Correlation functions in integrable Quantum Field Theory: Chern-Simons Research Lecture Series

Fedor Smirnov
LPTHE, CNRS

UC Berkeley
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Lecture 1: 27 September 2011

The plan:

1. Axiomatics of quantum field theory
2. How the axiomatics is implemented in integrable models
3. S-matrix, form factors, correlation functions

So basically I will give complete solutions for these models of quantum field theory, but for the last point there is a little problem in the description of short-distance (ultraviolet) behavior.

4. Relation to the second-order phase transitions, conformal field theory, perturbations of conformal field theory
5. One point functions. Some very important part of of ultraviolet behaviour which is not controlled by CFT is hidden in these.

Since I don't imply any preliminary knowledge, then this may sound like a foreign language. But you should not be afraid, because I will give some explanations. But unfortunately, of course this is a very big subject, and so my explanations will be very brief. I will then indicate a way you can learn more about it.

So, I start with a kind of dream *Slide: "..."*. There are several axiomatic formulations for the QFT, and I will briefly tell you about Lehmann-Symanzik-Zimmermann axiomatics. QFT is supposed to be about interaction between several fundamental particles. This axiomatization is applicable only for theories that consist entirely of massive stable particles. Of course, in nature, there are also photons, but we assume we know how to treat those (QED).

Quantum field theory is roughly speaking relativistic quantum mechanics. So if there is interaction it will necessarily involve infinitely many degrees of freedom. For the moment, we believe that our

theojf@math.berkeley.edu

world is relativistic, but right now some people claim to have found particles that move faster than light, but for the moment we will not treat that.

So we assume that we are in $3+1$-dimensional space. We have three dimensions of space, one dimension of time, and the metric near some reference point is

$$x^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2$$

Then at a reference point there is a *light cone* of points that are "light-like separated" from the point. Outside it are "space-like separated" and inside "time-like separated". Everything is supposed to be invariant under the *Poincaré group* $\mathbb{R}^4 \rtimes \mathrm{O}(3,1)$.

Slide: "…" Let us for the moment consider a theory with one kind of particle. Every particle carries a momentum, and we fix the mass m, so that $m^2 = p^2$. Then we also allow our particles to have internal degrees of freedom. Let's assume that we construct our Hilbert space out of some vacuum $|\mathrm{vac}\rangle$, and creation and annihilation operators $a^*_{\mathrm{in},\epsilon}(k)$, $a^\epsilon_{\mathrm{in}}(k)$ for $k = (p_1, p_2, p_3)$ and satisfy

$$[a^\epsilon_{\mathrm{in}}(k), a^*_{\mathrm{in},\epsilon'}(k')] = \delta^\epsilon_{\epsilon'} \delta^{(3)}(k - k').$$

and $a^\epsilon_{\mathrm{in}}(k)|\mathrm{vac}\rangle = 0$. This is bosonic.

Then we assume also that there is another set of operators $a^*_{\mathrm{out},\epsilon}(k)$, $a^\epsilon_{\mathrm{out}}(k)$. You suppose that you have an accelerator, and create particles going in at time $= -\infty$, and then they collide and something happens and particles come out at time $= +\infty$. So we suppose that $a^*_{\mathrm{out},\epsilon}(k)$, $a^\epsilon_{\mathrm{out}}(k)$ are related to $a^*_{\mathrm{in},\epsilon}(k)$ and $a^\epsilon_{\mathrm{in}}(k)$ by conjugating by some unitary operator, the *S-matrix*. We also assume $S|\mathrm{vac}\rangle = |\mathrm{vac}\rangle$. Finally, we assume that one-particle states are stable.

Then we have **long equation from slide, in which the sum starts at 2 because one-particle states are stable**.

Kolya: This S can be regarded as partition function for an infinite cylinder, if x-coordinates are compactified. **Fedor:** But they are not compactified. **Paul:** You are not in Euclidean signature. **Fedor:** I am not.

Every matrix element corresponds to some interaction, with all possible things, so theory is quite complicated, and doesn't always happen.

So this of course is not enough — we should assume something about how the particles interact. We assume that on the Fock space that I described there are *local operators*. Each local operator is a family of operators over all of space time, and we assume that they commute on spacelike separated points. Really a local operator is an operator-valued distribution, and the commutator is a distribution. This is an idea from Einstein: you cannot send a signal faster than the speed of light.

So among these operators, there must be some special operators. First, there must be an energy-momentum tensor $T_{\mu,\nu}(x)$, where μ,ν run from 0 to 3. These act on Fock space as before, and we start to combine them with S-matrix. We ask that $T_{\mu,\nu}$ be symmetric, that it satisfy a conservation law $\partial_\mu T_{\mu,\nu} = 0$, and that its integral is really energy and momentum. And it must be self-consistent. *Slide: "2. Locality…"*

Slide: "Interpolating…"

Finally, there must be a local field, whose weak limit as $t \to \pm\infty$ is described by **eqn on slide**.

If all these things are satisfied, then I have quantum field theory. I have particles, I ask that fields by in Fock space generated by particles, and conversely that particles are obtained as limits of local fields.

I should say that this is very much different than in quantum mechanics. In quantum mechanics, there is of course S-matrix, but S-matrix in QM is of course strong limit of some operator like e^{itH}, or you have to divide by free dynamics, but the strong limit of such operator exists and describes the S-matrix. That's why the quantum mechanics was easily absorbed by mathematics, because you can do it all with functional analysis, the theory of operations, and scattering theory is just the perturbation theory of continuous spectrum.

Paul: What topologies? On unitaries, strong and weak are the same. **Fedor:** I mean the limit of matrix elements.

Ty: I don't know this in detail, but I've heard of a theorem that obstructs defining a Fock space in interacting theory. **Fedor:** That theorem says that if S-matrix is nontrivial, than Fock representation for asymptotic particles and for local things are not equivalent. The matrix-element limit does not respect products.

Comments: Physics is *free* if $S = 1$. We are interested in theories that are not free. What knowledge are we interested in? S is analytic function of the momentum, and we can ask for analyticity questions. In principle, you have interpolating fields, which are local, so vanish somewhere, and so Fourier transform is analytic somewhere. So as number of particles grow, we have very complicated analytic properties.

Slide: "2. Integrable …" So now we simplify our life. We cannot manage what we want in 4 dimensions, so we move to 2. Then Lorentz group is just one-dimensional. The transformations that do not change $x_0^2 - x_1^2$ are of form $\left(\begin{smallmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{smallmatrix}\right)$.

It is convenient to parameterize the mass shell $p_0^2 - p_1^2 = m^2$ by one real number β, the *rapidity*, by $p_0 = m\cosh\beta$ and $p_1 = m\sinh\beta$.

Kolya: Usually when I hear "quantum field theory" you start with classical theory, and then the word "quantum" appears. What is the classical theory for S-matrix? **Fedor:** You can write down path integrals if you want. But originally QFT was looking about perturbative corrections to classical field theory, but then there are too many infinities, and this is an answer to the question "what do we really want from a QFT".

Slide: "Integrability.…"

We have conservation law $\partial_\mu T_{\mu\nu} = 0$. Then it's very convenient to use light-cone coordinates $0, \pm$, for which action of energy-momentum tensor is given by the simple formula **see slide**.

Now we make a simple assumption, that in addition to energy-momentum, there are more local conserved quantities. So we assume that there are $I_{\pm}^{(s)}$, all commuting, so that $I_{\pm}^{(s)} = \int_\Sigma T_{0,\pm}^{(s)}$ (Σ is some space-like slice), for some local operators $T^{(s)}$. Then $I^{(1)} = P$. We ask that these operators have "spin s", meaning that under $\left(\begin{smallmatrix}\cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta\end{smallmatrix}\right)$, the operator $I^{(s)} \mapsto e^{s\theta} I^{(s)}$. **Kolya:** One for each integer? **Fedor:** Or each odd integer, or each one not divisible by three.

Then it is clear that they must have spectrum that is additive against local densities, and so they must have **equation**. **Ty:** So this is like the analog in quantum mechanics of having a complete set of commuting observables? **Fedor:** Yes, but the locality is important.

Slide: "Implications for scattering" So what are the consequences? Morally, S-matrix is e^{itH}, and so all of these should commute with S. The implication is that the scattering must be elastic. This is quite a surprise: multiple particle production is a very characteristic condition of QFT. But experts were convinced by examples and computations that this kind of thing corresponds to vanishing of sum of diagrams, and in examples they do.

Kolya: In topological field theory, S-matrix vanishes. **Fedor:** That is a very difficult question.

So then S-matrix simplifies to **eqn**.

So now, let's take the 2-to-2 scattering, the first one. By Lorenz invariance, this elastic scattering depends only on the difference of the two rapidities. *Slide: "We start..."*

So now what can be said about this thing? First of all, this can be proved at some rigorous level, that $S_{\epsilon_1,\epsilon_2}^{\epsilon'_1,\epsilon'_2}(\beta)$ is meromorphic function of β. And if there are no bound states, then it is a regular function in the strip $0 \leq \mathrm{Im}\beta \leq \pi$. Then, there is unitarity: $\bar{S}(\beta) = S(-\beta)$ and $S\bar{S} = 1$.

Then there's something I can't comment on too much, but it's very typical: in qft, to every particle, you have an antiparticle, and we combine into the same multiplets with the same numbers ϵ. Then the point is if you scatter two particles with momentum p_1, p_2, or particle and antiparticle with momentum $\bar{p}_2, p_1$, then the S-matrices for these two particles are related by analytic continuation with some matrix c. *Slide: "3. Crossing symmetry"*.

Slide: "4. Factorizability..."

Then comes the most important party, that is not easy to explain, so I give some literature. **see slide**

What is the idea here? The idea is: suppose you have this purely elastic scattering. Then what is in- and out-going particles? Because you're doing local field theory, it's not quite a global notion. If you have interaction of three particles, first two interact, and then all the integrals of motion restrict this interaction to be elastic. And then every multi-particle process is determined as product of two-particle scatterings.

But this thing, if you assume that, and I recommend the paper by Zamolodchikov and Zamolodchikov, then you get restriction on possible two-particle scattering. See, consider preparing three particles with three momenta, far in the past. In QM, according to Heisenberg principle, if you know the momenta you don't know the positions. So you don't know the order, how the scatterings happen in order. You cannot even say it. So the only possibility out of this contradiction is that actually three-particle scattering is independent of the order. So this gives the Yang-Baxter equation, which at that time was called the "equation of triangles". *Slide: ""*

You see, already I started with general words, and eventually I finish with some very precisely formulated mathematical problem. If I want to find S-matrix to describe some physical equation, then it must satisfy the Yang-Baxter equation, the crossing symmetry (analytic continuation), and unitarity. And thenm you hope you've found S-matrix for some integrable model.

So this is how we proceed when doing mathematical physics. We start with general physical arguments, and then reduce to something we can compute. So our starting point — of course, keep the general locality arguments in mind — is really just three equations.

Slide: "Remark." Of course, S-matrix that satisfies all these requirements, it is not unique, and we can always introduce what are called "CDD multipliers". So there is a huge ambiguity, and it would be nice to reconstruct completely the QFT — we are talking now about solving an inverse scattering problem — can we reconstruct the full QFT from S-matrix? Local operators, including energy-momentum tensors, and so on, to finish the construction of QFT.

In fact, this is possible. *Slide: "3. Form factor bootstrap"* What does it mean to construct operator acting on Fock space? You know operator if you know all its matrix elements. Then **eqn from slide**. So we know the matrix elements.

But actually there is more than that. The crossing symmetry applies to things like that, so actually **eqn from slide** can be obtained by analytic continuation.

What does this mean? You apply operators to the vacuum, and then see what kind of particles you can compute.

The point is: the elastic scattering is a property of one-shell scattering, but this is off-shell, so multiple creation is allowed.

Then what is assumed about these matrix elements? First of all, I put in operators **eqn**, and it's assume that if we first order them $\beta_1 < \cdots < \beta_n$, then I can continue analytically, and the result will be meromorphic function of all variables, and this meromorphic function as I shall show is subject to a number of requirements. Then if I want to repropduce general matrix element I take **?** with some number of multiplicities, shift by πi, and then there are some details that I will explain, but it is not a big deal to reconstruct.

Slide: "Form factor axioms"

These form factors are defined by several axioms, which are formulated in my book, and they are:

0. Analytic condition: the poles are only in a few places.

1. If I commute two particles **...**

2. If I shift the last rapidity by $2\pi i$ it becomes the first one.

All of these have physical explication, but I will not go into detail, and explain a different way of verifying it.

You see, for given S-matrix, all of these are vector-valued equation. For given S-matrix, it's rather hard to solve these things. Because 2. makes the last one into the first one, and then with S-matrix and 1. you can move it back. So you get analytic function whose values there are here are related by some horrible matrix, and this is the Riemann-Hilbert problem.

Kolya: A comment: these two equations #1,#2, they form example of what's called KZ equation, and they appear very naturally in quantum affine Lie algebra, but it first appeared here. **Fedor:** Yes.

Slide: "3. Annihilation pole." Finally, there is the third axiom which allows to compute the residue of the form factor at the pole corresponding to annihilation of two particles.

Suppose once again that I have factorizable S-matrix, and suppose I can find solutions to these axioms. Then what can I say?

Theorem *If I have two operators — operator is identified with something from form factors — then they are local: they commute with each other at spacelike points.*

Slide: "2. Asymptotical theorem"

Slide: "Correlation functions."

I will give example of how the formulas look like, but before, let me explain some important issue. What are the quantities that we want to compute in qft? One set of important quantities are correlation functions or Green function or whatever you call them. You take two operators setting at 0 and x, and you compute their correlation between $|\text{vac}\rangle$ and $|\text{vac}\rangle$. I will be particularly interested when this x is spacelike.

One has to realize: I said that the main relativistic principle for qft is that two events that are spacelike separated are independent — this requirement is encoded in the requirement that $[O_1(0), O_2(x)] = 0$. But $|\text{vac}\rangle$ is something: it adds some important flavor to the theory. Causality doesn't mean that $\langle\text{vac}|O_1(0)O_2(x)|\text{vac}\rangle = 0$. Instead, it is **eqn**.

But then we do a clever trick. We use Lorenz transformation to shift so that exponent picks up a decreasing part. This only improves the convergence. The observables grow at infinity, but not fast, like $e^{s\beta}$. And then after transforming, the horrible exponent ($e^{-mr\sum\cosh\beta_j}$) makes everything converge.

This is very nice, if r is finite — r is the distance between the operators. But if $r \to 0$, then in the integrals the convergence is getting slower and slower. So for large r, obviously one-particle is most important, then 2-particle, and so on. But as $r \to 0$, all of them become equally important.

In qft this is the most interesting situation, the high energy part. This describes something you are really interested in, and there is a serious question about computing the asymptotics of this correlation function when $x \to 0$.

The first idea is that you have to somehow resum the series when all of them start to contribute. But this idea looks a little bit stupid here. **Kolya:** Why? **Fedor:** We have a formula, it's complicated, how can you resum them? Scaling dimensions you can compute, but not much more.

On the other hand, the short-distance behavior for qft must be described, for this 2-dimensional qft, must be in principle describable by conformal field theory, as I will explain next time or time after next. So it would be interesting to develop some other approach to computing the short-distance asymptotics, coming from cft.

So this is what I will do, but not immediately. First I will explain what is cft, and then how it is related to massive qft, and then how to compute all these massive asymptotic behavior. But next time, I will just do some mathematics, and explain how in some particular case how these form factors look like. To explain the real level of complexity.

Kolya: Will you say anything about semiclassical limit of these results? **Fedor:** No. **Kolya:** Because it is complicated, or because it is not done? **Fedor:** In some cases it is more or less complicated. If I have time, if you want to know about semiclassical limit I can do it. **Kolya:** Maybe on Tuesday after the lecture? **Fedor:** Yes. For example, I will have in cft case all sorts of three-point functions, and so on, and these do have semiclassical limit. For massive theories, roughly speaking, you have to solve on the plane some special solution to classical equation with some singularity at the origin.

Question from the audience: In the same vein: it's not obvious to see where the quantum equation of motion just from looking at the form factor? **Fedor:** This is a good question: yes, it is doable. But this is only very recently that we could figure out how to write down exactly all the equations of motions. **Kolya:** By this you mean differential equations for the local operators? **Fedor:** Yes. You expect the same number as for the classical case, but you must compute some constants, and this can be done.

You can say that the slides for this lecture are partly available: 21-36.

Lecture 2: 29 September 2011

Last time we decided that

$$\langle \mathrm{vac}|\mathcal{O}_1(x)\mathcal{O}_2(0)|\mathrm{vac}\rangle = \sum \frac{1}{n!} \int \mathrm{d}\beta_1 \cdots \int \mathrm{d}\beta_n \, f_1(\beta_1, \dots, \beta_n) \, e^{-mr\sum \cosh\beta_j}$$

This is for spacelike separated points ($x^2 = -r^2$), but it looks like imaginary time.

Harold: Can you remind me the definition of the form factors? **Fedor:** They are $\langle \mathrm{vac}|\mathcal{O}(0)|\beta_1 \dots \beta_n\rangle$.

Today we investigate our main example, the *sine-Gordon model*.

The action is:

$$\mathcal{A}^{\mathrm{sG}} = \int \left[\frac{1}{16\pi} (\partial_\mu \varphi)^2 + \frac{\mu^2}{\sin \pi\beta^2} 2\cos(\beta\varphi(x)) \right] \mathrm{d}^2 x$$

This action makes sense after quantization. We normalize the action so that $[\varphi(x), \partial_0\varphi(y)] = \delta(x-y)$. See, we can call $\phi = \beta\varphi$, and $\frac{1}{\beta^2}$ is Planck's constant.

A reference: L. Faddeev, L. Takhtajan, 1974 . They investigate this on the classical level.

There is periodicity $\phi \mapsto \phi + 2\pi$. Then there are *solitons*: a smooth step up with height 2π. **Kolya:** "soliton" means? **Fedor:** They propagate without changing the shape.

Then there is scattering. Suppose that I have soliton, and far away antisoliton, moving in opposite direction (looking roughly like a bump). This is at time $-\infty$, and then at $+\infty$ they have moved past each other, and the only difference is that there will be some phase shift. Even in multi-soliton scattering, everything (at the classical level) is just two-particle interaction.

Ok, so the idea to quantize, is that the solitons should correspond to particles. This is somewhat unusual: usually, in quantization, particles correspond to small variations of ϕ, and so these are huge at the classical level.

In the paper by L. Faddeev and V. Korepin, 1978, they found that this quantization does exist. **Kolya:** Perturbative quantization? Some formal version of the path integral?

So here, we want S-matrix, and it can by done, but there is some trouble. People naturally assumed that on the quantum level they scatter in the same way, with no reflection, but this is not true quantum mechanically. At the quantum level, there is reflection, but it cannot be observed semiclassically. It only happens as e^{-1/β^2}. This is like tunelling in quantum mechanics. To describe it classically requires taking complex solutions to equations of motion.

So this was described by A. Zamolodchikov , 1977. We take two solitons with rapidity β_1, β_2, and we combine them into one multiplet, and denote $\eta_\pm$ the particle and antiparticle. We write $B_j = e^{\beta_j}$. This is natural because it encodes energy-momentum. In addition, we denote $b_j = e^{\frac{2\nu}{1-\nu}\beta_j}$, and then $\nu = 1-\beta^2$, so that quasiclassical is $\nu \approx 1$.

Ok, so then S-matrix:

$$S_{1,2}(\beta_1 - \beta_2) = S_0(\beta_1 - \beta_2)\, \tilde{S}_{1,2}(b_1/b_2)$$

$$S_0(\beta) = \exp\left(-i \int_0^\infty \frac{\sin(2k\nu\beta)\sinh((2\nu-1)\pi k)}{k\cosh(\pi\nu k)\sinh(\pi(1-\nu)k)} \mathrm{d}k \right)$$

$$S_{1,2}(b_1/b_2) = \frac{1}{2}(I\otimes I + \sigma^3\otimes\sigma^3) + \frac{b_1 - b_2}{b_1 q^{-1} - b_2} \frac{1}{2}(I\otimes I - \sigma^3\otimes\sigma^3) + \frac{\sqrt{b_1 b_2}}{b_1 q^{-1} - b_2 q}(\sigma^+\otimes\sigma^- + \sigma^-\otimes\sigma^+)$$

This is well-known expression. It is supposed to be S-matrix, and sure enough it satisfies unitarity, and also crossing and so on from last time.

Kolya: Can you explain the logic? I have classical field theory, and I look for quantum field theory that reproduces the behavior in the classical limit? **Fedor:** Yes. For me, quantum theory *is* S-matrix. Where it comes from, I don't care. **Harold:** Where do I see the qualitative behavior of solitons? **Fedor:** This is the S-matrix for interaction of two solitons, with rapidity β_1, β_2. Since the soliton and antisoliton have the same mass, I combine them into the same multiplet. **Kolya:** We now switch: this S-matrix defines some theory, and we will show that semiclassical limit reproduces the classical limit.

The big contribution of Zamolodchikov is the introduction of the last term. Naively semiclassically S-matrix should be diagonal.

We now introduce some more notation:

$$\Phi_\alpha(x) = e^{i\alpha \frac{\nu}{2\sqrt{1-\nu}}\varphi(x)}$$

These satisfy slightly funny statistics, and so we will modify the axioms from last time slightly. One of them:

$$S(\beta_j - \beta_{j+1}) f_{\mathcal{O}_\alpha}(\beta_1 \dots \beta_j \beta_{j+1} \dots \beta_{2n}) = f(\dots \beta_{j+1}\beta_j \dots) \tag{*}$$

This does not change. See, the form factor requires even number of particles and antiparticles. But:

$$f_{\mathcal{O}_\alpha}(\beta_1 \dots \beta_n + 2\pi i) = e^{-\frac{\pi i \nu}{1-\nu}\alpha} f(\dots) \tag{**}$$

f is not a function, it is a tensor product.

Kolya: This vector $\beta_1 \dots \beta_{2n}$ is in $(\mathbb{C}^2)^n$. The form factor is function valued in this space, multi-valued. **Fedor:** No, meromorphic.

$$2\pi i \,\mathrm{res} f = (1 - e^{-\frac{\pi i \nu}{1-\nu}\alpha} S \dots S) f_{\mathcal{O}_\alpha}(\beta_1 \dots \beta_{2n-2}) \otimes S_{2n-1,2n} \tag{***}$$

$s_{ij} = e_i^+ \otimes e_j^- + e_i^- \otimes e_j^+$

We need to solve this system of equations. We work for few years, and then:

$$f_{\mathcal{O}_\alpha}(\beta_1 \dots \beta_{2n}) = \sum_{\substack{\{1,\dots,2n\} = I^- \cup I^+ \\ \#I^- = \#I^+ = n}} w^{\epsilon_1 \cdots \epsilon_{2n}}(\beta_1 \dots \beta_{2n}) \mathcal{F}_{\mathcal{O}_\alpha}(\beta_{I^-} | \beta_{I^+})$$

Here I is the index set, and we sum over ways to break it into two pieces, and ϵ records which one. $\epsilon_j = \pm$ if $\beta_j \in I^\pm$.

$$S_{i,i+1} w^{\cdots \epsilon_i \epsilon_{i+1} \cdots}(\dots \beta_i \beta_{i+1} \dots) = w^{\cdots \epsilon_{i+1}\epsilon_i \cdots}(\dots \beta_{i+1}\beta_i \dots)$$

So let me write σ so that $S = e^\sigma$ and $s = e^{\frac{2\nu}{1-\nu}\sigma}$. And let me introduce a function satisfying the following equation:

$$\chi(\sigma + 2\pi i) p(sq^4) = \chi(\sigma) p(sq^2) \qquad p(s) = \prod_{j=1}^{2n} (s - b_j)$$

Here $q = e^{\pi i/(1-\nu)}$. This is just a difference equation, so it is easy to solve. **Kolya:** There are infinitely many solutions. **Fedor:** If I ask it to be regular for $0 > \Im(\sigma) > -\pi$, then this more or less makes it unique.

As a prize for solving this equation, you get that this function satisfies another equation:

$$\chi(\sigma + \frac{1-\nu}{\nu}\pi i)\, P(SQ) = \chi(\sigma)\, P(-S) \qquad P(S) = \prod (S - B_j)$$

$$\chi(\sigma) \underset{\sigma\to-\infty}{\simeq} e^{-2n\frac{\sigma}{1-\nu}} x^+(s) X^+(S), \qquad x^+(s) = 1 + \sum_{j=1}^{\infty} s^j x_j^+$$

$$\chi(\sigma) \underset{\sigma\to+\infty}{\simeq} x^-(s) X^-(S)$$

$$I_\alpha(\beta_1 \dots \beta_{2n}) = \int_{\mathbb{R}\setminus 0} \chi(\sigma|\beta_1 \dots \beta_{2n}) e^{\frac{\nu\alpha}{1-\nu}\sigma} \mathrm{d}\sigma$$

This is just Laplace transform. The way it is written, it is obvious that the integral is defined for $0 < \Re(\alpha) < 2n/\nu$. This integral can be continued in α to entire complex plane, with only simple poles, at points like $\alpha = 2n + 2m + (2n+\ell)\frac{1-\nu}{\nu}$ and $\alpha = -2m - \ell\frac{1-\nu}{\nu}$, with $\ell, m \geq 0$. This is an important legacy that we get from our great predecessors that if you see a function, you have to continue it analytically.

So I will consider Laurant polynomials $\ell(s)$ and $L(S)$, and for such polynomials I will define a pairing $(\ell, L)_\alpha$ by two things:

- Bilinear.
- If $\ell(s) = s^m$ and $L(S) = S^k$, then $(\ell, L)_\alpha = I_{\alpha+2m+\frac{1-\nu}{\nu}k}$.

So I just put them under the integral. And the form factors is constructed out of this pairing.

A few words. What are you used to in usual classical mathematics? That you may pair differential forms and cycles by an integral. After the quantization, they become on the same footing: both of them become like differential forms, but in different variables. So in the classical limit, one of these guys explodes, and then it becomes an integral, so you should think of the pairing as the pairing between forms and cycles. **Kolya:** Which is forms and which is cycles? **Fedor:** This is a difficult question. There are two classical limits, one with $\nu \to 1$ and the other with $\nu \to 0$, and they switch. **Kolya:** This $\nu \to 0, 1$, this is some kind of duality, which can be interpreted at "T-duality." **Fedor:** Come on. Don't use these funny words.

SO, if we take antisymmetric polynomials $\ell_1 \wedge \dots \ell_n = \ell^{(n)}$, and similarly for L, then the definition is

$$(\ell^{(n)}, L^{(n)})_\alpha = \det(\ell_i, L_j)_\alpha$$

This is just free antisymmetric product on space of polynomials.

So then we have special functions $\ell_{I^-\sqcup I^+,j}(s)$ (explicit formula, but doesn't matter), and:

$$\mathcal{F}_{\mathcal{O}_\alpha}(\beta_{I^*}|\beta_{I^-}) = (\ell_{I^-\sqcup I^+,1} \wedge \cdots \wedge \ell_{I^-\sqcup I^+,n}, L^{(n)})_\alpha$$

and they automatically satisfy the main equations (*,**) to satisfy.

So we have satisfied almost everything, but we still need the residues (***).

Then we want symmetric polyonimals $L^{(n)}_{\mathcal{O}_\alpha}(S_1,\dots,S_n|B_1,\dots,B_{2n})$, and these must satisfy:

$$L^{(n)}_{\mathcal{O}_\alpha}(S_1,\dots,S_{n-1},B|B_1,\dots,B_{2n-2},B,-B) = B\prod_{j=1}^{n-1}(B^2S_p^2)L^{(n-1)}_{\mathcal{O}_\alpha}(S_1\dots S_{n-1}|B_1\dots B_{2n-2}) \tag{****}$$

If you know conformal field theory, you find all of the solutions to these by counting solutions to simple equations. The simplest one is:

$$L^{(n)}_{\Phi_\alpha}(S_1,\dots,S_n) = \langle\Phi_\alpha\rangle S\wedge S^3\wedge\cdots\wedge S^{2n-1}$$

And I did not say, but there are some exact forms here — the pairing vanishes on some special forms. So using that we can make all the polynomials $L^{(n)}$ to be odd in the variables. Then the rest of the polynomials correspond to some fermionc structure. The one above is: all places filled from 1 to $2n-1$. Or you create a hole somewhere and add a particle somewhere else.

Kolya: So this is like the representation of the Clifford algebra.

So, n is fixed, but you can take it huge, and then start doing a simple thing, but as it goes to small n it will all mix up.

Free fermions

So, there is a special case, where $\nu=\frac{1}{2}$. Then $S=-1$, and the form factor is

$$f(\beta_1\dots\beta_{2n})_{+\dots+-\dots-} = \left(\frac{2\sin\frac{pi\alpha}{2}}{\pi}\right)^n e^{\frac{1}{2}\sum\beta^+-\beta^-}\frac{\prod_{i<j}\sinh\frac{1}{2}(\beta_i-\beta_j)}{\prod\sinh(\beta_1^+-\beta_1^-)}$$

Where then there are some very particular differential equations, called *Painlevé III*, and they have a tau function τ, and then:

$$\frac{\langle\Phi_{\alpha_1}(x)\Phi_{\alpha_1}(0)}{\langle\Phi_\alpha(0)\rangle} = \frac{\langle\phi_{\alpha_1}\rangle\langle\Phi_{\alpha_2}\rangle}{\langle\Phi_{\alpha_2}\rangle}\tau((\frac{1}{2}Mr)^2)$$

and from this we can satisfy the short time asymptotics:

$$\frac{\langle\Phi_{\alpha_1}(x)\Phi_{\alpha_1}(0)}{\langle\Phi_\alpha(0)\rangle} = r^{\frac{\alpha_1\alpha_2}{2}}\left(1+\frac{\alpha_1\alpha_2}{4\alpha^2}\left\{(Mr)^2+\frac{4(Mr)^{2(1+\alpha)}}{(\alpha+2)^2}s-\frac{4(Mr)^{2(1-\alpha)}}{(2-\alpha)^2}s^{-1}+\dots\right\}\right)$$

where s has some explicit description in gamma functions. I will show in my talks how to do this in general.

Lecture 3: 30 September 2011

Our goal is to compute these correlation functions *Slide: "$\langle \mathrm{vac}|O_1(x)O_2(0)|\mathrm{vac}\rangle$"*. For large distances, you can compute these by stationary phase. But for small distances there is a problem. As an example, we consider the case of Free Fermions when S-matrix is trivial. We have $S = -I$, and we have encoded the statistics into the S-matrix. Then form facts are given by a simple formula **from slide**. We put out in front a vacuum-to-vacuum normalization factor that I will explain later.

Then how to compute this? *Slide: "Recall that ..."* Then we are doing a simple model, but the operators are complicated, and so the correlation functions are not trivial.

The statement is that the leading term of the short-distance asymptotics of the two-point function is very simple if the one-point functions are given by the following expression.

If I introduce logarithmic derivative *Slide: "We have..."* then I get the Painleve equation. This equation has the property that there are two regular singularities — this is a family of equations, for parameter $\theta = \alpha_1 - \alpha_2$ — and there might be other poles but there are two singularities that do not move. Then you can compute the asymptotics in a straightforward way. **see slide** This is second-order differential equation, so to get complete asymptotics I need two constants, r and s. It's not hard to see what fractional degrees the asymptotics will contain. The first few terms will be very important to us.

So what do we want? We want the short-order asympyotics. The honest way is to compute the actual integral, but this is very hard. It's ok if you know what you want to get. But we are mathematical physics, so our goal is not to prove theorems, but to get formulas. **Kolya:** And to understand structures. **Fedor:** In mathematical physics, there are things that are called "theorems", but they are very bad theorems. Like, for example: entropy grows. This is anything, but a mathematical theorem. Instead, our goal is to guess the asymptotics.

Slide: "Euclidean case"

We decided that the the short-distance asymptotics is equivalent to the Euclidean case, and so we do some sort of "Wick-rotation". If this is correct, then formally the two-point correlation function we would write some sort of functional integral. And in Euclidean case you can really make sense of the functional integral, on the contrary to the Minkowski case where all you have is oscillating integral and stationary phase. **eqn from slide**

One of the most important things in modern mathematical physics is the analogy between Euclidean qft and classical statistical mechanics. If you consider classical statistical mechanics, you define a partition function — it is the first thing you want to compute. Then you would say that D-dimensional qft is similar to D-dimensional statistical mechanics if you identify action with a hamiltonian. But in statistical mechanics, it's usual that your system actually is defined on a lattice. For example, think of Ising model. Then if you have on this lattice two observables (for example, you fix two spins), then such a correlation function normally will decay exponentially with the distance between the two points.

Kolya: Why? **Fedor:** If you consider Ising model, then you have $H = -J \sum \sigma_i \sigma_{i+\epsilon}$, where σ_i is a ± 1-valued variable at each point i, then obviously the minimum of this Hamiltonian is when all the σs have the same direction. And this will be realized at 0-temperature. When temperature grows, other configurations will start to contribute. But it is still that the correlation between two spins that are rather small. At every finite distance, there is still some order, but for positive temperature for large distance the order doesn't happen. **Kolya:** You assume that the volume is infinite? **Fedor:** Yes. Well, no. Except in some degenerate cases, we assume that for large lattice nothing depends on the boundary.

So this $\xi(T)$ is a correlation length that depends on the temperature. It can be 10 sites, or 20 sites. But then it doesn't make sense to take continuum limit. Because you want to consider the case on large scales, and then there is no correlation. So that's why naively there is no way to take a continuous limit from the lattice model of statistical mechanics. Except! There is one important exception.

Question from the audience: What you might mean to say: at 0 temperature, there is analogy, but not at finite temperature? **Fedor:** No, at 0 temperature it is even worse. The analogy is at critical temperature.

So, at 0 temperature, you start with all the feromagnets turned down, but as you increase the temperature, some of them start to turn up. And there will be, probably, a critical temperature when the substance looses its feromagnetism. And at the critical temperature, the scaling distance diverges. Then correlations decay following a power law, not exponentially, and so then we can take a continuum limit. Also, in this case, the boundary conditions matter. **Kolya:** That one slide contains a good book of material.

Slide: "Lattice model ..." Suppose we are very close to the critical temperature, but not exactly at it. Then there will be different scales. At some scales, the theory will be already critical, because the distances will be smaller than the correlation length. Our real goal, the distances that are smaller than the correlation lengths, we rescale something, and then the correlations will already decay exponentially, as is typical for massive Euclidean qft. But most important, there is an intermediate region — this is another theorem, but it is not very good, and it is what we believe. **Kolya:** It is a conjecture, and one of the last Fields Medal was given to Stansilas Smirnov for showing that at a particular point on this lattice in the Ising model, the limit is massive (or conformal) field theory. **Fedor:** Yes. But there is a very important field theory which is continuous, called conformal field theory, and it is between the two things, massive field theory and the critical lattice model. The massive ft is on slightly larger scales, and CFT is supposed to describe its short-distance behavior . On the other hand CFT is supposed to describe the long-distance behaviour of the critical lattice model.

A few words: *Slide: "Conformal field theory"* We hope some day to describe better the conformal field theory. But a few words about it. First of all, general in Euclidean field theory there is important observable called "stress tensor". Suppose there there is some background, some metric on the plane on which you put your model. So you deform the lattice. Then you can differentiate with respect to this metric, and you expect to get some very special local observable that describes the response of your theory to such metric deformations. (I use complex coordinates).

Then generally you must satisfy

$$\partial_{\bar z} T = \partial_z \Theta$$

Kolya: These are Noether equations. **Fedor:** Yes. Note: the slide is wrong.

So then by definition of conformal field theory, it is when the trace Θ of energy-momentum is 0. So this has huge consequence: it means that T is holomorphic function, and $\bar T$ is antiholomorphic.

Slide: "Operator product expansions" Now every other operator in conformal field theory is independent of rescaling, up to scaling dimensions $\Delta, \bar\Delta$. Then in these conformal field theories, in many cases at least, you can describe actually all the local operators — you can count them somehow. Then suppose you have two observables, one at this point and one at that point, then you can describe all the observables, and so you expect an operator product expansion. **Kolya:** This is asymptotic expansion? You need the power series in z. **Fedor:** Definition of conformal field theory is that this series are convergent, in the weak sense in qft: you put it inside an expectation value.

So, how do I count these operators? I count them according to highest weights of two Virasoro algebras, for each chirality. You start acting, and you create all the fields.

Slide: "Perturbed CFT" So now, to return to our picture, we have CFT, and then massive FT outside. And we switch the point of view, so that massive field theory is a perturbation of cft. We write the action for pcft as follows **see slide**. Then for renormalization you ask the total scaling dimension $\Delta + \bar{Delta}$ to be less than 2. See the book "Renormalization and phase transitions." Then the uv corrections and so on, they will only change the coupling constant g, they won't add new constants. Since I don't want to break rotation invariants, I put the same constants here and there, and so g must scale as this dimension.

Then the first idea for doing this perturbed field theory, the perturbation will lead to massive field theory, and so you want to compute something like the two-point functions, and you write a series filled with original cft correlation functions that you can compute, by pushing the exponent down. **Kolya:** If I were being complete pedantic, I would correct the last line on the slide $g \rightsquigarrow -g/2\hbar$. **Fedor:** Well, $\hbar = 1$, but $-g$, yes.

Then what can happen is that in the limits, some of these terms might diverge. You use renromalization theory.

In this case, it is not too bad. The idea of how to get out of the difficulty is in a nice paper by Zamolodchikov. **Kolya:** This is not Sasha Zamolodchikov, of KZ equations, but his twin brother.

Slide: "This is wrong because ..." So, let's assume that we are not looking for complete description of correlation functions, but of short-distance description. Then what we need is this formula **from slide**, and if you think about the model form point of view of cft, you take the massless particle in cft and then perturb it by massive term. Then constant g is essentially just this mass.

Then you see that from that point of view this expression is in drastic contradiction from what we may have from the expansion from the previous slide. Because one of them has g^{2n}, and the other has fractional exponents. The true asymptotics must contain these fractional exponents, so what Zamolodchikov explains, is that the true thing does have the correlation functions. The point is that it is very naive to assume that asymptotics have power series in g expansion, but rather in a fractional power of g. **Question from the audience:** I always thought you had UV cut-off, and this changes it. **Fedor:** We do not have UV divergence in this problem. UV divergence is inherent in qft, because we don't have short-distance description. But IR cut-off is 1 meter, and we know what happens at 1 mater, so if you have IR divergence, you're doing it wrong.

Ok, so we decided that in conformal field theory, we know how to count the operators. How do you do it in massive field theory? The idea is, there are the same numbers. If you have no divergences, still you may have finite counter terms, so it seems that the operators are not defined uniquely, but if you assume that all the operators are all incommensurable (irrational with respect to each other) then by dimensional reasons you can eliminate all these counter terms, and there is one-to-one correspondence between cft before and after perturbation.

So then you can use these multiple integrals from last time to change the OPE. In the new expressions, these Cs aren't just constants, but they are functions in g, given by formal power series expansion, and the coefficients, I won't explain, but they can be regularized. And cft is powerful enough to provide this type of OPE for the perturbed theory.

So then, what was wrong with our computation of 2-point function? If I want to compute it, it reduces to computation of 1-point functions, and these guys are not defined by cft. This is the data that is out of reach of cft, and they must contain other powers (non-integers) of g. You see, in cft on the plane, all the one-point functions for all operators (except the operator 1), are 0, because you can always rescale $O(ax) = |a|^{-\Delta-\bar{\Delta}}O(x)$, and if you put here 0 for x, there is a contradiction. So there are no 1-point functions, but after deformation you can get 1-point functions. How? This is clear. Look at Ising model. At 0 temperature, all the spins are pointing down, so average value of spins is -1. Then as you increase the temperature, at critical temperature all of a sudden the average spin is 0. But now start at cft = critical temperature, and decrease the temperature, and you see the creation of expectation value of 1-point function.

So, now, in computation of 2-point functions, you need in addition the 1-point function data. And these depend on the background — they depend on the IR boundary of the theory. They are not universal to the theory, but depend on the geometry of your theory at ∞.

Kolya: Usually when you read textbooks in cft, it explains that it is entirely determined by OPE. But actually this is not true, it depends on the 1-point functions? **Fedor:** Yes. When I said that 1-point function is 0, I meant on the plane. On a genus-g curve, the 1-point functions encode the moduli of the curve.

Slide: "The structural functions ..." So Then the 1-point functions are as $g^{\#}G$, and this is the cause of non-analytic behavior, and this is in perfect agreement with our earlier story. On the other hand, there is a generalization, in which we can put the system in some external geometry. For example, the infinite cylinder with circumphrence $2\pi R$. Then $gR^{2-2\Delta}$ is dimensionless, and so G_k

is some function of this dimensionless number, and in general you cannot say anything about this function.

BREAK

So now we want to connect this up to our favorite model. Namely, we want to apply this procedure to sine-Gordon model. Often, when you do qft, you often hear that you should impose certain regularization procedure to keep Lorentz invariance. For us, we will be more open: we are on the lattice, so we have already broken invariance. For us, what is important is the integrability condition.

Slide: "Example. Eight-vertex model"

Here is a simple theory. At each edge of the lattice you put an arrow, and then you consider the so-called "eight-vertex model". Then the action is local: $e^{-\beta H} = \prod e^{-\beta H_{ij}}$. The great difference with six-vertex model is that we do allow all arrows to go in or all to go out. See, there are 16 configurations, and there in general is no integrability, but with the partity retained it is solvable.

Slide: "We consider..." It is convenient to parameterize a, b, c, d by elliptic functions. This is not the most general case, but suffices for us. Then the "temperature" is this k, and at critical temperature is $k = 0$, and at this temperature $d = 0$, and the model becomes six-vertex. Then also we had the scaling theory, and we rescale the lattice simultaneous to $k \to 0$, and then the scaling theory is precisely sine-Gordon, which is cft with central charge equal to 1.

So, now, what we want to do, strangely, I want to start with this critical lattice model. Our goal is to compute the one-point functions, and these functions we have no hint how to compute them from cft, so the computation must be based on something else, from integrability. So I want to think about this critical model, which is exactly solvable, and think how to compare it to cft.

Slide: "Expectation values for six vertex model" I want to think about this on the cylinder, and hope to reproduce the results on the plane as the radius goes to ∞. How can I mimic the insertion of local operator on this cylinder? What I do is, on this ring, I sum up everybody here with usual Blotzman weights, but on this horizontal line, I want to put something else. Then most importantly I put some breaks, and fix the arrows, and I compute with these fixed boundary conditions and some disorders. Then I hope that I can find a scaling limit where this thing will reproduce correlation function in cft.

I will explain this later, but my hope is to have a lattice version of cft, and then I can compute everything with 1-point functions. See, Kolya correctly said that in cft on the cylinder, then the 1-point function is not zero, because there is additional scale. But what I want to mimic here is that 1-point function will go to 0 as the radius of the cylinder gets bigger and bigger.

All this is based on five papers which are written with Boos, Jimbo, Miwa, and Takeyama.

I want to mimic a cft, which has main property is scale invariance. An operator which is in 2,3,4 sites, fedorcannot be scale-independent, so I need to take combinations in clever way to have a conformal limit.

Slide: "Exact definition." So these are combined into R-matrix. The weirdest thing is that scattering matrix in two dimensional and in statistical mechanics are related to same mathematics. So I have R-matrix, and formally I consider two spaces.

Slide: "Introduce..." Then I take product of R-matrices, and I get a thing defined on a lattice. *Slide: "Our main object ..."* Our partition function has the following formula **from slide** where $S(0) = \frac{1}{2}\sum_{j=-\infty}^{0}\sigma_j^3$. Then this local operator, which lives on the finite number of sites — on a tensor product of finite number of $\mathbb{C}^2$s, and so I take traces, and that is the meaning of the formula. It is the most general thing I can imagine, and it is what I want to compute. I normalize it by partition function without insertion of the operator.

Now, you see, what happens. We have this operator, acting in Matsubara direction, and it is repeated infinitely many times, and it is called "Matsubara transfer matrix". It can be shown in rather wide conditions that this matrix has a simple spectrum. And that's why obviously if I repeat it many many times, it will reduce just to the projector onto the maximal eigenvalue. So that's why what is interesting to me are the maximum eigenvalue of this matrix to the left with one twist and to the right with a different one. So that's why there are these two important eigenvalues, one maximal to one and the other to another. And I assume that they are not orthogonal. This is not too big assumption, because it is generic condition.

So I have this massive object, these operators. And I want to have something like in cft, where I can organize them somehow. I want to obtain them as module of some algebra. In cft, I have primary fields, and then act by Virasoro, and I obtain all the fields. So now I will formulate the results of our construction. I hope I will be able to explain the construction itself, but I'm not sure.

There is a purely algebraic construction, based on some rather complicated application of quantum groups, and the result is the following. First of all, by definition, the spin of this operator — *spin* of an operator is \$$\mathcal{O} = [S, \mathcal{O}]$, where $S = \frac{1}{2}\sum_{j=-\infty}^{\infty}\sigma_j^3$. Then it is very important to introduce other operators with tail $\alpha - s$ and spin s, and consider direct sum. This is very important in cft, where you have operators that act not in one Verma module, but between them. So in this space, I introduce annihilation and creation operators. These are operators acting on operators, not on states.

So, what are these operators? *Slide: "These are one-parameter ..."* I cannot explain. But they depend on spectral parameter. The main tragedy of our work is that all essential quantities depend not on ζ but on ζ^2. Anyway, at special point $\zeta^2 = 1$ is when R-matrix becomes the permutation. So in general, the creation operators, they're Taylor series going infinitely in in the positive direction. The annihilation operators are Laurent series going in negative direction. Then these operators act on my space of quasi-local operators, which I will explain soon in more details, but they act on this space $\mathcal{W}^{(\alpha)}$. Then this guy is in the center of the algebra.

Slide: "The rest of the operators ..." And these bs are fermions, and here are the nontrivial anticommutators — the rest are zero. And every component of them act on these spaces, for example **see slide**

Now, take any of these two b_p, c_p annihilation operators . For any quasi-local operator you have a notion of length: it is the number of sites where it is really nontrivial. Then for any x, these guys acting on x will vanish if p is greater than the length of x. Because annihilation operators make the local operator shorter.

And these creation operators, essentially what they do is to take an operator, and make it longer, but they do this in a controllable way.

Then there are several theorems, and since I didn't not explain how I construct these operators, they I cannot supply proofs. Actually the operators t_p^* correspond to the adjoint action of the local integrals of motion, in particular t_1^* just shifts along the lattice. If I put arbitrary number of t^*s , and then b^* and c^* in the same number, and act on this primary field, then this will be a basis of the local operators. So, up to now we are very happy. And then the point is that we want to compute this functional Z^κ on this space of operators which are created from primary fields by applying t^*, b^*, c^*. And our theorem is that if you apply t^*, it is the same as multiplying by this function ρ which I will explain. And if you apply b^*, there there is an integral, and a curve Γ in $\mathbb{C}$ which goes around 1 and ζ^2 is outside — then I trade creation operator and annihilation operator. And this of course allows me to compute Z^κ on every operator created in this way.

If I act by these fermions, then for example I have $b^*(\zeta_1)c^*(\zeta_2)$. First I apply one, and the b^* turns into $c(\zeta)$, and then it commutes with c^* and so using these three identities I can compute Z^κ on any quasilocal operator obtained from primary fields by acting by my algebra. This is very like in conformal field theory: you have primary fields, Virasoro algebra, Ward identities, and you can compute any descendent correlation function, e.g. 1-point function on the cylinder, which is the same as 3-point function in cft, and you can compute any descendent and express the result in terms of 3-point functions of primary fields.

Slide: "Since…" Since we have these relations, I can obtain the following formula that if I take some number of t^*s, some number of b^*s, and some number of c^*s, then I get the formula **from slide**. And at this point, we get very excited, because look, this is exactly as in cft. I am trying to do the same thing as in cft but on the lattice, and then I understand that if I am trying to compute qft, then I don't just have operator at site 14, but a sum of operators. So I need to compute not just one operator, but a sum of them going to ∞.

See, I want $\sum(\zeta^2-1)^{p-1}\mathcal{O}_p^*$, and this will not just be one operator, but a sum of them, with length controlled by the parameter (ζ^2-1).

Slide: "Switching on the Matsubara direction." **skipped about 20 slides** So for the moment I will just say that there is an algebraic construction of these operators, and if you compute it — some brave people computed it up to 10 sites, but let me instead explain something about the functions. First, this function ρ. We decided that we have these two transfer matrices **circle with an x or an o**. Then we have maximum eigenvalues for $T(1) = T_{\mathrm{Mat}}(1, x)$. But actually they come in commutative families, by Yang-Baxter equation. So if I have an eigenvalue of $T(1)$, I have it for every spectral parameter. **Theo:** As spectral parameter changes, the maximal eigenvalue changes. **Fedor:** Yes, of course, but I take the maximal eigenvalue of $T(1)$, and ask how it changes in ζ. That defines ρ **see slide**.

Then the way we prove this theorem, it has some analogy with algebraic geometry, or more precisely quantized algebraic geometry. You know, in classical case, algebraic integrable models is a branch of algebraic geometry of Riemann surfaces.

So that's how we characterize this function ω. Suppose you have a Riemann surface Σ of genus g. Then you have two bases in cohomology $a_1, \dots, a_g$ and $b_1, \dots, b_g$. Then you construct a differential form ω on $\Sigma \times \Sigma$, by asking that

$$\omega(x,y) = \left(\frac{1}{(x-y)^2} + O(1)\right) \mathrm{d}x \mathrm{d}y, \quad \text{and} \quad \int_{a_j} \omega(x,y) = 0$$

and then it has remarkable thing from Riemann bilinear identity that it is symmetric You can write it in terms of Riemann theta-function:

$$\omega(x,y) = \mathrm{d}_x \mathrm{d}_y \log \Theta \left(\int_x^y \omega + \Delta \right),$$

where Δ is the Riemann constant.

So then the way how to prove these identities, how do you prove in the classical case? You check that the left hand side vanishes when you integrate, and you look for the singularity. This is more or less the way how it works here.

On the other hand, *Slide: "Destri-Devega equations"* there is another way of describing this function, which is very useful for the cft limit that you want to do. Let me say a few words: I consider the eigenvalue of $T(\zeta, \kappa)$. In addition to the usual transfer matrix, in Matsubara direction I have n sites, and at every site I have $\mathbb{C}^2$, and then let me introduce one auxiliary $\mathbb{C}^2$, and then I consider $\mathrm{tr}_a\, T_{a,M}$. Then there are other transfer matrices that you can consider, and for all representations they commute, and in particular for q-oscillator, the transfer matrix is call Q. Then the two operators, they satisfy this equation, called "Baxter equation". They are polynomials in ζ^2 for both of them, and $Q(\zeta)$ (well, it's not exactly a polynomial, but it is premultiplied by $\zeta^{\kappa - s}$), and this completely characterizes the spectrum.

Now, it is convenient to introduce the following function, just the ratio of these two guys. Then obviously $a(\zeta, \kappa) = -1$ corresponds either to zeros of Q or of T. Using this, one can easily derive following equation, and it looks like complete tautology, but it is important equation.

Oh, we did not say:

$$\Delta_\zeta f(\zeta) = f(\zeta q) - f(\zeta q^{-1})$$

$$\psi(\zeta, \alpha) = \frac{\zeta^\alpha}{\zeta^2 - 1}$$

And importantly, the contour goes *clockwise*.

Slide: "Introduce" Then I define another function and the convolution $\star$. Then I define a resolvent with kernel K_α, and this is a type of integral equation for this "dressed resolvent." Then there are two kernels using this operation. *Slide: "The function ω"*.

So everything actually is defined, and we have reached many goals at the same time, because first of all we define this local operators living in the space direction, and their definition has nothing to do with Matsubara. And now I put it on the cylinder, and compute the partition function with this insertion, and the result is expressed by this function which is completely defined by two eigenvalues in Matsubara. So the analogy that I explained with OPE, which depends only on local properties, and the 1-point function depends on global things. The analogy is transparent here: on the lattice we have the same over all behavior. Next time I will explain these things.

Lecture 4: 4 October 2011

Slide: "Our main object..." **Before the lecture begins, Fedor also draws on the chalkboard a picture that appeared on the slides previously. I will try to describe it. You take a square lattice, with periodic boundary in one direction so that it is a cylinder. You then snip a few of the edges in a row, and also add some decorations. Oh, just look at the slides.**

Let me repeat what we did last time. We consider for any operator an expression **from slide**. Here $\mathbf{n}$ is the number of states in the Matsubara direction, and S is more or less infinite — we start with a trace, and then take the limit as the time direction becomes infinite. Then we suppose that κ is the maximal eigenvalue at one end, and $\kappa+\alpha$ at the other end, with eigenvectors $|\kappa\rangle, |\kappa+\alpha\rangle$, and we suppose that $\langle\alpha+\kappa|\kappa\rangle$.

Slide: "These are" And so, these creation operators, we consider them as power series in the parameter (ζ^2-1). Creation is supported in positive powers, annihilation in negative, and $\mathbf{t}^*$ lies in the center. Then $\mathbf{b}, \mathbf{b}^*, \mathbf{c}, \mathbf{c}^*$ are Fermions. And important thing is that the modes act between certain prescribed spaces, which control the length. We have an estimate of how much the lengths can change **see slide**.

Slide: "Among them..." Then $\mathbf{t}_1^*$ is a special shift along the lattice, and if I act by the "primary field" in this way (number of $\mathbf{c}^*$ and $\mathbf{b}^*$ is the same, because one of them raises and the other lowers the spin), then this is a basis of this space.

Kolya: You are constructing the Fock space of these operators. Do you actually prove that as $n\to\infty$, these transfer matrices produce these operators? **Fedor:** No. These are local operators of the type $q^{2\alpha S(0)}\mathcal{O}$, where $S(0)=\frac{1}{2}\sum_{j=-\infty}^{0}\sigma_j^3$, and $\mathcal{O}$ lives on finitely many sites. **Kolya:** Are you constructing these operators as in the continuum case, or do you start with the lattice? **Fedor:** I start with the lattice, and then I consider a space of operators, and then I consider operators on operators. **Kolya:** This is somewhat different from CFT, where you constructed things from axioms? **Fedor:** It is exactly like cft. This is analogous to the Virasoro algebra. **Kolya:** In cft, you do not have microscopic description of the fields. **Fedor:** No, in that way it's different. I want cft in the limit.

Ok, so for these operators, I have purely algebraic construction, and you can compute on the computer (up to ten sites) how they act. The important thing is that if I take my partition function on this operator, then it is given **as on slide**. And so that's why I'm able to compute this function on every operator of the type **drawn on board**. So this you can develop in $\zeta^2 - 1$ and obtain all the operators, and they expand from the "generating function" formula. Then $\rho =$ ratio between two transfer matrices **on slide**.

Slide: "Destri-Devega equations". The function ω is more complicated. I introduce Baxter operator, and then — my logic is as in qft, where I did something locally, and then the 3-point function is defined by global data — and it is convenient to replace Bethe ansatz with a certain *Destri-Devega equation*. **Kolya:** But Q is an operator. **Fedor:** For the eigenvalues. Γ goes around all the roots.

Slide: "Introduce" Then the point is how I define ω. I define a convolution with a certain measure, and then a resolvent and some kernels, and *Slide: "The function ω"* ω then has the following expression. Note: it contains Δ^{-1} so it is transcendental function. The first part is nice function of ξ, ζ, but the other part is transcendental. **Kolya:** Nice means? **Fedor:** It is of form $\zeta^\alpha \xi^{-\alpha}$ times a rational function of ζ^2, ξ^2, with poles only at 0s of $T(\zeta)T(\xi)$.

So the point is, if I want to define operators which are nice in the conformal limit, I cannot take operators that live on ten, or fifteen sites. Rather, I must take operators that are sums, and live on all sites, so and that they are nice for conformal rescaling. So I have now realized my dream: the operator $\mathbf{b}^*(\zeta)$ exactly rescales with $(\zeta^2 - 1)$, and this is what I want. Moreover, this is supported by the fact that if I take stupid limit as $n \to \infty$, where n is number of sites in the Matsubara direction, then the DD equation becomes linear equation and you can solve everything to $\rho = 1$ and $\omega = 0$, and this supports the analogy with cft, because in the cft on the plane you want all the descendents to be 0.

Slide: "The parameters…" What I want to do? I want to identify the 3-point function in cft as: at the two ends of the cylinder are boundary conditions κ and $\kappa + \alpha$, and I insert α primary field and its descendents somewhere. But this is somehow stupid and we hate it that $(\kappa+\alpha) = (\kappa)+(\alpha)$, and I would like to emancipate this last term. This is for $c = 1$ cft. I would prefer to perturb it, in sine-Gordon theory, where I split cos into two terms, and perturb one of them **as on slide**. Then this is conformal field theory with central charge $c = 1 - 6\frac{\nu^2}{1-\nu}$, and this is a nice way to understand sG theory.

Slide: "Important generalization" You remember, there is a nasty transcendental piece in the theory, and we can introduce other operators by applying a transformation, and then there will be no transcendental piece for the new guys. These are nice in the sense above, and important property which was used when I was deriving the main determinant formula is that when the operator acts on $\mathcal{W}$ it behaves as **on slide**.

Slide: "Changing boundary conditions" If I want to mimic this going from $c = 1$ to $c < 1$, then I want to mimic Feigin-Fuchs-Dotsenko-Fateev construction. What you do is introduce a charge at infinity, and compensate it by introducing screening operators. So I introduce $Y^{(s)}$ in Matsubara direction, and it immediately breaks the balance of spins, and so X must have spin. So how to understand this? X has s more $\mathbf{c}^*$s than $\mathbf{b}^*$s. But then the rational guy has negative modes, which we call the "screen", and this makes it impossible to prove our main theorem. But this is good rather than bad, because we now define new functional with "screening" operators. And since this is a finite Grassman algebra, and I apply one more operator, all the negative modes will go. So for this generalized function, the same way of computation is possible. *Slide: "The determinant formula…"*. The only thing is that the maximal eigenvalues, they are usually of spin zero, and now they are of spin $-s$, and so ρ will change, but nothing else does.

Now if I look for example on this Q operator, it is generally of the same form, and as my Baxter equation if I take this eigenvalue, then formally the equation looks the same if I replace κ by κ'. This is not exactly the same thing, because for spin 0 the eigenvalue I have $n/2$ roots, and now I have $n/2 - s$. But remember, when I defined this functional depending on s, I start to get more or less the same thing as if I change κ'.

Ok, so what do I want. *Slide: "Back to CFT"*. I am sorry for the strange parameterization, but I return to cft with central charge $c = 1 - 6\frac{\nu^2}{1-\nu}$. So take this cylinder and insert somewhere a local operator of spin 0. *Slide: "Along with …"* Then the Virasoro algebra acts by this formula **slide, with picture**. And by ope expansion, I have **missed**. We call this thing (around the point x) the "local coordinate", and in the Matsubara direction "global coordinates". *Slide: "In global coordinates"* In each coordinate system I can write energy-momentum tensor. Now you can using either conformal Ward identities or if you prefer representation theory language, whichever your preferred way is, whatever you do first of all there are primary fields, and then the boundary conditions correspond to the asymptotic conditions **see slide**. *Slide: "Using all …"* Take arbitrary descendent of the field $\phi(\alpha)$, and compute this ratio, it is a pure algebraic operation so I don't give much details, but I get these expressions **from slide**.

Slide: "Integrable structure of CFT". Now I should say a few words about integrable structure of cft. This is what we hope to understand. Normally we create local operators by Virasoro algebra, but integrable structure — it is related, but it is different — the one I describe is prepared for perturbation like in sine Gordon. So there is an infinite set of local densities, and they get more and more complicated, and for every density you associate a local integral acting on local operator, by taking the density and integrating around the local operator. **see slide**. But since you are on the cylinder, there are other integrals over these global cycles, and the first nontrivial integral is $L_0 - c/24$. **Kolya:** You call them integrals because they commute? **Fedor:** Yes, although the densities don't.

Slide: "The primary fields …" Now, suppose I want to apply to a local operator and compute my favorite functional. First, I should say that the primary fields are eigenvectors. Then I get difference of two eigenvalues **from slide**. **Kolya:** I understand this formula if everything is given by T-ordering. **Fedor:** Yes, then it is just deformation, and this is a good way to think about.

And it's not very surprising that this formula looks exactly as how our operator $\mathbf{t}^*$ acts, because on the lattice $\mathbf{t}^*$ is already commutator with local integrals of motion.

So let's take the basis in the Verma module like this: **see slide**. Then one can argue that there are no relations, and this is true.

Slide: "Scaling limit …" So now we ask how to make the scaling limit in this picture? In qft, it's hard to decide what is direct limit of operators. It's better to do it for physical quantities, like this functional Z. So I will think about scaling limit in the Matsubara direction. I consider $\nu \in (\frac{1}{2}, 1)$, and $\nu = 1$ is classical theory. Then Bethe roots for small κ start to be distributed densely in the positive real axis. I am interested in some piece that is rather close to 0. In such a piece, the roots that are not just next to zero — rather big, but still close — behave **as on slide**. n is Matsubara direction. Horizontal direction is already infinite.

Slide: "The following limit exists". Then I start to construct the following limit. I introduce a parameter a, with na a fixed number $2\pi R$. So as $n \to \infty$, $a \to 0$. Then the roots, I want to keep them finite in a new parameter. So these limits exist, and they still satisfy the Baxter equation, except that there are some multipliers that go to 1, so it becomes more universal. The important thing: I told you that when I modify ζ, I introduce s boundary condition, and it satisfies same equation as if $\kappa \rightsquigarrow \kappa'$, but only number of Bethe roots changes, but I don't care because this number is infinity anyway. So now I have very symmetric picture: this is analytic continuation in κ, and now I can consider κ, κ' as free parameters. Ok, you say, they still differ by integer number, but all answers are analytical.

Slide: "Similarly…" In the same way, the following limit exists **from slide**. Then the question it, what can be said about the following scaling limits? This T now — it used to be a polynomial for finite f, but now it becomes an entire function with 0s concentrated on $\mathbb{R}_{>0}$, and it has the following asymptotical behavior **as on slide**. The coefficients are exactly the eigenvalues. If I have a field $\Phi_{\kappa+1}$ and act by I_{2k-1} then I get $I_{2\kappa+1}(\kappa)\Phi_{\kappa+1}$. So in cft you have expressions for this local integrals of motion, and you compare them, and both formulas are ugly but they are the same thing asymptotically.

Important thing here is that the parameter λ is such that $\lambda^{-1/\nu}$ has dimension of Length.

Slide: "…" And then, as for my function ω, for the moment the only thing I'll say is that one can prove that such a function exists **from slide**. Now our logic is completely inverted. We see that in the Matsubara direction, all of our functions ρ, ω there is a nice limit, but our operators were defined nicely on the lattice. So the way to define them in the continuum limit is to follow the same argument. So what we do is, to perform the scaling limit in the horizontal direction at all, the limits exist, and we see immediately that the asymptotics are **as on slide**.

Then the question is: from these asymptotics, I decide that they define some operators. I want to understand what they are doing.

Slide: "The Verma module …" I decide that in the fermionic Verma module, they act in **way on slide**.

These are words, but we want formulae, not words. So we want to check it some how, when does this mess make sense? We can check when $\kappa = \kappa'$, because then you don't care about the $\mathbf{i}$s, and also in conformal case for technical reasons it is easier. And at the end it will be important for some more serious problems. You see, if I act by this $\mathbf{t}^*$, I produce a ρ, and it also contains the operator of shifts along the lattice, so if $\rho \neq 1$ then I don't have translation invariance, and for many reasons I would like to have it. So for the moment let me say, I compute what I can. In

$\kappa \neq \kappa'$ case I cannot compute it.

BREAK

Kolya: Let's take a step back. What do you want to show in this scaling limit? **Fedor:** I take this cylinder, and I assume that there are scaling limits for the operators $\mathbf{b}^*, \mathbf{c}^*$ and so on. **Kolya:** But, without details. **Fedor:** I want to make a 3-point function on the sphere.

So, fortunately when $\kappa \to \infty$ there is a nice limit. It looks strange — everything depends on $q^\kappa = e^{2\pi i\kappa}$. But we take analytic continuation. Then the smallest root λ_1 of Q, it goes like κ^ν with some constant that can be computed. Then you can introduce a parameter t such that the minimal root corresponds to 1.

Slide: "Long computations..." Then starts a very complicated investigation, and this is a great piece of mathematical physics that I was very surprised we were able to manage it, but the result is **as on slide**. It is a kind of Melin transform. This is asymptotical formula in κ. The $\tilde{S}$ is given by ratio of Gamma functions, and the Θ you can obtain order-by-order (in $1/\kappa$) expansion, and we can go up to about 10 with the computer easily.

So I want to talk about the first two orders. How the function behaves at $\lambda = \infty$? I just have to close my contours to the upper half plane, and pick up all the residues that I have. The residues come only from some of the Gamma functions, and they will be like $\lambda^{-(2j-1)/\nu}$. Then if I want to compute the first coefficient, I have to put both $l, m = i/2\nu$ (the first pole), and the property of the expansion is that all other terms vanish at this value. *Slide: "All together ..."* Then I find that there are some constants, which are residues of some Gamma functions, and an expression that is exactly what we had before: I take $\ell_{-2}\Phi_\alpha$ on the cylinder, with κ at each end. This is quite amazing formula, and was the best day of our last two years. Then we have these funny constants that depend only on α, and the other part is polynomial in κ. So we conclude that we have a formula **as on slide**. Why modulo $\mathbf{i}$? We do not control the left action of integrals of motion, because of our choice of geometry.

Slide: "Generally," So then we come to a conjecture, which is checked up to level 8. **formula of slide**. By "even" and "odd" I mean how it transforms under changing β to γ. If you are very careful, you will say: wait a moment, the number of descendants grows as the number of partitions, and this is number of unknowns, whereas number of equations only grows linearly. So if I go to my favorite case where I put primary fields, then it is overdetermined up to level 6 and underdetermined at level 8. But actually I put some extra boundary conditions, so that I get overdetermined system that I can solve.

So in some sense we are rather happy now, and indeed our fermionic basis goes from the lattice to the cft, and we can identify step-by-step its action with the action of usual Virasoro generators. The interesting thing is, what's this constant about? It consists of the Gamma functions **from the previous slide**. If you know the literature, you will see that this is very similar to what you have in sine-Gordon 1-point functions.

First of all, we were doing the following scaling limit. We want to rescale around some small eigenvalues, but we can do oppositely and try to keep the big eigenvalues. This is the second chirality.

Slide: "Returning to sine-Gordon model."

So now we return to our main problem, the sine-Gordon model. I told you that generically if you have perturbation of cft, you can write the ope. In particular, for sG model, if I take exponents the way I like them, then you have **formula on slide**. Then, dropping the integrals of motion, I have to compute **fraction from slide**. For the momemt, I will compute for $m = 0$, because for the moment it's the only one I know how to compute. Then I will say how to shift.

Slide: "Mimicking sG model on the lattice." This is more or less the same construction. The honest way is to say that sG model is made out of 8-vertex model, but it's hard to work with 8-vertex model, because if somebody seriously works in 8-vertex once in his life, as I did, then he will never return. Because of the mess of elliptic functions, you go crazy. So instead, you try to make 6-vertex model inhomogeneous, by putting different spectral parameters. Then just consider the same functional as I have before, but replacing transfer matrix by an alternating one. Then there is no problem to introduce creation and annihilation fermions, and they naturally split into two pieces. **see slide**.

Slide: "Local operators ..." As for the creation operator, I make once again a small Bogolubov transformation. Essentially, what I do is to take Taylor series around the points $\zeta = \pm\zeta_0^{\pm 1}$. Why I do that? I have two sublattices, one for ζ_0 and one for ζ_0^{-1}, and each is homogeneous but they interact. I want to change them a bit so that as $\zeta \to \infty$, then I have some analytical functions in ζ_0. So this is the reason for the transformation.

Now I again consider the scaling limit. I introduce some a, and take a limit as the massive soliton $M = 4a^{-1}\zeta_0^{-1/\nu}$ is fixed. *Slide: "We get rid of ..."*. Then these operators have limits. These are two chiral field theories.

Slide: "Thermodynamical functions." We want to compute already for sG our functions ω and so on. Then the DDV equation becomes **as on slide**. *Slide: "Introducing"* Then function ω is also easy to compute, and is given by this Mellin transform **from slide**. This function Θ is actually satisfies a compact integral equation, and in mathematical physics to reduce to compact integral equation is to solve the problem./

Then there are poles, and you can close to upper half plane or lower half plane, and get coefficients either to $\lambda^{\pm(2j-1)/\nu}$.

Slide: "Final result." If you do all of that, you get the following formula. You get the 1-point function, which is obtained from two chiral sets of fermions, and we know from previous how they are expressed in terms of Virasoro generators — the formula expresses this as some kind of determinant. **see slide**. **Ty:** This might be a silly question, but I know that sG is a bosonization of Thurin model. Is there relation between the lattice fermions and the Thurrin fermions? **Fedor:** No. My fermions act on the space of operators, but even there they have integer spins.

So then you have very nice and simple formula for this vacuum expectation value. As $R \to \infty$, so that we are on the plane, then the first part drops out and everything is given by determinant of

theojf@math.berkeley.edu

diagonal matrix. You can check it in all known cases, and it works perfectly well. The known cases: there are several of them, but for example for $R \to \infty$ the simples descendent in $\langle \ell_{-2}\bar{\ell}_{-2}\Phi_\alpha \rangle$, and this was computed by Fateev, Fradkin, Lukyanov, Zamolodchikov, Zamolodchikov, and for that it agrees.

Then there is one more unpleasant thing. For the moment, I computed only a part of this series **from earlier slide**. I had these shifts, and I computed for now only the ratio of descendencet of Φ_α by Φ_α itself. So I should explain how to shift. And how to shift, it needs some more sophisticated knowledge of cft, but I can give a result.

By definition, for any descendent the number of elements for I^+ and I^- must coincide, and same for $\bar{I}$. But if you look at sG model by itself, the only real requirement is that number of β^*s correspond to number of γ^*s. So we are kind of missing something. A more profound study of conformal case *Slide: "However, the only..."* shows that if I want to consider a shifted field, then I should consider some fixed multi-index, where $I_{\text{odd}}(m) = \{1, 3, \dots, 2m-1\}$, then I have **formulas on slide**.

So that's why, we see we can compute all of the 1-point functions with the α shifted. And all of them are given by these formulae. For example, I can compute the ratio $\langle \Phi_{\alpha+2m\frac{1-\nu}{\nu}} \rangle / \langle \Phi_\alpha \rangle$, and this must agree with the famous formula for the 1-point function when $R \to \infty$. And it does.

So what is our achievement? We said we wanted to know all 1-point functions for sG model, because we want to know ultraviolet behavior, and this must be computed for all shifted primary fields and their descendents. But we decided that Virasoro algebra isn't very good, because we don't know how to deform it from the CFT case, but instead we found that same Verma module for Virasoro algebra can be built form some other fermionic algebra, and in this basis we can compute just with determinants.

theojf@math.berkeley.edu