exact sequence

$$0 \to \Gamma(X_R, \mathscr{F}) \to W_R \overset{a(\gamma^{-1}\gamma_0)}{\to} W_R \to H^1(X_R, \mathscr{F}) \to 0.$$

The final step is to show that the above exact sequence implies that there exists a finitely generated R-module W_0, and w_0 an automorphism of W_0 such that

$$0 \to \Gamma(X_R, \mathscr{F}) \to W_0 \overset{w_0}{\to} W_0 \to H^1(X_R, \mathscr{F}) \to 0,$$

where $\det(w_0) = \tau_{\mathscr{O}^{\oplus r}}(\gamma^{-1}\gamma_0, w_0)$ Indeed, let W_0 be the largest free R-module generated by the image of $ua(\gamma^{-1}\gamma_0) - \mathrm{id}$, where u is a lift of $a(\gamma_0^{-1}\gamma)$ to an automorphism of $\mathscr{K}^{\oplus r}$. Then one can take for w_0 the restriction of $ua(\gamma^{-1}\gamma_0)$ to W_0. The result follows from the definitions. □

Notice that $\mathscr{L}_{\chi_0}$ is $\mathrm{SL}_r(F_X)$-equivariant[13]. Now one can use a result by [6, 7] which calculates the cohomology of $\mathscr{L}_{\chi_0}$. It states the following:

$$\Gamma(\mathscr{G}\mathrm{r}_{\mathrm{SL}_r}, \mathscr{L}_{\chi_0}^{\otimes n}) \simeq V_n,$$

as $\hat{\mathfrak{sl}}_r$-modules.

To compute the cohomology of $\mathscr{L}$ one has to use a formal result about descent of line bundles on algebraic stacks, in this case ind-schemes. It says that if $\mathscr{L}_{\chi_0}$ is an $\mathrm{SL}_r(F_X)$-equivariant line bundle on $\mathscr{G}\mathrm{r}_{\mathrm{SL}_r}$, then[14]

$$\Gamma(\mathscr{G}\mathrm{r}_{\mathrm{SL}_r}, \mathscr{L}_{\chi_0})^{\mathrm{SL}_r(F_X)} \simeq \Gamma(\mathrm{SL}_r(F_X)\backslash\mathscr{G}\mathrm{r}_{\mathrm{SL}_r}, \mathscr{L}),$$

where $\mathscr{B}\mathrm{un}_{\mathrm{SL}_r} \simeq \mathrm{SL}_r(F_X)\backslash\mathscr{G}\mathrm{r}_{\mathrm{SL}_r}$.

Finally, one has to relate the global sections of $\mathscr{L}$ over $\mathscr{B}\mathrm{un}_{\mathrm{SL}_r}$ with that over the (coarse) moduli space of semisimple objects, i.e. $\mathrm{Bun}^{\mathrm{ss}}_{\mathrm{SL}_r}$. The crucial fact is the following:

Proposition 3. *One has an isomorphism between the spaces of global sections*[15]

$$\Gamma(\mathscr{B}un^{\mathrm{ss}}_{SL_r}, \mathscr{L}) \simeq \Gamma(\mathscr{B}un_{SL_r}, \mathscr{L}).$$

[12] Namely a splitting of the co-lattice $W_R = \mathscr{K}^{\oplus r}/\mathscr{O}^{\oplus r}$ associated to $\mathscr{O}^{\oplus r}$.

[13] Indeed, $\mathrm{SL}_r(\mathscr{O})$ still acts on $\mathscr{G}\mathrm{r}_{\mathrm{SL}_r}$ by left multiplication, so any line bundle over $\mathscr{G}\mathrm{r}_{\mathrm{SL}_r}$ obtains an $\mathrm{SL}_r(F_X)$-equivariant structure by restriction.

[14] We omit a certain subtlety here, that says that one does not actually have an $\mathrm{SL}_r(F_X)$-equivariant structure on the level of R-points for any k-algebra R, but only for the ones finitely generated. However, this is enough for our purposes because since $\mathrm{SL}_r(F_X)$ is reduced (cf. Proposition 1. (iv)) it is equivalent to produce a $\mathrm{SL}_r(F_X)$-equivariant section at all R-points (R finitely generated over k), at k-points, or at the level of Lie algebras (cf. [1, Proposition. 7.4.]).

[15] This is morally the statement that $\mathrm{Bun}^{\mathrm{ss}}_{\mathrm{SL}_r}$ has codimension 2 inside $\mathrm{Bun}_{\mathrm{SL}_r}$. The reason we say morally is that one can not make sense of this directly (cf. the proof of the proposition) for algebraic stacks.

Proof. Let's remember one way of concretely constructing the moduli stack $\mathscr{B}\mathrm{un}_{\mathrm{SL}_r}$. By Mumford regularity, there exists a sufficiently big integer $N \geq 0$, such that for any coherent sheaf $\mathscr{F}$ (in particular vector bundle) over X, $\mathscr{F}(Nx)$ has vanishing higher cohomologies and is generated by its global sections. Let for $\mathscr{F}$ a rank r vector bundle, consider such and N and let $h(N) = \dim(\Gamma(X, \mathscr{F}(Nx)))$. Then one can look at

$$K_N \to \mathscr{Q}\mathrm{uot}_X(\mathscr{O}_X^{\oplus h(N)}),$$

an open subscheme of the functor of points[16]

$$\mathscr{Q}\mathrm{uot}_X(\mathscr{O}_X^{\oplus h(N)})(R) \equiv \left\{ \mathscr{O}^{\oplus h(N)} \xrightarrow{q} \mathscr{G} \mid \mathscr{G} \text{ is flat over } R \right\},$$

defined by $\ker(q)$ is a subbundle of rank r. This gives a cover of $\mathscr{B}\mathrm{un}_{\mathrm{SL}_r}$ by taking the quotient with respect to the $GL_{h(N)}$-action on $\mathscr{O}_X^{\oplus h(N)}$. Stricly speaking this construct the stack $\mathscr{B}\mathrm{un}_{GL_r}$, one can get to $\mathscr{B}\mathrm{un}_{\mathrm{SL}_r}$ by restricting K_N to the subscheme H_N where the determinant of the universal bundle is non-zero. So it's enough to show that the open subscheme $H_N^{\mathrm{ss}} \to K_N$, which is defined as those quotient sheaves whose kernel is a semistable vector bundle, has codimension at least 2.

Consider the functor of points $H_N^{(s,d)}$ which associates triples $(\mathscr{F}, \mathscr{E}, \delta)$ where $\mathscr{F}$ is a rank r vector bundle, $\mathscr{E} \to \mathscr{F}$ is a subbundle of rank $0 < s < r$ and degree $d > 0$, and δ a trivialization of $\Lambda^s(\mathscr{E})$. The union

$$\bigcup_{(s,d)} H_N^{(s,d)}$$

is by construction the complement of H_N^{ss}. A simple dimension (cf. [1, Lemma. 8.2.]) estimate gives the result.

Thus, one can calculate $\Gamma(\mathscr{B}\mathrm{un}_{\mathrm{SL}_r}^{\mathrm{ss}}, \mathscr{L})$ by looking at the $\mathrm{GL}_{h(N)}$-invariant sections of $\Gamma(H^{\mathrm{ss}}, \mathscr{L})$. Thus the later, by Hartog's lemma, is equivalent to the $\mathrm{GL}_{h(N)}$-invariant part of $\Gamma(H^{ss}, \mathscr{L})$, which in turn is equivalent to $\Gamma(\mathscr{B}\mathrm{un}_{\mathrm{SL}_r}, \mathscr{L})$. □

Finally to compare $\Gamma(\mathscr{B}\mathrm{un}_{\mathrm{SL}_r}, \mathscr{L})$ with the global sections over the moduli space $\mathrm{Bun}_{\mathrm{SL}_r}^{\mathrm{ss}}$, one notices that the later can be realized as the GIT quotient of H_N^{ss} by $GL(h(N))$. So by the usual theory of GIT one has that

$$\Gamma(\mathrm{Bun}_{\mathrm{SL}_R}^{\mathrm{ss}}, \mathscr{L}) \simeq \Gamma(H_N^{\mathrm{ss}}, \mathscr{L})^{GL(h(N))},$$

where the righthand side is identified with the sections over the moduli stack by the above proposition.

[16] This is know to be representable by a projective scheme by Grothendieck[8].

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