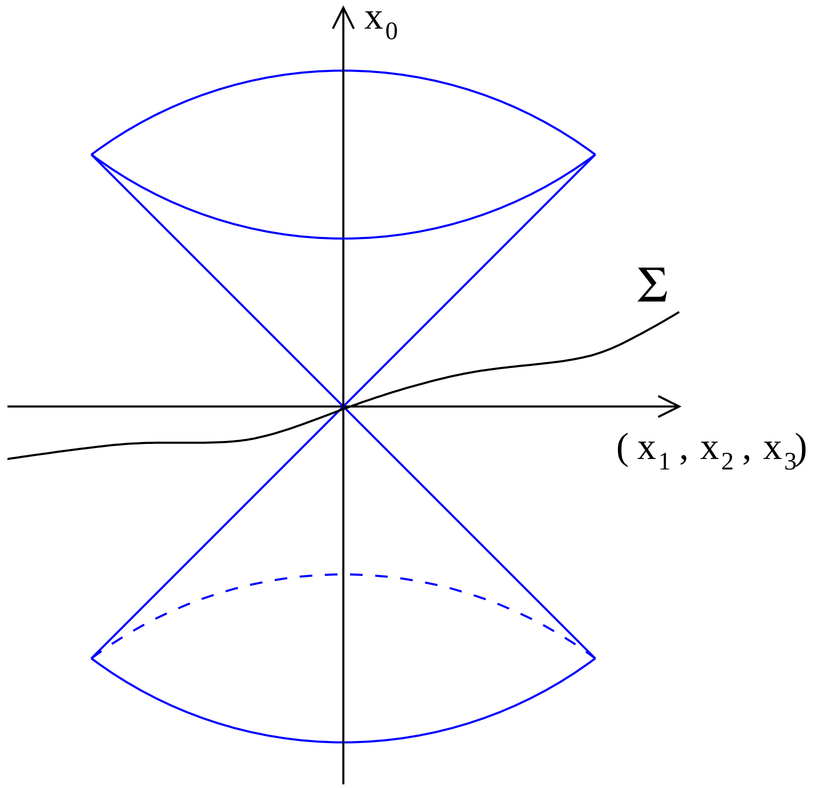
LECTURES ON CORRELATION FUNCTIONS IN INTGRABLE MODELS OF QUANTUM FIELD THEORY.

Fedor Smirnov

1. Lehman-Symanzik-Zimmermann axiomatics for QFT.



$$x = (x_0, x_1, x_2, x_3), \quad x^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2 \, .$$

Poincaré group: $\mathbb{R}^{1,3} \rtimes O(1,3)$.

1. The space of states.

The space of states is the Fock space of particles. Every particle carries the momentum $p = (p_0, p_1, p_2, p_3)$ satisfying

$$p^2 = p_0^2 - p_1^2 - p_2^2 - p_3^2 = m^2 \, .$$

It may have external degrees of freedom counted by $\epsilon = 1, \cdots, N$. They are Lorenz and isotopic. For simplicity I consider only Lorenz scalars. There is unique state called vacuum: $|\mathrm{vac}\rangle$. The creation-annihilation operators $a^*_{\mathrm{in},\epsilon}(k)$, $a^\epsilon_{\mathrm{in}}(k)$, $k = (p_1, p_2, p_3)$ satisfy

$$[a^\epsilon_{\mathrm{in}}(k), a^*_{\mathrm{in},\epsilon'}(k')] = \delta^\epsilon_{\epsilon'} \delta^{(3)}(k - k') \, .$$

Annihilation operators kill the vacuum

$$a^\epsilon_{\mathrm{in}}(k)|\mathrm{vac}\rangle = 0 \, ,$$

creation operators create n-particle states.

There is another set of operators $a^*_{\mathrm{out},\epsilon}(k)$, $a^\epsilon_{\mathrm{out}}(k)$. The two sets are related by unitary operators called S-matrix:

$$a^*_{\mathrm{out},\epsilon}(k) = S a^*_{\mathrm{in},\epsilon}(k) S^* \,, \qquad S\,|\mathrm{vac}\rangle = |\mathrm{vac}\rangle\,.$$

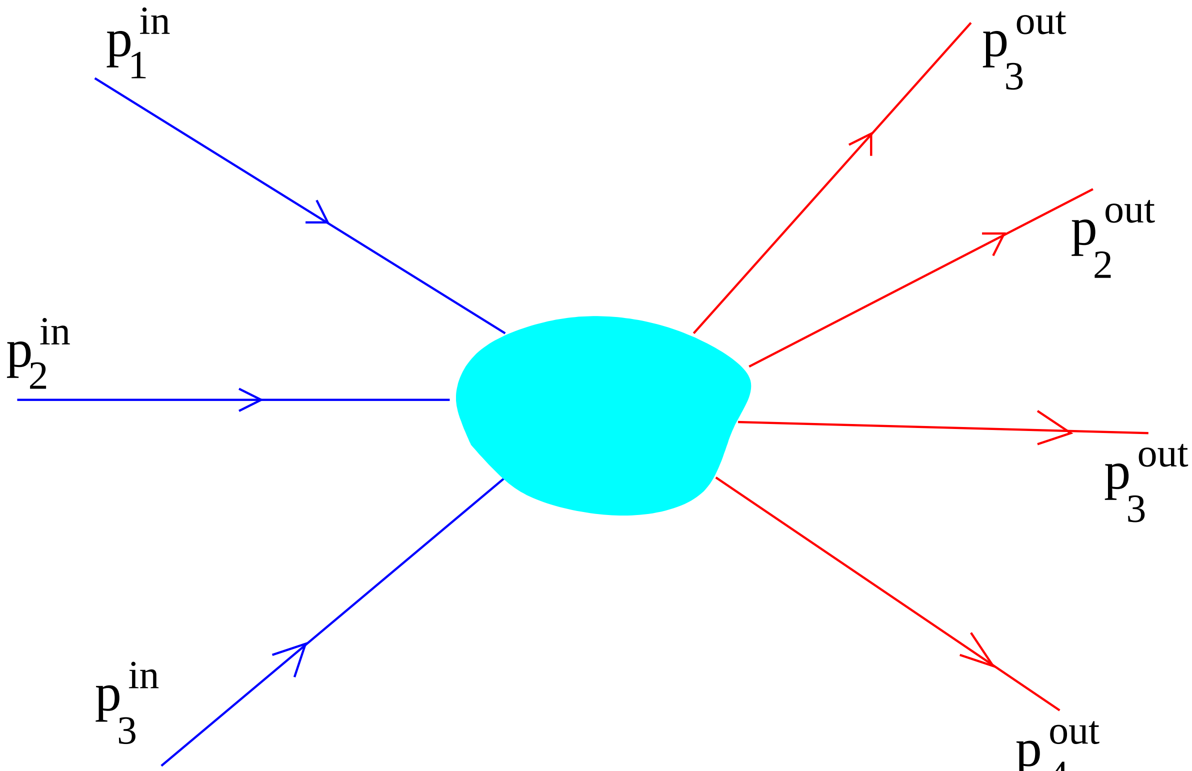
More precisely

$$S = I + \sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\int d^3k_1 \cdots \int d^3k_m \int d^3k'_1 \cdots \int d^3k'_n$$

$$\cdot\, \delta^{(3)}\Big(\sum k_j - \sum k'_j\Big)\ \ S(k_1,\cdots,k_m|k'_1,\cdots,k'_n)^{\epsilon_1,\cdots,\epsilon_m}_{\epsilon'_1,\cdots,\epsilon'_n}$$

$$\cdot\, a^*_{\mathrm{in},\epsilon_1}(k_1)\cdots a^*_{\mathrm{in},\epsilon_m}(k_n) a^{\epsilon'_1}_{\mathrm{in}}(k'_1)\cdots a^{\epsilon'_m}_{\mathrm{in}}(k'_m)$$

Every matrix element corresponds to scattering process



2. Locality. There are local operators $O(x)$. Locality means

$$[O_1(x), O_2(0)] = 0, \quad x^2 < 0\,.$$

Among these operators there is the energy-momentum tensor $T_{\mu,\nu}(x)$ such that

$$T_{\mu,\nu}(x) = T_{\nu,\mu}(x), \quad \partial_\mu T_{\mu,\nu}(x) = 0, \quad P_\mu = \int_\Sigma T_{\mu,0}(x)\,,$$

and

$$[P_\mu\ ,\ a^*_{\substack{\text{in}\\ \text{out}}}(k)] = p_\mu\ a^*_{\substack{\text{in}\\ \text{out}}}(k), \qquad P_\mu|\text{vac}\rangle = 0\,.$$

Self-consistency:

$$\mathcal{O}(x) = e^{iP_\mu x_\mu} O(0) e^{-iP_\mu x_\mu}\ .$$

Interpolating field $\varphi_\epsilon(x)$:

$$\mathrm{w-}\lim_{x_0 \to \mp\infty} \varphi_\epsilon(x) = \varphi_{\substack{\mathrm{in}\\ \mathrm{out}},\epsilon}(x)$$
$$= \int \left(e^{ip_\mu x_\mu} a^*_{\substack{\mathrm{in}\\ \mathrm{out}},\epsilon}(k) + e^{-ip_\mu x_\mu} a^\epsilon_{\substack{\mathrm{in}\\ \mathrm{out}}}(k) \right) \frac{d^3 k}{(2\pi)^3 \sqrt{k^2 + m^2}} \,.$$

Comments.

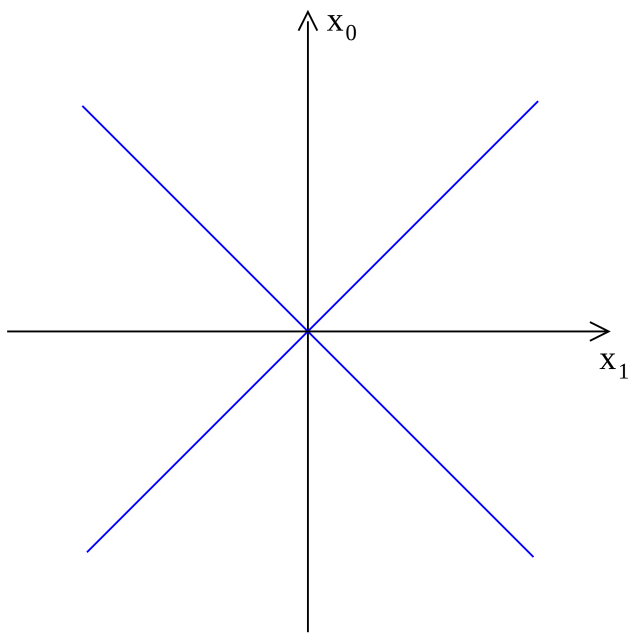
1. The theory is called free if $S = I$. The goal is to find a theory with non-trivial S-matrix.

2. Analyticity. The locality implies rather rich and complicated analytical properties of the S-matrix.

3. Mathematically the difference between the scattering theory in Quantum Mechanics and Quantum Field Theory is due to the difference between strong and weak limits. Hence the problem with the perturbation theory in QFT.

2. Integrable models in two dimensions.

Consider the two-dimensional space-time:



Parametrization of the energy-momentum of particles:

$$p_0^2 - p_1^2 = m^2, \quad p_0 = m\cosh\beta, \quad p_1 = m\sinh\beta\,.$$

Integrability. We had the conservation law $\partial_\mu T_{\mu,\nu}(x) = 0$. The light-cone components of the energy momentum tensor $P_\pm = P_0 \pm P_1$ have the eigenvalues on the asymptotical states:

$$P_\pm a^*_{\mathrm{in},\epsilon_1}(\beta_1)\cdots a^*_{\mathrm{in},\epsilon_n}(\beta_n)|\mathrm{vac}\rangle = m\sum e^{\pm\beta_j}\ a^*_{\mathrm{in},\epsilon_1}(\beta_1)\cdots a^*_{\mathrm{in},\epsilon_n}(\beta_n)|\mathrm{vac}\rangle\,.$$

Integrability implies existence of local operators $T^{(s)}_{\mu,\pm}(x)$ satisfying the conservation law $\partial_\mu T^{(s)}_{\mu,\pm}(x) = 0$. Such that

$$[I^{(s)}_\pm, I^{(s')}_\pm] = 0, \quad I^{(s)}_\pm = \int\limits_\Sigma T^{(s)}_{0,\pm}(x)\,, \quad I^{(1)}_\pm = P_\pm\,.$$

Further

$$I^{(s)}_\pm a^*_{\mathrm{in},\epsilon_1}(\beta_1)\cdots a^*_{\mathrm{in},\epsilon_n}(\beta_n)|\mathrm{vac}\rangle = m^{(s)}\sum e^{\pm s\beta_j}\ a^*_{\mathrm{in},\epsilon_1}(\beta_1)\cdots a^*_{\mathrm{in},\epsilon_n}(\beta_n)|\mathrm{vac}\rangle\,.$$

Implications for scattering. S-matrix commutes with all $I_{\pm}^{(s)}$. It is impossible to satisfy infinite number of equations

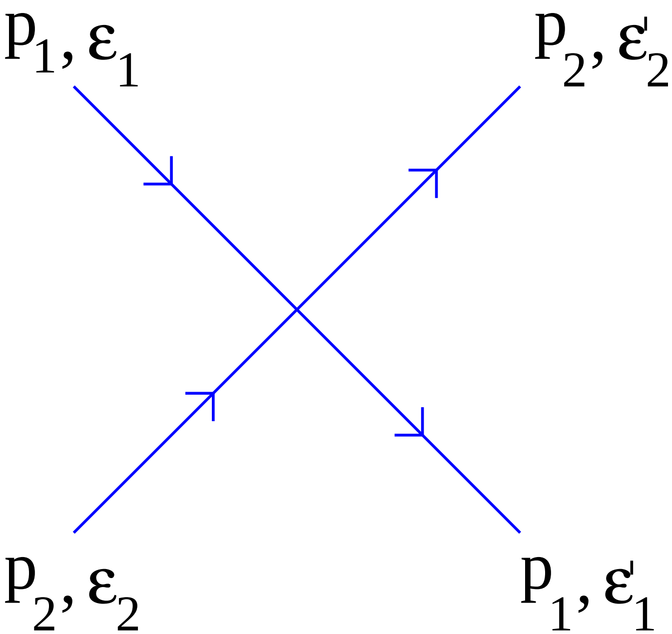
$$\sum_{j=1}^{m} e^{\pm s\beta_j} = \sum_{j=1}^{n} e^{\pm s\beta_j'} \,,$$

except the trivial solution $m = n$, $\{\beta_j\} = \{\beta_j'\}$.
Hence the first conclusion. The scattering is purely elastic:

$$S = I + \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \int \beta_1 \cdots \int d\beta_m \; S(\beta_1, \cdots, \beta_m)_{\epsilon_1', \cdots, \epsilon_n'}^{\epsilon_1, \cdots, \epsilon_m}$$

$$\cdot\, a_{\mathrm{in},\epsilon_1}^{*}(\beta_1) \cdots a_{\mathrm{in},\epsilon_m}^{*}(\beta_m) a_{\mathrm{in}}^{\epsilon_1'}(\beta_1) \cdots a_{\mathrm{in}}^{\epsilon_m'}(\beta_m)$$

We start with the two-particle S-matrix. Graphically it is represented as

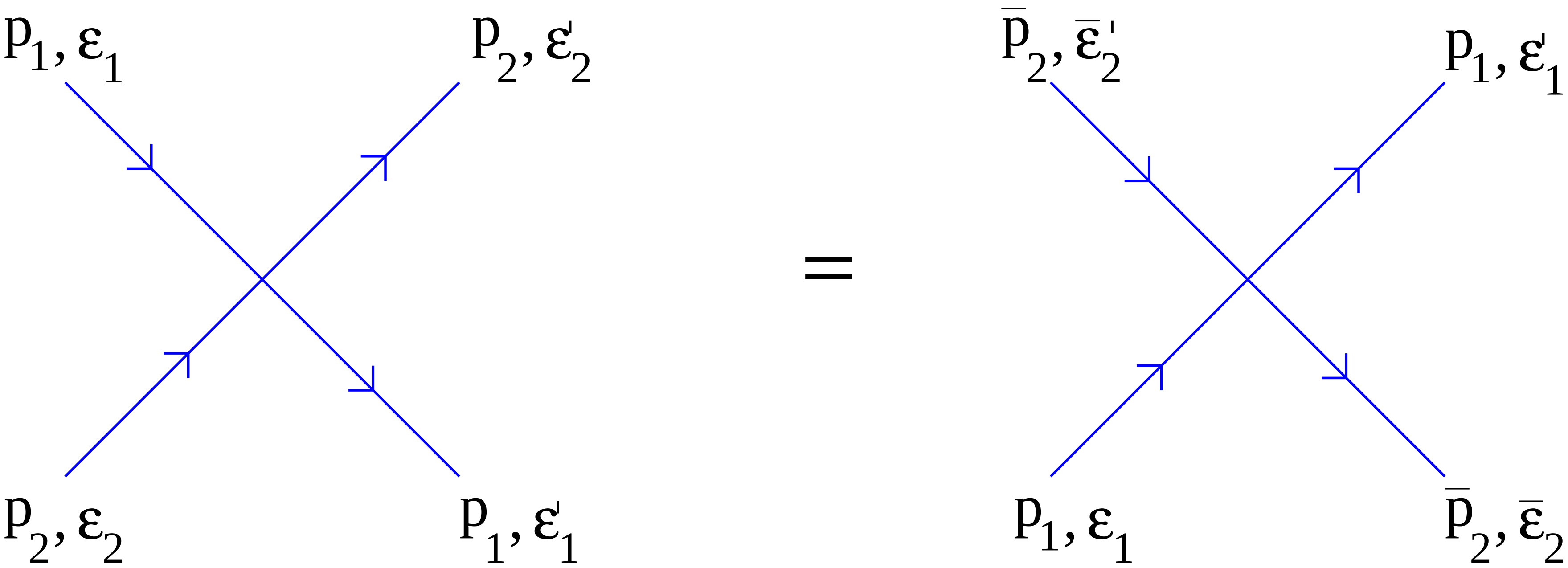


$$= S^{\varepsilon'_1 \varepsilon'_2}_{\varepsilon_1 \varepsilon_2}(\beta_1 - \beta_2)$$

1. Analyticity. $S^{\epsilon'_1,\epsilon'_2}_{\epsilon_1,\epsilon_2}(\beta)$ is meromorphic function of β regular at $0 \leq \mathrm{Im}(\beta) \leq \pi$.
2. Unitarity. Unitarity implies two relations

$$\overline{S^{\epsilon'_1,\epsilon'_2}_{\epsilon_1,\epsilon_2}(\beta)} = S^{\epsilon'_2,\epsilon'_1}_{\epsilon_2,\epsilon_1}(-\beta),\ \beta \in \mathbb{R}; \quad S^{\epsilon'_1,\epsilon'_2}_{\epsilon''_1,\epsilon''_2}(\beta) S^{\epsilon''_2,\epsilon''_1}_{\epsilon'_2,\epsilon'_1}(-\beta) = \delta^{\epsilon_1}_{\epsilon'_1}\delta^{\epsilon_2}_{\epsilon'_2}\ .$$

3.Crossing symmetry.



This is written as

$$S^{\epsilon_2', \epsilon_1'}_{\epsilon_2, \epsilon_1}(\pi i - \beta) = c_{\epsilon_2, \epsilon_2''}\ S^{\epsilon_1', \epsilon_2''}_{\epsilon_1, \epsilon_2'''}(\beta)\ c^{\epsilon_2''', \epsilon_2'}\ .$$

4. Factorizability of scattering. The most important restriction on the two-particle S-matrix comes from consideration of multi-particle scattering.

Multi-particle scattering reduces to sequence of two-particle ones.

I. Ya. Aref'eva, V.E.Korepin. *S-matrix for Sin-Gordon theory.* Pisma JETF, **20** (1974)

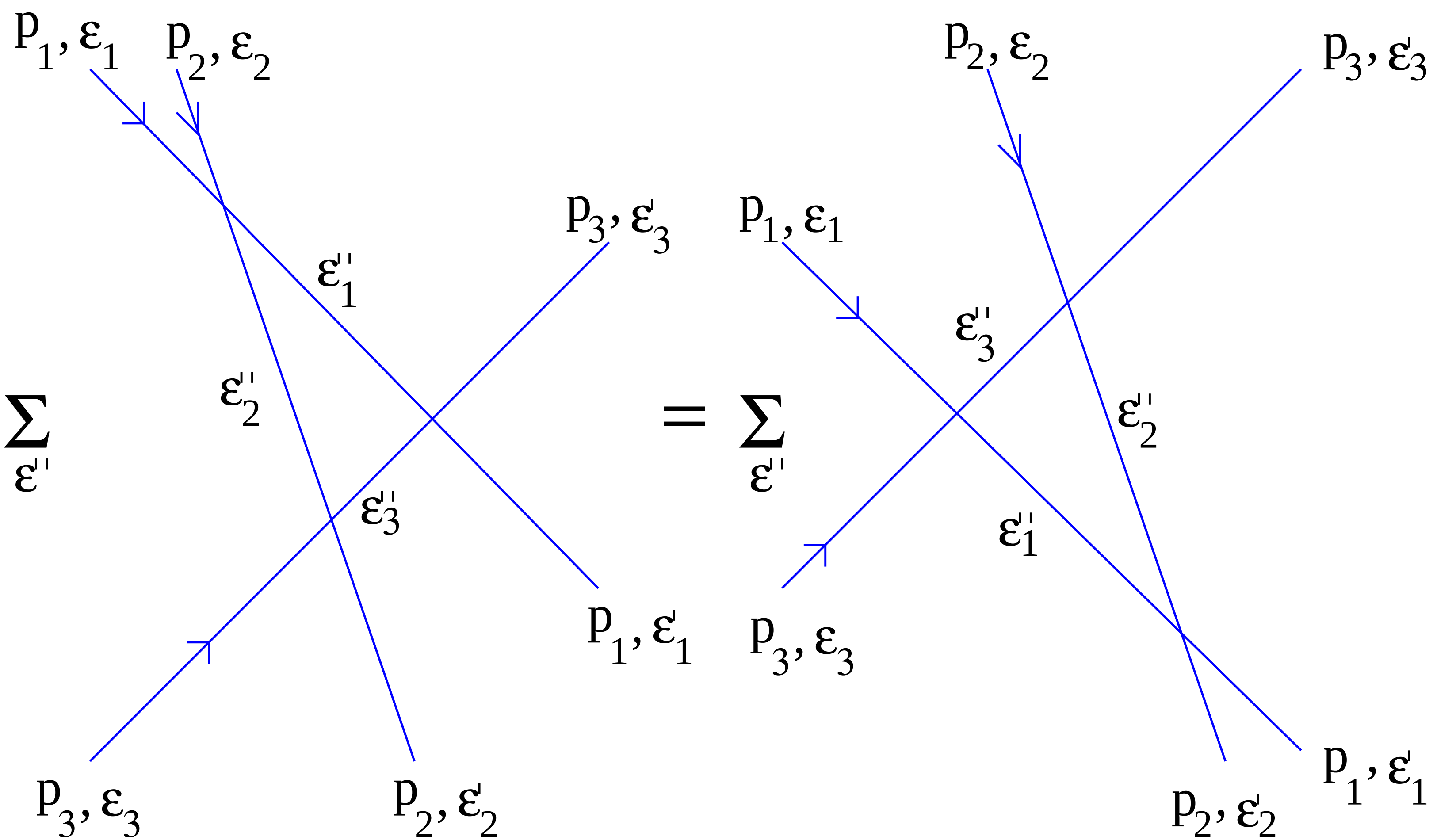
D. Iagolnitzer, *Factorization of the multiparticle S matrix in two-dimensional space-time models.* Phys. Rev. D **18** (1978)

A.B. Zamolodchikov, Al.B Zamolodchikov *factorized scattering in two dimensions of certain relativistic quantum field theory models.* Annals of Physics, **120** (1979)

Consistency relation (Yang-Baxter equation).

$$S^{\epsilon_1,\epsilon_2}_{\epsilon_1'',\epsilon_2''}(\beta_1-\beta_2)S^{\epsilon_1'',\epsilon_3}_{\epsilon_1',\epsilon_3''}(\beta_1-\beta_3)S^{\epsilon_2'',\epsilon_3''}_{\epsilon_2',\epsilon_3'}(\beta_2-\beta_3)$$
$$=S^{\epsilon_2,\epsilon_3}_{\epsilon_2'',\epsilon_3''}(\beta_2-\beta_3)S^{\epsilon_1,\epsilon_3''}_{\epsilon_1'',\epsilon_3'}(\beta_1-\beta_3)S^{\epsilon_1'',\epsilon_2''}_{\epsilon_1',\epsilon_2'}(\beta_1-\beta_2)\,.$$

Graphical illustration to Yang-Baxter equation.



Remark. Many solutions to these bootstrap equtions are known. It should be said that they are subject to Castillejo- Dalitz-Dyson (CDD) ambiguities. Namely, we can multiply by

$$\prod_j \frac{i\sinh\alpha_j - \sinh\beta}{i\sinh\alpha_j + \sinh\beta}\,.$$

These ambiguities are eliminated for particular models by considering perturbation theory, but still the situation is unsatisfactory unless we know how to reconstruct the local theory from the S-matrix.

3. Form factor bootstrap.

Consider a local operator $O(x)$. It is completely defined by its matrix elements

$$\langle \text{vac}|a_{\text{in}}^{\epsilon_1}(\beta_1)\cdots a_{\text{in}}^{\epsilon_m}(\beta_m)\ O(x)\ a^*_{\text{in},\epsilon'_1}(\beta'_1)\cdots a^*_{\text{in},\epsilon'_k}(\beta'_k)|\text{vac}\rangle$$
$$= e^{ix_\mu\left(\sum p_\mu(\beta_j)-\sum p_\mu(\beta'_j)\right)}\langle \text{vac}|a_{\text{in}}^{\epsilon_1}(\beta_1)\cdots a_{\text{in}}^{\epsilon_m}(\beta_m)\ O(x)\ a^*_{\text{in},\epsilon'_1}(\beta'_1)\cdots a^*_{\text{in},\epsilon'_k}(\beta'_k)|\text{vac}\rangle\ .$$

It is sufficient to find the form factors

$$f(\beta_1,\cdots,\beta_n)^{\epsilon_1,\cdots,\epsilon_n}\ =\ \langle \text{vac}|\ a_{\text{in}}^{\epsilon_1}(\beta_1)\cdots a_{\text{in}}^{\epsilon_n}(\beta_n)\ O(0)\ |\text{vac}\rangle\ .$$

We set $\beta_1<\cdots<\beta_n$ and then continue analytically. The result is assumed to be meromorphic. General matrix elements are obtained by analytical continuation $\beta_j\to\beta_j+\pi i$ for last k rapidities ($n=m+k$), and by lowering indices by the matrix c.

Form factor axioms.

F.A. Smirnov
Form factors in completely integrable models of quantum field theory.
World Scientific (1992) 208 p.

0. Analyticity. $f(\beta_1,\cdots,\beta_n)^{\epsilon_1,\cdots,\epsilon_n}$ is meromorphic function of all its arguments. As function of β_n is has in the strip $0\leq \mathrm{Im}(\beta_n)\leq 2\pi$ only simple poles at $\beta_n=\beta_j+\pi i$, $j=1,\cdots,n-1$.

1. Symmetry.

$$S^{\epsilon_j,\epsilon_{j+1}}_{\epsilon'_j,\epsilon'_{j+1}}(\beta_j-\beta_{j+1})f(\beta_1,\cdots,\beta_j,\beta_{j+1},\cdots,\beta_n)^{\epsilon_1,\cdots,\epsilon'_j,\epsilon'_{j+1},\cdots,\epsilon_n}$$
$$=f(\beta_1,\cdots,\beta_{j+1},\beta_j,\cdots,\beta_n)_{\epsilon_1,\cdots,\epsilon_{j+1},\epsilon_j,\cdots,\epsilon_n}\,.$$

2. Riemann-Hilbert problem.

$$f(\beta_1,\cdots,\beta_{n-1},\beta_n+2\pi i)^{\epsilon_1,\cdots,\epsilon_{n-1},\epsilon_n}=f(\beta_n,\beta_1,\cdots,\beta_{n-1})^{\epsilon_n,\epsilon_1,\cdots,\epsilon_{n-1}}\,.$$

3. Annihilation pole.

$$2\pi i \ \mathrm{res}_{\beta_n=\beta_{n-1}+\pi i} f(\beta_1,\cdots,\beta_{n-2},\beta_{n-1},\beta_n)^{\epsilon_1,\cdots,\epsilon_{n-2},\epsilon_{n-1},\epsilon_n}$$
$$= f(\beta_1,\cdots,\beta_{n-2})^{\epsilon_1,\cdots,\epsilon_{n-2}} c^{\epsilon_{n-1},\epsilon_n}$$
$$- S^{\epsilon_{n-1},\epsilon_1}_{\epsilon'_{n-1},\epsilon'_1}(\beta_{n-1}-\beta_1)\cdots S^{\epsilon_{n-1},\epsilon_{n-2}}_{\epsilon'_{n-1},\epsilon'_{n-2}}(\beta_{n-1}-\beta_{n-2}) f(\beta_1,\cdots,\beta_{n-2})^{\epsilon'_1,\cdots,\epsilon'_{n-2}} c^{\epsilon'_{n-1},\epsilon_n}$$

This is the origin of simple poles for general matrix elements. The way of understanding these poles can be explained.

Theorems.

1. Local commutativity theorem. *Suppose the form factors of two operators* $\mathcal{O}_{1,2}(x)$ *obey the axioms. Then the operators are local*

$$[O_1(x)\ ,\ O_2(0)] = 0 \quad \text{for} \quad x_\mu^2 < 0\,.$$

2. Asymptotical theorem. *Suppose the operator $\varphi_\epsilon(x)$ satisfies the axioms and has non-vanishing one-particle form factor $f(\beta)_\epsilon \neq 0$ .Then*

$$\mathrm{w}-\lim_{x_0\to\mp\infty}\varphi_\epsilon(x)=\varphi_{\substack{\mathrm{in}\\\mathrm{out}},\epsilon(x)}\,.$$

3. Energy-momentum theorem. *Suppose we have the operators $T_{\mu,\nu}(x)$ such that their form factors $f_{\mu,\nu}$ are of the form*

$$\begin{aligned}&f_{\mu,\nu}(\beta_1,\cdots,\beta_n)^{\epsilon_1,\cdots,\epsilon_n}\\&=m^2\sum(e^{\beta_j}-(-)^\mu e^{-\beta_j})\sum(e^{\beta_j}-(-)^\nu e^{-\beta_j})g(\beta_1,\cdots,\beta_n)^{\epsilon_1,\cdots,\epsilon_n}\,,\end{aligned}$$

and $g(\beta_1,\cdots,\beta_n)$ satisfy all the axioms except for additional simple pole at two-particle form factor

$$2\pi i\,\mathrm{res}_{\beta_2=\beta_1+\pi i}\,g(\beta_1,\beta_2)^{\epsilon_1,\epsilon_2}=c^{\epsilon_1,\epsilon_2}\,.$$

Then $T_{\mu,\nu}(x)$ can be taken for the energy-momentum tensor.

Correlation functions.

Let $x^2 = -r^2 < 0$, then considering for simplicity Lorenz scalar operators we have

$$G(r) = \langle \mathrm{vac}|O_1(x)O_2(0)|\mathrm{vac}\rangle$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int d\beta_1 \cdots d\beta_n f_1(\beta_n, \cdots, \beta_1)_{\epsilon_1,\cdots,\epsilon_n} f_2(\beta_1, \cdots, \beta_n)^{\epsilon_n,\cdots,\epsilon_1} e^{-mr\sum \cosh \beta_j} .$$

We begin with $e^{im(x_0 \sum \cosh \beta_j - x_1 \sum \sinh \beta_j)}$, then we make the Lorenz transformation $\beta_j \to \beta_j - \mathrm{arcsinh}(x_1/r)$ arriving at $e^{-imr\sum \sinh \beta_j}$, then we shift $\beta_j \to \beta_j - \pi i/2$. The latter is possible since

$$e^{-imr \sinh(\beta - i\theta)} = e^{-imr \sinh \beta \cos \theta - mr \cosh \beta \sin \theta} .$$

Two-point space-like correlation function is the same as two-point Euclidean correlation function.

The difficulty with describing the short-distance behaviour is obvious.

Sine-Gordon model.

$$\mathcal{A}^{\mathrm{sG}} = \int \Big[\frac{1}{16\pi}(\partial_\mu \boldsymbol{\varphi}(x))^2 + \frac{\boldsymbol{\mu}^2}{\sin \pi\beta^2} 2\cos(\beta\boldsymbol{\varphi}(x))\Big] d^2x\,.$$

I shall use the parameter

$$\nu = 1-\beta^2\,, \qquad 1 > \nu > 0\,.$$

Semi-classical domain $\nu \to 0$.
The spectrum of the model consists of soliton-antisoliton with mass M and for $1/2 < \nu < 1$ of $\left[\frac{\nu}{1-\nu}\right] - 1$ bound states (breathers) with masses $2M\sin\left(\pi\frac{1-\nu}{2\nu}j\right)$, $j = 1, \cdots \left[\frac{\nu}{1-\nu}\right] - 1$.

Free fermion point $\nu = \frac{1}{2}$.

Two soliton S-matrix.

I use the notations

$$\mathfrak{b}_j = e^{\frac{2\nu}{1-\nu}\beta_j}, \quad \mathfrak{q} = e^{\pi i \frac{1}{1-\nu}} .$$

$$S_{i,j}(\beta_i - \beta_j) = S_0(\beta_i - \beta_j)\widetilde{S}_{i,j}(\mathfrak{b}_i/\mathfrak{b}_j) ,$$

$$S_0(\beta) = \exp\left(-i \int_0^\infty \frac{\sin(2k\nu\beta)\sinh((2\nu-1)\pi k)}{k\cosh(\pi\nu k)\sinh(\pi(1-\nu)k)} dk \right) ,$$

and

$$\widetilde{S}_{i,j}(\mathfrak{b}_i/\mathfrak{b}_j) = \tfrac{1}{2}(I_i \otimes I_j + \sigma_i^3 \otimes \sigma_j^3) + \frac{\mathfrak{b}_i - \mathfrak{b}_j}{\mathfrak{b}_i\mathfrak{q}^{-1} - \mathfrak{b}_j\mathfrak{q}} \cdot \tfrac{1}{2}(I_i \otimes I_j - \sigma_i^3 \otimes \sigma_j^3)$$
$$+ \sqrt{\mathfrak{b}_i\mathfrak{b}_j}\frac{\mathfrak{q}^{-1} - \mathfrak{q}}{\mathfrak{b}_i\mathfrak{q}^{-1} - \mathfrak{b}_j\mathfrak{q}} \cdot (\sigma_i^+ \otimes \sigma_j^- + \sigma_i^- \otimes \sigma_j^+) .$$

Local fields.

We shall consider the "primary fields" $\Phi_\alpha(x) = e^{i\alpha\ \frac{\nu}{2\sqrt{1-\nu}}\ \varphi(x)}$, and their relatives ("descendants"). The axioms are slightly modified:

Symmetry axiom.

$$S_{j,j+1}(\beta_j - \beta_{j+1}) f_{\mathcal{O}_\alpha}(\beta_1, \cdots, \beta_j, \beta_{j+1}, \cdots, \beta_{2n}) = f_{\mathcal{O}_\alpha}(\beta_1, \cdots, \beta_{j+1}, \beta_j, \cdots, \beta_{2n}),$$

Riemann-Hilbert problem axiom.

$$f_{\mathcal{O}_\alpha}(\beta_1, \cdots, \beta_{2n-1}, \beta_{2n} + 2\pi i) = e^{-\frac{\pi i \nu}{1-\nu}\alpha\sigma^3_{2n}} f_{\mathcal{O}_\alpha}(\beta_{2n}, \beta_1, \cdots, \cdots, \beta_{2n-1}).$$

Residue axiom.

$$2\pi i \ \mathrm{res}_{\beta_{2n}=\beta_{2n-1}+\pi i} f_{\mathcal{O}_\alpha}(\beta_1, \cdots, \beta_{2n-2}, \beta_{2n-1}, \beta_{2n}) =$$
$$\Big(1 - e^{-\frac{\pi i \nu}{1-\nu}\alpha\sigma^3_{2n}} S_{2n-1,1}(\beta_{2n-1} - \beta_1) \cdots S_{2n-1,2n-2}(\beta_{2n-1} - \beta_{2n-2})\Big)$$
$$\times f_{\mathcal{O}_\alpha}(\beta_1, \cdots, \beta_{2n-2}) \otimes s_{2n-1,2n},$$

where $s_{i,j} = e^+_i \otimes e^-_j + e^-_i \otimes e^+_j$.

We look for the form factors in the form

$$f_{\mathcal{O}_\alpha}(\beta_1,\cdots,\beta_{2n}) = \sum_{\epsilon_1,\cdots,\epsilon_{2n}=\pm} w^{\epsilon_1,\cdots,\epsilon_{2n}}(\beta_1,\cdots,\beta_{2n}) \ \cdot \ \mathcal{F}_{\mathcal{O}_\alpha,n}(\beta_{I^-}|\beta_{I^+})\,,$$

where $I^\pm = \{j \mid 1 \le j \le 2n,\ \epsilon_j = \pm\}, \quad \beta_{I^\pm} = \{\beta_j\}_{j\in I^\pm}$.
The basis satisfies

$$S_{i,i+1}(\beta_i-\beta_{i+1})w^{\epsilon_1,\cdots,\epsilon_i,\epsilon_{i+1},\cdots,\epsilon_{2n}}(\beta_1,\cdots,\beta_i,\beta_{i+1},\cdots,\beta_{2n})$$
$$= w^{\epsilon_1,\cdots,\epsilon_{i+1},\epsilon_i,\cdots,\epsilon_{2n}}(\beta_1,\cdots,\beta_{i+1},\beta_i,\cdots,\beta_{2n})\,,$$

and $\mathcal{F}_{\mathcal{O}_\alpha,n}(\beta_{I^-}|\beta_{I^+})$ is symmetric function of β_{I^-} and β_{I^+}

Integrals. The list of notations

$$S = e^{\sigma}, \;\; B_j = e^{\beta_j}, \;\; Q = e^{\pi i \frac{1-\nu}{\nu}}, \;\; A = e^{\pi i \alpha}$$

$$\mathfrak{s} = e^{\frac{2\nu}{1-\nu}\sigma}, \;\; \mathfrak{b}_j = e^{\frac{2\nu}{1-\nu}\beta_j}, \;\; \mathfrak{q} = e^{\pi i \frac{1}{1-\nu}}, \;\; a = e^{\pi i \frac{\nu}{1-\nu}\alpha}\,.$$

Consider the meromorphic function$\chi(\sigma|\beta_1, \cdots, \beta_{2n})$, which for real β_j is regular as function of σ for $0 > \mathrm{Im}(\sigma) > -\pi$, and which satisfies

$$\chi(\sigma + 2\pi i)p(\mathfrak{s}\mathfrak{q}^4) = \chi(\sigma)p(\mathfrak{s}\mathfrak{q}^2)$$
$$\chi(\sigma + \tfrac{1-\nu}{\nu}\pi i)P(SQ) = \chi(\sigma)P(-S)\,,$$

where

$$P(S) = \prod_{j=1}^{2n}(S - B_j)\,, \quad p(\mathfrak{s}) = \prod_{j=1}^{2n}(\mathfrak{s} - \mathfrak{b}_j)\,.$$

We have the asymptotics

$$\chi(\sigma|\beta_1,\cdots,\beta_{2n}) \simeq_{\sigma\to\infty} e^{-2n\frac{1}{1-\nu}\sigma} x^+(\mathfrak{s}|\mathfrak{b}_1,\cdots,\mathfrak{b}_{2n}) X^+(S|B_1,\cdots,B_{2n}) \,,$$
$$\chi(\sigma|\beta_1,\cdots,\beta_{2n}) \simeq_{\sigma\to-\infty} x^-(\mathfrak{s}|\mathfrak{b}_1,\cdots,\mathfrak{b}_{2n}) X^-(S|B_1,\cdots,B_{2n}) \,,$$

where

$$x^+(\mathfrak{s}|\mathfrak{b}_1,\cdots,\mathfrak{b}_{2n}) = 1 + \sum_{k=1}^{\infty} x_k^+(\mathfrak{b}_1,\cdots,\mathfrak{b}_{2n})\mathfrak{s}^{-k} \,,$$

$$X^+(S|B_1,\cdots,B_{2n}) = 1 + \sum_{k=1}^{\infty} X_k^+(B_1,\cdots,B_{2n}) S^{-k} \,,$$

$$x^-(\mathfrak{s}|\mathfrak{b}_1,\cdots,\mathfrak{b}_{2n}) = \mathfrak{q}^n \prod_{j=1}^{2n} \mathfrak{b}_j^{-\frac{1}{2}} \Bigl(1 + \sum_{k=1}^{\infty} x_k^-(\mathfrak{b}_1,\cdots,\mathfrak{b}_{2n})\mathfrak{s}^{k}\Bigr) \,,$$

$$X^-(S|B_1,\cdots,B_{2n}) = \prod_{j=1}^{2n} B_j^{-1} \Bigl(1 + \sum_{k=1}^{\infty} X_k^-(B_1,\cdots,B_{2n}) S^{k}\Bigr) \,.$$

Our main object is the Laplace transform:

$$I_\alpha(\beta_1,\cdots,\beta_{2n})=\int\limits_{\mathbb{R}-i0}\chi(\sigma|\beta_1,\cdots,\beta_{2n})e^{\frac{\nu\alpha}{1-\nu}\sigma}d\sigma\,.$$

Literally the integral is defined for $0<\mathrm{Re}(\alpha)<\frac{2n}{\nu}$, but it can be analytically continued to entire complex plane of α as meromorphic function with simple poles at $\alpha=2n+2m+(2n+l)\frac{1-\nu}{\nu}$ and $\alpha=-2m-l\frac{1-\nu}{\nu}$ for $,m\geq 0$.

Let us see how it goes. Define $m^{(1)}(\mathfrak{s})$ by

$$p(\mathfrak{s}\mathfrak{q}^{-2})=m^{(1)}(\mathfrak{s})+a^{-2}p(\mathfrak{s})n^{(1)}(\mathfrak{s}\mathfrak{q}^{-4})-p(\mathfrak{s}\mathfrak{q}^{-2})n^{(1)}(\mathfrak{s})\,,$$

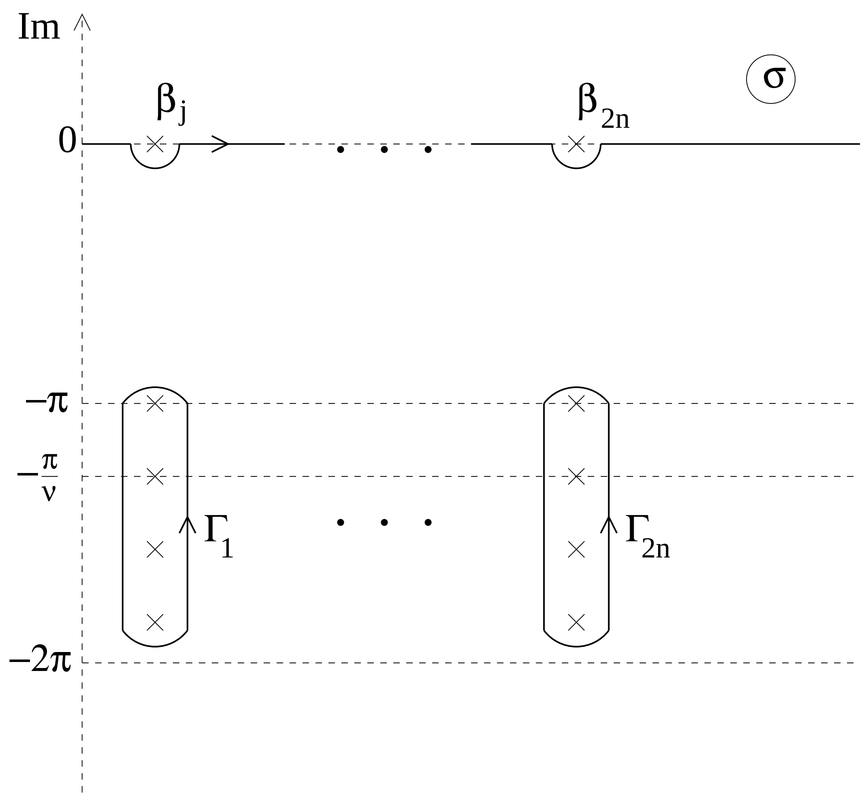
where

$$n^{(1)}(\mathfrak{s})=\frac{1}{a^{-2}\mathfrak{q}^{4n}-1},\quad m^{(1)}(\mathfrak{s})=\sum_{j=0}^{2n-1}m^{(1)}_j\mathfrak{s}^j\,.$$

For $0 < \mathrm{Re}(\alpha) < \frac{2n}{\nu}$ we have the identity

$$I_\alpha = \int\limits_{\mathbb{R}-i0} \chi(\sigma) e^{\frac{\nu\alpha}{1-\nu}\sigma} \frac{m^{(1)}(\mathfrak{s})}{p(\mathfrak{s}\mathfrak{q}^{-2})} d\sigma + \sum_j \int\limits_{\Gamma_j} \chi(\sigma) e^{\frac{\nu\alpha}{1-\nu}\sigma} n^{(1)}(\mathfrak{s}) d\sigma\,,$$

which allows to continue to $\frac{2n}{\nu} \leq \mathrm{Re}(\alpha) < \frac{2n}{\nu} + 2$.



Pairing. For two Laurent polynomials $\ell(\mathfrak{s})$ and $L(S)$ we define pairing $(l, L)_\alpha$ by

1. The pairing is bilinear.
2. If $\ell(\mathfrak{s}) = \mathfrak{s}^m$, $L(S) = S^l$ then

$$(\ell, L)_\alpha = I_{\alpha+2m+\frac{1-\nu}{\nu}l}\,.$$

$\mathfrak{q}$-exact forms

If $D_a[z](\mathfrak{s}) = a^{-2}p(\mathfrak{s})z(\mathfrak{s}) - p(\mathfrak{s}\mathfrak{q}^2)z(\mathfrak{s}\mathfrak{q}^4)$, then $(D_a[z], L)_\alpha = 0$.

Q-exact forms.

If $D_A[Z](S) = Z(S)P(S) - AZ(SQ)P(-S)$, then $(\ell, D_A[Z])_\alpha = 0$.

Consider k Laurent polynomials $\ell_1(\mathfrak{s}), \cdots \ell_k(\mathfrak{s})$. We define

$$(\ell_1 \wedge \cdots \wedge \ell_k)(\mathfrak{s}_1, \cdots \mathfrak{s}_k) = \sum_{\pi} (-1)^{\pi} \ell_1(\mathfrak{s}_{\pi(1)}), \cdots \ell_k(\mathfrak{s}_{\pi(k)}) \, .$$

Similarly we define $L^{(k)}(S_1, \cdots, S_k)$. Then

$$(\ell^{(k)}, L^{(k)})_{\alpha} = \det \left((\ell_i, L_j)_{\alpha} \right)_{i,j=1,\cdots,k} \, .$$

Returning to form factors. For a partition $I^- \sqcup I^+ = \{1, \cdots, 2n\}$ such that $\sharp(I^-) = \sharp(I^+)$, define the polynomials

$$p_{I^-}(\mathfrak{s}) = \prod_{j \in I^-} (\mathfrak{s} - \mathfrak{b}_j), \quad p_{I^+}(\mathfrak{s}) = \prod_{j \in I^+} (\mathfrak{s} - \mathfrak{b}_j) \, ,$$

so that we have $p(\mathfrak{s}) = p_{I^+}(\mathfrak{s}) p_{I^-}(\mathfrak{s})$.

Define

$$c_{I^-\sqcup I^+}(\mathfrak{t},\mathfrak{s}) = \frac{a\mathfrak{t}}{\mathfrak{t}-\mathfrak{q}^4\mathfrak{s}}p(\mathfrak{q}^2\mathfrak{s}) - \frac{a^{-1}\mathfrak{t}}{\mathfrak{t}-\mathfrak{s}}p(\mathfrak{s})$$

$$+\,p_{I^+}(\mathfrak{t})p_{I^-}(\mathfrak{s})\left(\frac{a^{-1}\mathfrak{t}}{\mathfrak{t}-\mathfrak{s}} - \frac{a^{-1}\mathfrak{t}}{\mathfrak{t}-\mathfrak{s}\mathfrak{q}^2}\right) + p_{I^-}(\mathfrak{t}\mathfrak{q}^{-2})p_{I^+}(\mathfrak{q}^2\mathfrak{s})\left(\frac{a^{-1}\mathfrak{t}}{\mathfrak{t}-\mathfrak{s}\mathfrak{q}^2} - \frac{a\mathfrak{t}}{\mathfrak{t}-\mathfrak{q}^4\mathfrak{s}}\right)\,.$$

$$:= \sum_{i=0}^{n-1} \mathfrak{t}^{n-i}\ell_{I^-\sqcup I^+,i}(\mathfrak{s})\,,$$

and

$$\ell^{(n)}_{I^-\sqcup I^+}(\mathfrak{s}_1,\cdots,\mathfrak{s}_n) = \left(\ell_{I^-\sqcup I^+,0}\wedge\cdots\wedge\ell_{I^-\sqcup I^+,n-1}\right)(\mathfrak{s}_1,\cdots,\mathfrak{s}_n)\,.$$

Theorem. Consider Laurent polynomial of all variables $L^{(n)}(S_1, \cdots, S_n | B_1, \cdots, B_{2n})$, antisymmetric in S_j and symmetric in B_j. The Riemann-Hilbert axiom is satisfied by

$$\mathcal{F}(\beta_{I^-}|\beta_{I^+}) = \prod_{j\in I^+} \mathfrak{b}_j^{\frac{1-\alpha}{4}} \prod_{j\in I^1} \mathfrak{b}_j^{-\frac{1+\alpha}{4}} \frac{1}{\prod_{i\in I^-, j\in I^+} (\mathfrak{b}_i - \mathfrak{b}_j)} \cdot \Big(\ell_{I^-\sqcup I^+, i} \ , \ L^{(n)} \Big)_\alpha \ .$$

Third axiom is equivalent to the simple recurrent relation

$$L^{(n)}_{\mathcal{O}_\alpha}(S_1, \cdots, S_{l-1}, B|B_1, \cdots, B_{2n-2}, B, -B)$$
$$= B \prod_{p=1}^{n-1} (B^2 - S_p^2) \cdot L^{(n-1)}_{\mathcal{O}_\alpha}(S_1, \cdots, S_{n-1}|B_1, \cdots, B_{2n-2})$$

In particular,

$$L^{(n)}_{\Phi_\alpha}(S_1, \cdots, S_n) = \langle \Phi_\alpha \rangle \cdot S \wedge S^3 \wedge \cdots \wedge S^{2n-1} \, .$$

Detailed discussion of all the solution can be found in Jimbo, Miwa, Smirnov "Fermionic structure in sine-Gordon model: form factors and null-vectors" (2011).

2. Example: free fermions and Painlevé.

For $\nu = 1/2$ the S-matrix trivialises:

$$S_{1,2}(\theta) = -I\,,$$

there are now breathers, and the form factors for solitons are simple:

$$f_\alpha(\theta_1,\cdots,\theta_n,\theta_{n+1},\cdots\theta_{2n})_{\frac{1}{2},\cdots\frac{1}{2},-\frac{1}{2},\cdots-\frac{1}{2}}$$

$$= \left(\frac{2\sin\frac{\pi\alpha}{2}}{\pi}\right)^n e^{\frac{1}{2}\alpha\sum\limits_{j=1}^{n}(\theta_j-\theta_{n+j})} \frac{\prod\limits_{i<j}^{2n}\sinh\frac{1}{2}(\theta_i-\theta_j)}{\prod\limits_{i=1}^{n}\prod\limits_{j=n+1}^{2n}\sinh(\theta_i-\theta_i)}\,.$$

The form factor series turn into the Fredholm determinant which is the tau-functioon for Painleve III. To cut the long story short

$$\frac{\langle\Phi_{\alpha_1}(\mathbf{x})\Phi_{\alpha_2}(0)\rangle^{\mathrm{sG}}}{\langle\Phi_{\alpha}(0)\rangle^{\mathrm{sG}}}=\frac{\langle\Phi_{\alpha_1}(0)\rangle^{\mathrm{sG}}\langle\Phi_{\alpha_2}(0)\rangle^{\mathrm{sG}}}{\langle\Phi_{\alpha}(0)\rangle^{\mathrm{sG}}}\cdot\tau((\tfrac{1}{2}Mr)^2),\quad \mathrm{recall}\quad -\mathbf{x}^2=r^2\,.$$

We denote $\alpha=\alpha_1+\alpha_2\ ,\ \theta=\alpha_1-\alpha_2$. One-point function is

$$\langle\Phi_{\alpha}(0)\rangle^{\mathrm{sG}}=\Bigl(\frac{M}{2}\Bigr)^{\frac{\alpha^2}{4}}\exp\Bigl(\int_0^\infty\Bigl(\frac{\sinh^2(\frac{\alpha t}{2})}{\sinh^2 t}-\frac{\alpha^2}{4}e^{-2t}\Bigr)\frac{dt}{t}\Bigr)\,.$$

Set

$$\zeta(t)=t\frac{d}{dt}\log\tau(t)\,.$$

we have

$$\left(t\frac{d^2\zeta}{dt^2}\right)^2=4\frac{d\zeta}{dt}\left(\frac{d\zeta}{dt}-1\right)\left(\zeta-t\frac{d\zeta}{dt}\right)+\frac{\theta^2}{4}\left(\frac{d\zeta}{dt}\right)^2\,.$$

Asymptotics for $0 < \alpha < 2$:

$$\frac{\langle \Phi_{\alpha_1}(\mathbf{x}) \Phi_{\alpha_2}(0) \rangle^{\mathrm{sG}}}{\langle \Phi_{\alpha}(0) \rangle^{\mathrm{sG}}}$$

$$\simeq r^{\frac{\alpha_1 \alpha_2}{2}} \left(1 + \frac{\alpha_1 \alpha_2 M^2}{4\alpha^2} \left[r^2 - \frac{4M^{\alpha}}{(2+\alpha)^2} s r^{2+\alpha} - \frac{4M^{-\alpha}}{(2-\alpha)^2} s^{-1} r^{2-\alpha} \right] + \cdots \right),$$

where

$$s = 2^{\alpha} \frac{\Gamma(1 - \frac{\alpha}{2})^2}{\Gamma(1 + \frac{\alpha}{2})^2} \frac{\Gamma\left(1 + \frac{\alpha_1}{2}\right)}{\Gamma\left(1 - \frac{\alpha_1}{2}\right)} \frac{\Gamma\left(1 + \frac{\alpha_2}{2}\right)}{\Gamma\left(1 - \frac{\alpha_2}{2}\right)}.$$

Euclidean case.

In Euclidean case we take seriously the functional integral. For example the two-pont function is

$$\langle \mathcal{O}_1(z,\bar z)\mathcal{O}_2(0)\rangle = \frac{\int e^{-\frac{1}{\hbar}\mathcal{A}[\varphi]}\mathcal{O}_1(z,\bar z)\mathcal{O}_2(0)\prod_{w,\bar w}\mathcal{D}\phi(w,\bar w)}{\int e^{-\frac{1}{\hbar}\mathcal{A}[\varphi]}\prod_{w,\bar w}\mathcal{D}\phi(w,\bar w)}\,.$$

There is an important analogy between D-dimensional Euclidean QFT and D-dimensional classical statistical mechanics. Partition function:

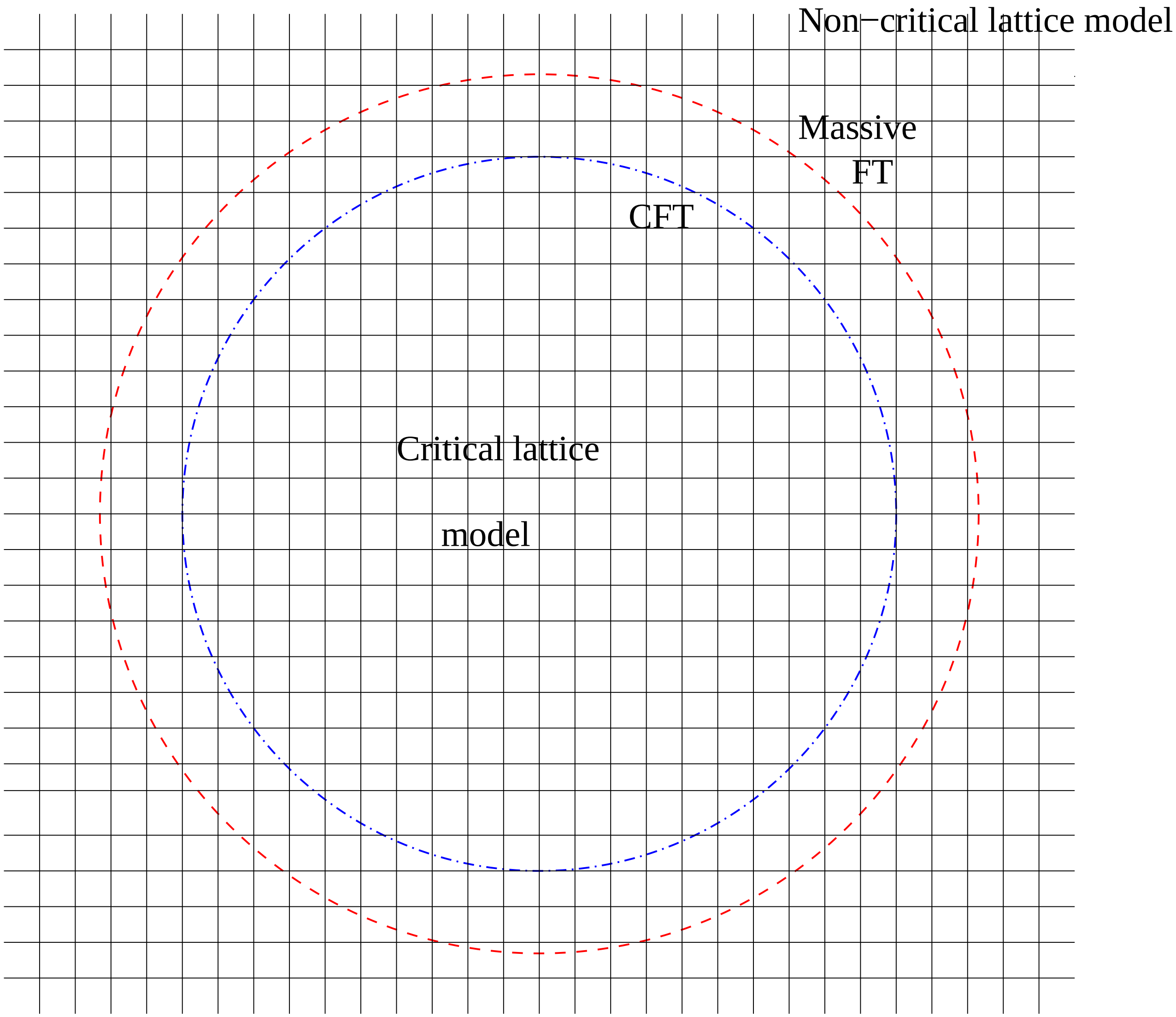
$$Z = \sum_{\text{configurations}} \left(e^{-\frac{1}{T}\mathcal{H}}\right).$$

The rule is $\mathcal{A} \leftrightarrow \mathcal{H}$. If we try the lattice regularization. Typically

$$\langle \sigma(r)\sigma(0)\rangle = \frac{1}{Z}\sum_{\text{configurations}} \left(e^{-\frac{1}{T}\mathcal{H}}\sigma(r)\sigma(0)\right) \simeq_{r\to\infty} e^{-\frac{r}{\xi(T)}}\,.$$

Scaling limit at T_c where $\xi(T)\simeq (T-T_c)^{-\nu}$ (another ν).

Lattice model near the point of the second order phase transition.



Conformal field theory.

Energy-momentum tensor describe variations with respect to external metric

$$\frac{\delta}{\delta g^{a,b}(z,\bar{z})} Z \Big|_{g^{a,b}\ \mathrm{Euclidean}} = \int T_{a,b}(z,\bar{z}) e^{-\frac{1}{\hbar}\mathcal{A}[\varphi]} \prod_{w,\bar{w}} \mathcal{D}\phi(w,\bar{w})\,.$$

There are three components of T for which we use the complex notations:

$$T(z,\bar{z}) = T_{z,z}(z,\bar{z}),\ \ \bar{T}(z,\bar{z}) = T_{\bar{z},\bar{z}}(z,\bar{z}),\ \ \Theta(z,\bar{z}) = T_{z,\bar{z}}(z,\bar{z})\,.$$

They satisfy the conservation

$$\partial_{\bar{z}} T(z,\bar{z}) = \Theta(z,\bar{z}), \quad \partial_z \bar{T}(z,\bar{z}) = \Theta(z,\bar{z})\,.$$

By definition for CFT $\Theta(z,\bar{z}) = 0$. Hence $T(z,\bar{z}) = T(z)$, $\bar{T}(z,\bar{z}) = \bar{T}(\bar{z})$.

Operator product expansion. Every observable $\mathcal{O}$ is characterized by scaling dimensions $\Delta, \bar{\Delta}$, in particular

$$\mathcal{O}(az, a\bar{z}) = a^{-\Delta-\bar{\Delta}}\mathcal{O}(z, \bar{z})\,.$$

Suppose we have complete set of local observables, then we must have OPE

$$\mathcal{O}_i(z,\bar{z})\mathcal{O}_j(0) = \sum_k z^{-\Delta_i-\Delta_j+\Delta_k}\,\bar{z}^{-\bar{\Delta}_i-\bar{\Delta}_j+\bar{\Delta}_k}C^k_{i,j}\mathcal{O}_k(0)\,.$$

The set of constants $C^k_{i,j}$ completely characterizes the CFT.

Perturbed CFT.

Al. B. Zamolodchikov Two-point correlation function in scaling Lee-Yang model.

$$\mathcal{A}^{\mathrm{PCFT}} = \mathcal{A}^{\mathrm{CFT}} + g\int \phi(z,\bar{z})d^2z, \quad d^2z = \frac{idz\wedge d\bar{z}}{2},$$

where $\phi(z,\bar{z})$ has scaling dimension (Δ,Δ), being relevant $\Delta < 1$.

$$g = [\mathrm{Length}]^{2\Delta-2}.$$

Naïve attempt of computing the short distance asymptotics of the two-point function for PCFT:

$$\int \mathcal{O}_1(z,\bar{z})\mathcal{O}_2(0)e^{-\frac{1}{\hbar}\mathcal{A}^{\mathrm{PCFT}}}\prod_{w,\bar{w}}\mathcal{D}\phi(w,\bar{w})$$

$$=\sum_{n=0}^{\infty}\frac{g^n}{n!}\int d^2z_1\cdots\int d^2z_n\langle\mathcal{O}_1(z,\bar{z})\mathcal{O}_2(0)\phi(z_1,\bar{z}_1)\cdots\phi(z_n,\bar{z}_n)\rangle_{\mathrm{CFT}}.$$

This is wrong because the integrals are IR divergent (they are UV convergent for $\Delta < 1/2$). But this is rather good than bad because this series contradict even to

$$\frac{\langle \Phi_{\alpha_1}(\mathbf{x})\Phi_{\alpha_2}(0)\rangle^{\mathrm{sG}}}{\langle \Phi_{\alpha}(0)\rangle^{\mathrm{sG}}}$$

$$\simeq r^{\frac{\alpha_1\alpha_2}{2}}\Big(1+\frac{\alpha_1\alpha_2 M^2}{4\alpha^2}\Big[r^2-\frac{4M^{\alpha}}{(2+\alpha)^2}sr^{2+\alpha}-\frac{4M^{-\alpha}}{(2-\alpha)^2}s^{-1}r^{2-\alpha}\Big]+\cdots\Big)\,,$$

in this case $g = M/2$.

The main conclusion is that the perturbation theory must be used rather for OPE than for the correlation functions. For irrational dimensions local operators can be identified, and we have

$$\mathcal{O}_i(z,\bar{z})\mathcal{O}_j(0)=\sum_k z^{-\Delta_i-\Delta_j+\Delta_k}\,\bar{z}^{-\bar{\Delta}_i-\bar{\Delta}_j+\bar{\Delta}_k}\,C^k_{i,j}(g(z\bar{z})^{1-\Delta})\mathcal{O}_k(0)\,,$$

where $C^k_{i,j}(x) = C^k_{i,j} + C^{k,(1)}_{i,j}x + C^{k,(2)}_{i,j}x^2+\cdots$.

The structural functions can be computed from PCFT, but the one-point functions

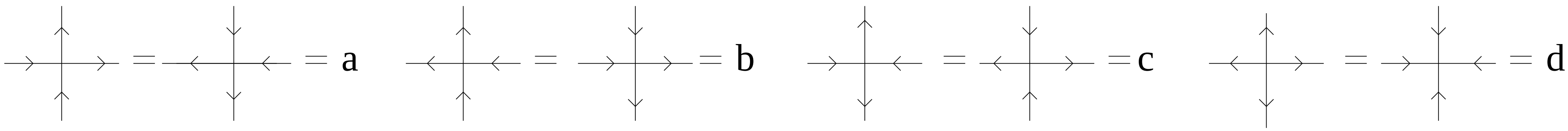
$$\langle \mathcal{O}_k(0) \rangle^{\mathrm{PCFT}} = g^{-\frac{\Delta_k + \bar{\Delta}_k}{2-2\Delta}} G_k$$

can not being non-analytical in g.
Generalization. We can impose some geometric environment, them OPE remain the same, and only G_k depend on the geometry. For example, for the cylinder of radius R we have

$$\langle \mathcal{O}_k(0) \rangle_R^{\mathrm{PCFT}} = g^{-\frac{\Delta_k + \bar{\Delta}_k}{2-2\Delta}} G_k(gR^{2-2\Delta}) \, .$$

Example.

Eight-vertex model



We consider homogeneous case when the Boltzmann weights are parametrized by two parameters: ν and k (which parametrizes the temperature).

$$a : b : c : d = \mathrm{sn}(\nu/2) : \mathrm{sn}(\nu/2) : \mathrm{sn}(\nu) : k(\mathrm{sn}(\nu/2))^2 \mathrm{sn}(\nu)\,.$$

Critical temperature $k = 0$. In this case $d = 0$,

$$a : b : c = \sin(\nu/2) : \sin(\nu/2) : \sin(\nu)\,.$$

This is homogeneous six-vertex model. Scaling theory is Euclidean sG:

$$\mathcal{A}^{\mathrm{sG}} = \int \left\{ \frac{1}{4\pi} \partial_z \boldsymbol{\varphi}(z,\bar{z}) \partial_{\bar{z}} \boldsymbol{\varphi}(z,\bar{z}) - \frac{\boldsymbol{\mu}^2}{\sin \pi\beta^2} 2\cos(\beta \boldsymbol{\varphi}(z,\bar{z})) \right\} \frac{idz \wedge d\bar{z}}{2}\,.$$

The CFT has central charge $c = 1$.

Expectation values for six vertex model.

Consider the partition function with defect:

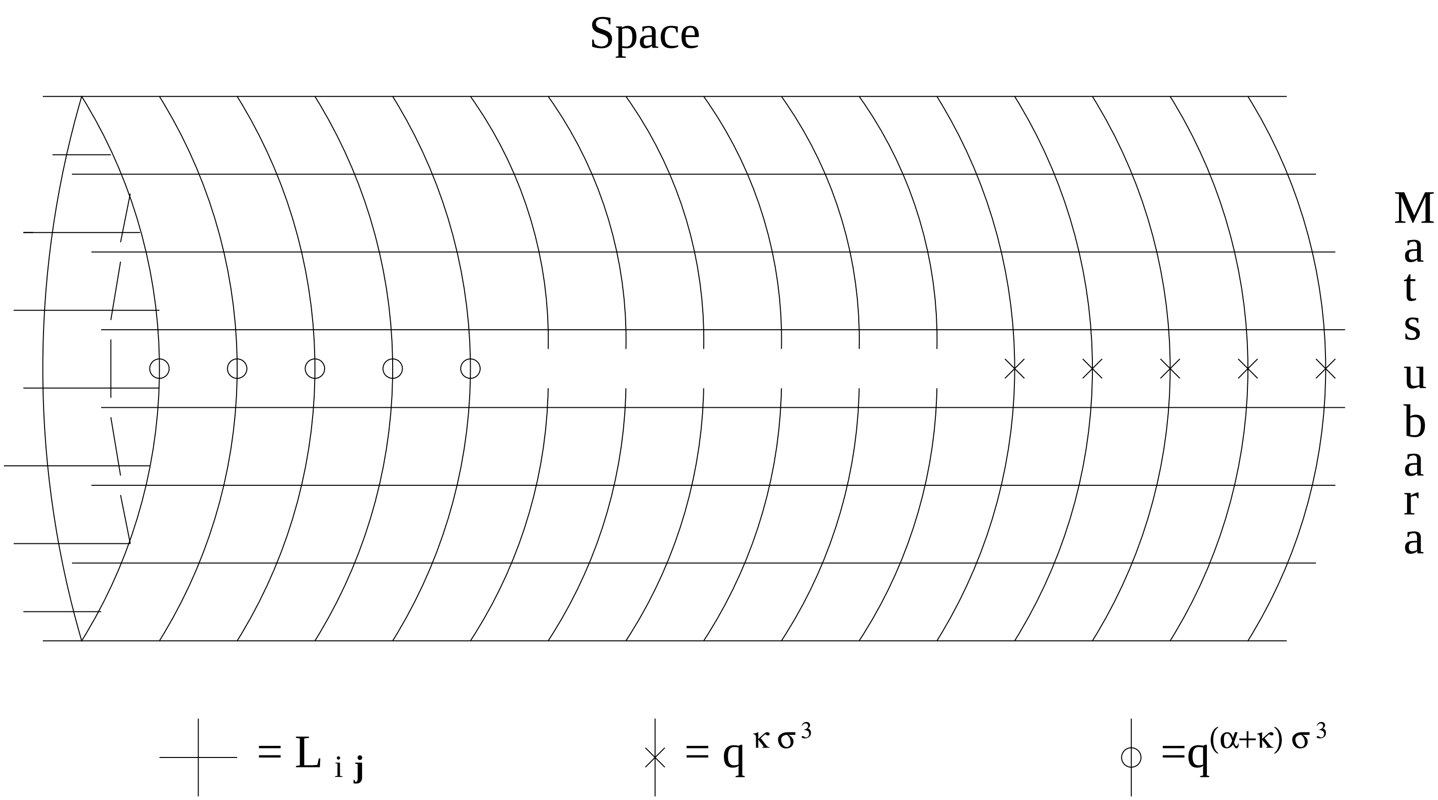


fig. 1

Exact definition. R-matrix

$$R_{1,2}(\zeta) = q^{\frac{1}{2}(\sigma_1^3\sigma_2^3+1)}\zeta - q^{-\frac{1}{2}(\sigma_1^3\sigma_2^3+1)}\zeta^{-1} + (q-q^{-1})(\sigma_1^+\sigma_2^- + \sigma_1^-\sigma_2^+)\,.$$

This R-matrix satisfies: Yang-Baxter equations:

$$R_{1,2}(\zeta_1/\zeta_2)R_{1,3}(\zeta_1/\zeta_3)R_{2,3}(\zeta_2/\zeta_3) = R_{2,3}(\zeta_2/\zeta_3)R_{1,3}(\zeta_1/\zeta_3)R_{1,2}(\zeta_1/\zeta_2)\,.$$

One more important property

$$R_{1,2}(1) = (q-q^{-1})P_{1,2}\,.$$

we shall formally use the space $\mathfrak{H}_{\mathrm{S}} = \bigotimes_{j=-\infty}^{\infty} \mathbb{C}^2$. Let us consider also the space $\mathfrak{H}_{\mathbf{M}} = \bigotimes_{\mathbf{j=1}}^{\mathbf{n}} \mathbb{C}^2$, where $\mathbf{M}$ stands for Matsubara.

Introduce

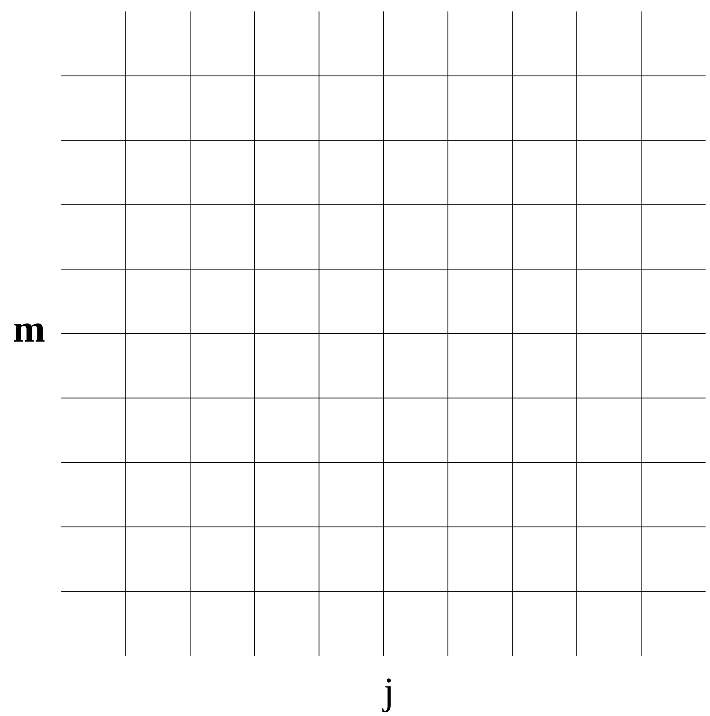
$$T_{j,\mathbf{M}}(\zeta) = R_{j,\mathbf{n}}(\zeta q^{-1/2}) \cdots R_{j,\mathbf{1}}(\zeta q^{-1/2})\,,$$

and

$$T_{j,\mathbf{M}} = T_{j,\mathbf{M}}(1)\,.$$

Further,

$$T_{\mathrm{S},\mathbf{M}} = \lim_{N\to\infty} T_{-N+1,\mathbf{M}} \cdots T_{N,\mathbf{M}}\,.$$



Our main object, the partition function with defect can be presented as

$$Z^{\kappa}_{\mathbf{n}}\Big\{q^{2\alpha S(0)}\mathcal{O}\Big\}=\frac{\mathrm{Tr}_{\mathrm{S}}\mathrm{Tr}_{\mathbf{M}}\Big(T_{\mathrm{S},\mathbf{M}}q^{2\kappa S+2\alpha S(0)}\mathcal{O}\Big)}{\mathrm{Tr}_{\mathrm{S}}\mathrm{Tr}_{\mathbf{M}}\Big(T_{\mathrm{S},\mathbf{M}}q^{2\kappa S+2\alpha S(0)}\Big)}\,.$$

Notice the importance of maximal eigenvalues. Non-degeneracy condition $\langle\kappa+\alpha|\kappa\rangle\neq 0$.

Our results.

Consider the space

$$\mathcal{W}^{(\alpha)}=\bigoplus_{s=-\infty}^{\infty}\mathcal{W}_{\alpha-s,s}\,,$$

On this space we defined the creation operators $\mathbf{t}^*(\zeta)$, $\mathbf{b}^*(\zeta)$, $\mathbf{c}^*(\zeta)$ and annihilation operators $\mathbf{b}(\zeta)$, $\mathbf{c}(\zeta)$.

These are one-parameter families of operators of the form

$$\mathbf{t}^*(\zeta) = \sum_{p=1}^{\infty} (\zeta^2 - 1)^{p-1} \mathbf{t}^*_p \,,$$

$$\mathbf{b}^*(\zeta) = \zeta^{\alpha+2} \sum_{p=1}^{\infty} (\zeta^2 - 1)^{p-1} \mathbf{b}^*_p \,, \ \mathbf{c}^*(\zeta) = \zeta^{-\alpha-2} \sum_{p=1}^{\infty} (\zeta^2 - 1)^{p-1} \mathbf{c}^*_p \,,$$

$$\mathbf{b}(\zeta) = \zeta^{-\alpha} \sum_{p=1}^{\infty} (\zeta^2 - 1)^{-p} \mathbf{b}_p \,, \ \mathbf{c}(\zeta) = \zeta^{\alpha} \sum_{p=1}^{\infty} (\zeta^2 - 1)^{-p} \mathbf{c}_p \,.$$

The operator $\mathbf{t}^*(\zeta)$ is in the center of our algebra of creation-annihilation operators,

$$[\mathbf{t}^*(\zeta_1), \mathbf{t}^*(\zeta_2)] = [\mathbf{t}^*(\zeta_1), \mathbf{c}^*(\zeta_2)] = [\mathbf{t}^*(\zeta_1), \mathbf{b}^*(\zeta_2)] = 0,$$

$$[\mathbf{t}^*(\zeta_1), \mathbf{c}(\zeta_2)] = [\mathbf{t}^*(\zeta_1), \mathbf{b}(\zeta_2)] = 0 \,.$$

The rest of the operators $\mathbf{b}$, $\mathbf{c}$, $\mathbf{b}^*$, $\mathbf{c}^*$ are fermionic. The only non-vanishing anti-commutators are

$$[\mathbf{b}(\zeta_1), \mathbf{b}^*(\zeta_2)]_+ = -\psi(\zeta_2/\zeta_1, \alpha)\,, \quad [\mathbf{c}(\zeta_1), \mathbf{c}^*(\zeta_2)]_+ = \psi(\zeta_1/\zeta_2, \alpha)\,,$$

where

$$\psi(\zeta, \alpha) = \frac{\zeta^\alpha}{(\zeta^2 - 1)}\,.$$

Each Fourier mode has the block structure

$$\mathbf{t}^*_p \;:\; \mathcal{W}_{\alpha-s,s} \;\to\; \mathcal{W}_{\alpha-s,s}$$

$$\mathbf{b}^*_p, \mathbf{c}_p \;:\; \mathcal{W}_{\alpha-s+1,s-1} \;\to\; \mathcal{W}_{\alpha-s,s}\,, \quad \mathbf{c}^*_p, \mathbf{b}_p \;:\; \mathcal{W}_{\alpha-s-1,s+1} \;\to\; \mathcal{W}_{\alpha-s,s}\,.$$

Further

$$\mathbf{x}_p(X) = 0, \quad p > \mathrm{length}\big(X\big), \qquad \mathbf{x} = \mathbf{b}, \mathbf{c}\,,$$

$$\mathrm{length}\big(\mathbf{x}^*_p\big(X\big)\big) \le \mathrm{length}\big(X\big) + p\,, \qquad \mathbf{x} = \mathbf{b}, \mathbf{c}, \mathbf{t}\,.$$

Among them, $\boldsymbol{\tau} = \mathbf{t}_1^*/2$ plays a special role. It is the right shift by one site along the chain. Consider the set of operators

$$\boldsymbol{\tau}^m \mathbf{t}^*_{p_1} \cdots \mathbf{t}^*_{p_j} \mathbf{b}^*_{q_1} \cdots \mathbf{b}^*_{q_k} \mathbf{c}^*_{r_1} \cdots \mathbf{c}^*_{r_k} \Big(q^{2\alpha S(0)} \Big),$$

where $m \in \mathbb{Z}$, $j, k \in \mathbb{Z}_{\geq 0}$, $p_1 \geq \cdots \geq p_j \geq 2$, $q_1 > \cdots > q_k \geq 1$ and $r_1 > \cdots > r_k \geq 1$. constitutes a basis of $\mathcal{W}_{\alpha,0}$. Main theorem relating Space and Matsubara

$$Z^\kappa \big\{ \mathbf{t}^*(\zeta)(X) \big\} = 2\rho(\zeta) Z^\kappa \{X\} \,,$$

$$Z^\kappa \big\{ \mathbf{b}^*(\zeta)(X) \big\} = \frac{1}{2\pi i} \oint_\Gamma \omega(\zeta, \xi) Z^\kappa \big\{ \mathbf{c}(\xi)(X) \big\} \frac{d\xi^2}{\xi^2} \,,$$

$$Z^\kappa \big\{ \mathbf{c}^*(\zeta)(X) \big\} = -\frac{1}{2\pi i} \oint_\Gamma \omega(\xi, \zeta) Z^\kappa \big\{ \mathbf{b}(\xi)(X) \big\} \frac{d\xi^2}{\xi^2} \,,$$

the functions ρ and ω are defined by Matsubara.

Since

$$\mathbf{c}(\zeta)(q^{2\alpha S(0)}) = 0, \quad \mathbf{b}(\zeta)(q^{2\alpha S(0)}) = 0\,,$$

we obtain

$$Z^{\kappa}\Big\{\mathbf{t}^*(\zeta_1^0)\cdots\mathbf{t}^*(\zeta_k^0)\mathbf{b}^*(\zeta_1^+)\cdots\mathbf{b}^*(\zeta_l^+)\mathbf{c}^*(\zeta_l^-)\cdots\mathbf{c}^*(\zeta_1^-)\big(q^{2\alpha S(0)}\big)\Big\}$$
$$= \prod_{p=1}^{k} 2\rho(\zeta_p^0) \times \det\big(\omega(\zeta_i^+, \zeta_j^-)\big)_{i,j=1,\cdots,l}\,.$$

Taking the Taylor coefficients in $(\zeta_i^{\epsilon})^2 - 1$ in both sides, one obtains the value of Z^{κ} on an arbitrary element of the fermionic basis.

The analogy with the CFT becomes transparent at this point.

Quantum loop algebra $U_q'(\widehat{\mathfrak{sl}}_2)$.

Quantum groups. Multiplication (with unit 1):

$$m: \ \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \,,$$

and comultiplication

$$\Delta: \ \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \,,$$

with the requirement that Δ is a homomorphism:

$$\Delta(xy) = \Delta(x)\Delta(y) \,.$$

Antipode is an anti-homomorphism $s: \ \mathcal{A} \to \mathcal{A}$, it is a deformation of inverse for Lie algebra. Counit is a homomorphism $\epsilon: \ \mathcal{A} \to \mathcal{A}$

$$m \circ (s \otimes id) \circ \Delta(x) = m \circ (id \otimes s) \circ \Delta(x) = \epsilon(x) \,.$$

Let σ be the permutation of two copies of $\mathcal{A}$ in the tensor product:

$$\sigma : \ \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}\,, \quad \sigma(x \otimes y) = y \otimes x\,,$$

and

$$\Delta' = \sigma \circ \Delta\,.$$

The quasi-triangularity requires the existence of a universal R-matrix, $\mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$ which intertwines two comultiplications:

$$\Delta' = \mathcal{R} \Delta \mathcal{R}^{-1}\,.$$

The universal R-matrix satisfies the Yang-Baxter equation:

$$\mathcal{R}_{1,2}\mathcal{R}_{1,3}\mathcal{R}_{2,3} = \mathcal{R}_{2,3}\mathcal{R}_{1,3}\mathcal{R}_{1,2}\,,$$

another important property is

$$(id \otimes s)\mathcal{R} = \mathcal{R}^{-1}\,.$$

$U'_Q(\widehat{\mathfrak{sl}}_2)$ is generated by e_i, f_i, h_i ($i = 0, 1$). We consider the case of central charge equal to zero: $h_1 = -h_0 \equiv h$. Two Borel subalgebras $U_q(\mathfrak{b}^+)$ and $U_q(\mathfrak{b}^-)$ are generated respectively by e_i, h and f_i, h. We have the commutation relations:

$$[e_i, f_j] = \delta_{i,j} \frac{t_i - t_i^{-1}}{q - q^{-1}} \,,$$

where $t_i = q^{h_i}$. The deformed Serre relations are

$$e_i^3 e_j + (q^2 + q^{-2} + 1)(e_i^2 e_j e_i - e_i e_j e_i^2) - e_j e_i^3 = 0 \,,$$
$$f_i^3 f_j + (q^2 + q^{-2} + 1)(f_i^2 f_j f_i - f_i f_j f_i^2) - f_j f_i^3 = 0$$

The comultiplication and antipode are given by

$$\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i, \quad \Delta(t_i) = t_i \otimes t_i \,,$$
$$s(e_i) = -t_i^{-1} e_i, \quad \Delta(f_i) = f_i t_i, \quad s(t_i) = t_i^{-1} \,.$$

The comultiplication looks quite simple, but the universal R-matrix intertwining Δ and Δ' is complicated. It can be written as follows:

$$\mathcal{R} = \overline{\mathcal{R}} q^{-\frac{h\otimes h}{2}} ,$$

$$\overline{\mathcal{R}} = 1 - (q - q^{-1}) \sum_{i=0}^{1} e_i \otimes f_j + \cdots \in U_q(\mathfrak{b}^+) \otimes U_q(\mathfrak{b}^-) ,$$

where the $\cdots$ stands for terms of higher degree in generators.
Representations. Let E, F, H be generators of $U_q(\mathfrak{sl}_2)$. The evaluation representation:

$$ev_\zeta(e_0) = \zeta F, \quad ev_\zeta(e_1) = \zeta E, \quad ev_\zeta(f_0) = \zeta^{-1} E, \quad ev_\zeta(f_1) = \zeta^{-1} F, \quad ev_\zeta(h) = H .$$

Choosing finite-dimensional representation of dimension $2s+1$ we obtain

$\pi_\zeta^{(2s)}$.

We have $(ev_{\zeta_1} \otimes \pi^{(1)}_{\zeta_2})(\mathcal{R}) = \tau(\zeta)L(\zeta)\,, \quad \zeta = \zeta_1/\zeta_2\,,$

$$L(\zeta) = \begin{pmatrix} 1-\zeta^2 q^{H+1} & -(q-q^{-1})\zeta F \\ -(q-q^{-1})\zeta E & 1-\zeta^2 q^{-H+1} \end{pmatrix} t_0^{\sigma^3/2}\,,$$

This will be used for

- Finite-dimensional of dimension $2s+1$:

$$Fv_j = v_{j+1}, \quad Hv_j = (-2s+2j)v_j, \quad t_0 = q^{-H}\,,$$

$$Ev_j = (q^j - q^{-j})(q^{2(s-2s-1)} - q^{-2(j-2s-1)})v_{j-1}, \quad j = 0, \cdots, 2s\,.$$

- Shifted Verma module with lowest weight Λ and shift m are denoted by $V_{\eta,m}(\Lambda)$. They are defined by

$$Fv_j = v_{j+1}, \quad Hv_j = (\Lambda+2j)v_j\,,$$

$$Ev_j = q^{-\Lambda+1}(q^{\Lambda-H-2}-1)(q^{\Lambda+H}-1)v_{j-1}, \quad j = 0, \cdots, \infty\,.$$

$$t_0 v_j = q^{-H-m} v_j\,, \qquad v_{-1} = 0\,.$$

Important generalization.

Bazhanov, Lukyanov, Zamolodchikov (1996).

Recall that $\overline{\mathcal{R}} \in U_q(\mathfrak{b}^+) \otimes U_q(\mathfrak{b}^-)$. Suppose we are given two algebras $A^{\pm}$ and homomorphisms $U_q(\mathfrak{b}^+) \to A^+$, $U_q(\mathfrak{b}^-) \to A^-$. I shall use the term L-operator for the image of the universal R-matrix under these maps. The q-oscillator algebra Osc is an associative algebra with generators $\mathbf{a}, \mathbf{a}^*, q^D$, and defining relations

$$q^D \mathbf{a}\, q^{-D} = q^{-1}\mathbf{a}, \quad q^D\, \mathbf{a}^* q^{-D} = q\, \mathbf{a}^*,$$
$$\mathbf{a}\,\mathbf{a}^* = 1 - q^{2D+2}, \quad \mathbf{a}^*\mathbf{a} = 1 - q^{2D}\,.$$

Representations of Osc relevant to us are $\rho^{\pm} : Osc \to \mathrm{End}(W^{\pm})$ defined by

$$W^+ = \oplus_{k\geq 0}\mathbb{C}|k\rangle, \quad W^- = \oplus_{k<0}\mathbb{C}|k\rangle,$$
$$q^D|k\rangle = q^k|k\rangle,\ \mathbf{a}|k\rangle = (1-q^{2k})|k-1\rangle,\ \mathbf{a}^*|k\rangle = (1-\delta_{k,-1})|k+1\rangle\,.$$

Trace

$$\mathrm{Tr}(q^{2\alpha D}XY) = \mathrm{Tr}(q^{2\alpha D}q^{2\alpha d(X)}YX) \quad (X, Y \in Osc, q^D X q^{-D} = q^{d(X)}X),$$
$$\mathrm{Tr}(q^{2\alpha D}q^{mD}) = \frac{1}{1-q^{2\alpha+m}} \quad (m \in \mathbb{Z}).$$

There is a homomorphism of algebras $o_\zeta : U_q\mathfrak{b}^+ \to Osc$ given by

$$o_\zeta(e_0) = \frac{\zeta}{q-q^{-1}}\mathbf{a}, \quad o_\zeta(e_1) = \frac{\zeta}{q-q^{-1}}\mathbf{a}^*, \quad o_\zeta(t_0) = q^{-2D}, \quad o_\zeta(t_1) = q^{2D}.$$

We define representations $o_\zeta^\pm : U_q\mathfrak{b}^+ \to \mathrm{End}(W^\pm)$ by

$$o_\zeta^+ = \rho^+ \circ o_\zeta, \quad o_\zeta^- = \rho^- \circ o_\zeta \circ \iota,$$

where ι denotes the involution $e_i \to e_{1-i}$, $t_i \to t_{1-i}$ of $U_q\mathfrak{b}^+$.

We define

$$(o^{\pm}_{\zeta} \otimes \pi_{\xi})\mathcal{R} = \sigma(\zeta/\xi) \cdot L^{\circ}_{Aj}{}^{\pm}(\zeta/\xi)\,,$$

Then by self-consistency one finds:

$$L^{+}_{A,j} = L_{A,j}(\zeta), \quad L^{-}_{A,j} = \sigma^{1}_{j} L_{A,j}(\zeta)\sigma^{1}_{j}\,.$$

$$L_{A,j}(\zeta) := \begin{pmatrix} 1-\zeta^2 q^{2D_A+2} & -\zeta \mathbf{a}_A \\ -\zeta \mathbf{a}^{*}_A & 1 \end{pmatrix}_j \begin{pmatrix} q^{-D_A} & 0 \\ 0 & q^{D_A} \end{pmatrix}_j , \,.$$

Notice the indices j, A, a!
R-matrices. Obvious:

$$L_{a,b}(\zeta_2/\zeta_1)L_{a,\mathbf{j}}(\zeta_2)L_{b,\mathbf{j}}(\zeta_1) = L_{b,\mathbf{j}}(\zeta_1)L_{a,\mathbf{j}}(\zeta_2)L_{a,b}(\zeta_2/\zeta_1)\,,$$

$$L_{A,a}(\zeta_2/\zeta_1)L_{A,\mathbf{j}}(\zeta_2)L_{a,\mathbf{j}}(\zeta_1) = L_{a,\mathbf{j}}(\zeta_1)L_{A,\mathbf{j}}(\zeta_2)L_{A,a}(\zeta_2/\zeta_1)\,.$$

Less obvious. $R_{A,B}(\zeta_1/\zeta_2)$ satisfying

$$R_{A,B}(\zeta_1/\zeta_2)L_{A,\mathbf{j}}(\zeta_1)L_{B,\mathbf{j}}(\zeta_2) = L_{B,\mathbf{j}}(\zeta_2)L_{A,\mathbf{j}}(\zeta_1)R_{A,B}(\zeta_1/\zeta_2)\,.$$

does exist. It is given by

$$R_{A,B}(\zeta) = P_{A,B}h(\zeta, u_{A,B})\zeta^{D_A+D_B},$$

where $u_{A,B} = \mathbf{a}_A^* q^{-2D_A}\mathbf{a}_B$, and $h(\zeta, u)$ is the unique formal power series in u satisfying

$$(1+\zeta u)h(\zeta, u) = (1+\zeta^{-1}u)h(\zeta, q^2u),$$
$$h(\zeta, u) = (1+\zeta^{-1}u)(1+q^{-2}\zeta u)h(q^{-2}\zeta, u)$$

and $h(\zeta, 0) = 1$.

Set $\zeta = \frac{\zeta_1}{\zeta_2}, \quad q^\Lambda = q\zeta$ .. Then

$$\{0\} = W_L^{(-1)} \subset W_L^{(0)} \subset W_L^{(1)} \subset \cdots \subset W_L^{(m)} \subset \cdots \subset W_{\zeta_1}^+ \otimes W_{\zeta_2}^- ,$$

$$\mathrm{w}{-}\lim_{m\to\infty} W_L^{(m)} = W_{\zeta_1}^+ \otimes W_{\zeta_2}^-,$$

such that $\iota_L : W_L^{(m)}/W_L^{(m-1)} \overset{\sim}{\to} V_{\sqrt{\zeta_1\zeta_2},2m}(\Lambda)$ ..

$$W_{\zeta_2}^- \otimes W_{\zeta_1}^+ = W_R^{(-1)} \supset W_R^{(0)} \supset \cdots \supset W_R^{(m)} \supset \cdots ,$$

$$\mathrm{w}{-}\lim_{m\to\infty} W_R^{(m)} = 0,$$

such that $\iota_R : W_R^{(m-1)}/W_R^{(m)} \overset{\sim}{\to} V_{\sqrt{\zeta_1\zeta_2},2m}(\Lambda)$.

There is no R-matrix, but there is a replacement. Introduce

$$U_{A,B}(\zeta) = \zeta \mathbf{a}_A^* + \mathbf{a}_B q^{2D_A}\,, \quad V_{A,B}(\zeta) = \zeta \mathbf{a}_B^* + \mathbf{a}_A q^{2D_B}\,,$$

$$Y_{A,B}(\zeta) = (\zeta q^2 - \mathbf{a}_A \mathbf{a}_B) q^{2D_A}\,, \quad Z_{A,B}(\zeta) = \zeta^{-1} q^{2D_B+2} - \mathbf{a}_A^* \mathbf{a}_B^* q^{-2D_A}\,.$$

$\mathcal{X}^L$	$\mathcal{X}^R$
$V_{A,B}(\zeta)$	0
0	$V_{B,A}(\zeta^{-1})$
$Y_{A,B}(\zeta)$	$Z_{B,A}(\zeta^{-1})$
$Z_{A,B}(\zeta)$	$Y_{B,A}(\zeta^{-1})$
$\sqrt{\zeta}(1-\zeta^{-1}q^{-2}Z_{A,B}(\zeta))\mathbf{a}_A$	$\sqrt{\zeta}\,U_{B,A}(\zeta^{-1})$
$\sqrt{\zeta}^{-1}U_{A,B}(\zeta)$	$\sqrt{\zeta}^{-1}(1-\zeta q^{-2}Z_{B,A}(\zeta^{-1}))\mathbf{a}_B$

$$\mathrm{Tr}_{A,B}\left\{\mathcal{X}^L(\zeta)\; L^+_{A,\star}(\zeta_1) q^{2\alpha D_A} L^-_{B,\star}(\zeta_2) q^{2\alpha D_B}\right\}$$

$$= \mathrm{Tr}_{A,B}\left\{\mathcal{X}^R(\zeta) L^-_{B,\star}(\zeta_2) q^{2\alpha D_B} L^+_{A,\star}(\zeta_1) q^{2\alpha D_A}\right\}\,,$$

One more important property.

$$L_{\{a,A\},j}(\zeta)(F_{a,A})^{-1}L_{a,j}(\zeta)L_{A,j}(\zeta)F_{a,A}$$

$$= \begin{pmatrix} 1 & 0 \\ \frac{q-q^{-1}}{\zeta-\zeta^{-1}}\,\sigma_j^+ & 1 \end{pmatrix}_a \begin{pmatrix} (\zeta^2-1)L_{A,j}(q\zeta)q^{-\sigma_j^3/2} & 0 \\ 0 & (\zeta^2q^2-1)L_{A,j}(q^{-1}\zeta)q^{\sigma_j^3/2} \end{pmatrix}_a ,$$

where $F_{a,A} = 1 - \mathbf{a}_A\sigma_a^+$.

Spectral properties in Matsubara direction.

Define

$$T_{\mathbf{x},\mathbf{M}}(\zeta) = L_{x,\mathbf{n}}(\zeta q^{-1/2} \cdots L_{x,\mathbf{1}}(\zeta q^{-1/2})\,,$$

and

$$T_{\mathbf{M}}(\zeta,\kappa) = \mathrm{Tr}_a\left(T_{\mathbf{a},\mathbf{M}}(\zeta)q^{\kappa\sigma_a^3}\right)\,, \quad Q^{\pm}_{\mathbf{M}}(\zeta,\kappa) = \zeta^{\pm(\kappa-\mathbf{S})}\mathrm{Tr}^+_A\left(T^{\pm}_{\mathbf{A},\mathbf{M}}(\zeta)q^{2\kappa D_A}\right)\,,$$

Then they satisfy Baxter equation

$$T_{\mathbf{M}}(\zeta,\kappa)Q^{\pm}_{\mathbf{M}}(\zeta,\kappa) = (\zeta^2q-1)^{\mathbf{n}}Q^{\pm}_{\mathbf{M}}(\zeta q^{-1},\kappa) + (\zeta^2q^{-1}-1)^{\mathbf{n}}Q^{\pm}_{\mathbf{M}}(\zeta q,\kappa)\,,$$

construction of annihilation operators.

Consider the operator $X_{[k,l]} \in \mathrm{End}\left(\mathbb{C}^{\otimes(l-k+1)}\right)$. Define

$$T_{a,[k,l]}(\zeta) = L^{\circ}_{a,l}(\zeta)\cdots L^{\circ}_{a,k}(\zeta)\,,$$

and the adjoint monodromy matrix

$$\mathbb{T}_a(\zeta,\alpha)(X_{[k,l]}) = T_{a,[k,l]}(\zeta)q^{\alpha\sigma^3_a}X_{[k,l]}T_{a,[k,l]}(\zeta)^{-1}\,,\ .$$

Define further

$$\mathbb{S}(X_{[k,l]}) := [S_{[k,l]}, X_{[k,l]}], \quad S_{[k,l]} := \tfrac{1}{2}\textstyle\sum_{j\in[k,l]}\sigma^3_j.$$

Then

$$(F_{a,A})^{-1}\left(\mathbb{T}_a(\zeta,\alpha)\mathbb{T}_A(\zeta,\alpha)(X_{[k,l]})\right)F_{a,A} = \begin{pmatrix}\mathbb{A}_A(\zeta,\alpha)(X_{[k,l]}) & 0\\ \mathbb{C}_A(\zeta,\alpha)(X_{[k,l]}) & \mathbb{D}_A(\zeta,\alpha)(X_{[k,l]})\end{pmatrix}_a,$$

Define

$$\mathbf{k}(\zeta,\alpha)(X_{[k,l]}) := \mathrm{Tr}_A\Big\{\mathbb{C}_A(\zeta,\alpha)\zeta^{\alpha-\mathbb{S}}\big(q^{-2S_{[k,l]}}\,X_{[k,l]}\big)\Big\}\,.$$

$\mathbf{k}(\zeta,\alpha)(X_{[k,l]})$ has poles of hight order at $\zeta^2 = 1, q^{\pm 2}$.
We shall use the involution

$$\phi(\mathbf{k})(\zeta,\alpha) = q^{-1}N(\alpha-\mathbb{S}-1)\circ\mathbb{J}\circ\mathbf{k}(\zeta,-\alpha)\circ\mathbb{J}\,,\quad N(x) = q^{-x}-q^{x}\,.$$

Define the operation

$$\Delta_\zeta f(\zeta) = f(\zeta q) - f(\zeta q^{-1})\,.$$

Definition. Exact q-one form is an expression of the form $\Delta_\zeta f(\zeta)$ with $f(\zeta)$ having poles at $\zeta^2 = 1$.

Using our algebra it can be shown.

$$\begin{aligned}&\mathbf{k}(\zeta_1,\alpha)\mathbf{k}(\zeta_2,\alpha+1)+\mathbf{k}(\zeta_2,\alpha)\mathbf{k}(\zeta_1,\alpha+1)\\&\quad=\Delta_{\zeta_1}\mathbf{m}^{(++)}(\zeta_1,\zeta_2,\alpha)+\Delta_{\zeta_2}\mathbf{m}^{(++)}(\zeta_2,\zeta_1,\alpha),\\&\mathbf{k}(\zeta_1,\alpha)\phi(\mathbf{k})(\zeta_2,\alpha+1)+\phi(\mathbf{k})(\zeta_2,\alpha)\mathbf{k}(\zeta_1,\alpha-1)\\&\quad=\Delta_{\zeta_1}\mathbf{m}^{(+-)}(\zeta_1,\zeta_2,\alpha)+\Delta_{\zeta_2}\mathbf{m}^{(-+)}(\zeta_2,\zeta_1,\alpha)\,,\end{aligned}$$

In RHS we have exact q-two forms.
Consider

$$\bar{\mathbf{c}}(\zeta,\alpha)(X_{[k,l]}):=\frac{1}{2\pi i}\int\limits_{\Gamma}\psi(\zeta/\xi,\alpha+\mathbb{S})\mathbf{k}(\xi,\alpha)(X_{[k,l]})\frac{d\xi^2}{\xi^2}\,,$$

$$\mathbf{c}(\zeta,\alpha)(X_{[k,l]}):=\frac{1}{4\pi i}\int\limits_{\Gamma}\psi(\zeta/\xi,\alpha+\mathbb{S})\left\{\mathbf{k}(q\xi,\alpha)+\mathbf{k}(q^{-1}\xi,\alpha)\right\}(X_{[k,l]})\frac{d\xi^2}{\xi^2}\,,$$

Then

$$\mathbf{c}(\zeta_1,\alpha)\mathbf{c}(\zeta_2,\alpha+1)+\mathbf{c}(\zeta_2,\alpha)\mathbf{c}(\zeta_1,\alpha+1)=0\,.$$

Define further

$$\mathbf{b}(\zeta, \alpha) = \phi(\mathbf{c})(\zeta, \alpha)\,.$$

Then

$$\mathbf{b}(\zeta_1, \alpha)\mathbf{b}(\zeta_2, \alpha - 1) + \mathbf{b}(\zeta_2, \alpha)\mathbf{b}(\zeta_1, \alpha - 1) = 0\,.$$

$$\mathbf{c}(\zeta_1, \alpha)\mathbf{b}(\zeta_2, \alpha + 1) + \mathbf{b}(\zeta_2, \alpha)\mathbf{c}(\zeta_1, \alpha - 1) = 0\,.$$

Is there more conceptual explanation for existence of these anti-commuatative families of operators?

Reduction. Trivial relation For $X_{[k,l]} = q^{2(\alpha+1)S_{[k,m-1]}} \otimes Y_{[m,l]}$ with $k < m < l$ we have

$$\mathbf{k}(\zeta,\alpha)(q^{2(\alpha+1)S_{[k,m-1]}} \otimes Y_{[m,l]}) = q^{2\alpha S_{[k,m-1]}} \otimes \mathbf{k}(\zeta,\alpha)(Y_{[m,l]})\,,$$

Non-trivial relation

$$\mathbf{k}(\zeta,\alpha)(Y_{[k,m]} \otimes I_{[m+1,l]}) = \mathbf{k}(\zeta,\alpha)(Y_{[k,m]}) \otimes I_{[m+1,l]} + \Delta_\zeta \mathbf{v}(\zeta,\alpha)(Y_{[k,m]} \otimes I_{[m+1,l]})\,,$$

These relations imply

$$\mathbf{c}(\zeta,\alpha)(q^{2(\alpha+1)S_{[k,m-1]}} \otimes Y_{[m,l]}) = q^{2\alpha S_{[k,m-1]}} \otimes \mathbf{c}(\zeta,\alpha)(Y_{[m,l]})$$
$$\mathbf{c}(\zeta,\alpha)(Y_{[k,m]} \otimes I_{[m+1,l]}) = \mathbf{c}(\zeta,\alpha)(Y_{[k,m]}) \otimes I_{[m+1,l]}\,.$$

Hence the inductive limit $l \to \infty, k \to -\infty$ is consistent.

Creation operators.

Simple but instructive example. Define

$$\mathbf{t}^*(\zeta,\alpha)(X_{[k,l]}) = \mathrm{Tr}_a \mathbb{T}_a(\zeta,\alpha)(X_{[k,l]})\,.$$

For trivial reasons it satisfies the left reduction relation:

$$\mathbf{t}^*(\zeta,\alpha)(q^{2\alpha S_{[k,m-1]}} \otimes X_{[m,l]}) = q^{2\alpha S_{[k,m-1]}} \otimes \mathbf{t}^*(\zeta,\alpha)(X_{[m,l]})\,.$$

Set

$$\widetilde{R}^{\vee}_{i,j}(\zeta^2) = \zeta^{\sigma^3_i/2} R_{i,j}(\zeta) P_{i,j} \zeta^{-\sigma^3_j/2}, \quad \widetilde{\mathbb{R}}^{\vee}_{i,j}(\zeta^2) = \zeta^{\mathbb{S}_i} \mathbb{R}_{i,j}(\zeta) \mathbb{P}_{i,j} \zeta^{-\mathbb{S}_j}\,,\,,$$

then

$$\mathbf{t}^*_{[k,l]}(\zeta,\alpha)(X_{[k,m]}) = \mathrm{Tr}_a\{\widetilde{\mathbb{R}}^{\vee}_{a,l}(\zeta^2)\widetilde{\mathbb{R}}^{\vee}_{l,l-1}(\zeta^2)\cdots\widetilde{\mathbb{R}}^{\vee}_{k+1,k}(\zeta^2)(q^{\alpha\sigma^3_k}\boldsymbol{\tau}(X_{[k,m]}))\}.$$

Define

$$\widetilde{\mathbb{R}}^{\vee}_{i,j}(\zeta^2) = 1 + (\zeta^2 - 1)\mathbf{r}_{i,j}(\zeta^2)\,,$$

and

$$\tilde{\mathbb{R}}^{\vee}(\zeta^2)(X_{[k,l]}) := \widetilde{\mathbb{R}}^{\vee}_{l,l-1}(\zeta^2)\cdots\widetilde{\mathbb{R}}^{\vee}_{k+1,k}(\zeta^2)(X_{[k,l]}).$$

Then

$$\begin{aligned}
&\mathbf{t}^*_{[k,l]}(\zeta,\alpha)(X_{[k,m]})\\
&\quad = 2\sum_{j=m}^{l-1}(\zeta^2-1)^{j-m}\mathbf{r}_{j+1,j}(\zeta^2)\cdots\mathbf{r}_{m+2,m+1}(\zeta^2)\tilde{\mathbb{R}}^{\vee}(\zeta^2)(Y_{[k,m+1]})\\
&\quad +(\zeta^2-1)^{l-m}\mathrm{Tr}_a\left\{\mathbf{r}_{a,l}(\zeta^2)\mathbf{r}_{l,l-1}(\zeta^2)\cdots\mathbf{r}_{m+2,m+1}(\zeta^2)\tilde{\mathbb{R}}^{\vee}(\zeta^2)(Y_{[k,m+1]})\right\}.
\end{aligned}$$

That is why the inductive limit $l\to\infty$ is well-defined for any fixed $(\zeta^2-1)^p$, and we have an operator

$$\mathbf{t}^*(\zeta) = \sum_{p=1}^{\infty}(\zeta^2-1)^{p-1}\mathbf{t}^*_p \quad : \quad \mathcal{W}_{\alpha,0}\to\mathcal{W}_{\alpha,0}\,.$$

The operator $\bar{\mathbf{c}}(\zeta)$ is not independent:

$$\bar{\mathbf{c}}(\zeta) = -\frac{1}{2\pi i}\int\limits_{\Gamma} \psi(\zeta/\xi, \boldsymbol{\alpha})\mathbf{t}^*(\xi)\mathbf{c}(\xi)\frac{d\xi^2}{\xi^2}\,,$$

Fermionic creation operators. Looking at

$$\mathbf{k}(\zeta,\alpha)(Y_{[k,m]}\otimes I_{[m+1,l]}) = \mathbf{k}(\zeta,\alpha)(Y_{[k,m]})\otimes I_{[m+1,l]} + \Delta_\zeta \mathbf{v}(\zeta,\alpha)(Y_{[k,m]}\otimes I_{[m+1,l]})\,,$$

one is tempted to consider

$$\mathbf{f}(\zeta,\alpha) = \Delta_\zeta^{-1}\mathbf{k}(\zeta,\alpha)\,.$$

What to do with the transcendental functions?

There is a singularity at $\zeta^2 = 1$, but by definition it comes with $\bar{\mathbf{c}}$. There are singularities at $\zeta^2 = q^{\pm 2}$, but by definition their half sum comes with $\mathbf{c}(\zeta)$. We define once forever

$$\Delta_\zeta^{-1}\psi(\zeta,\alpha)= \tfrac{1}{2\nu} VP \int_0^\infty \psi(\zeta/\eta,\alpha)\frac{\eta^{1/\nu}}{\eta^{1/\nu}+1}\frac{d\eta^2}{2\pi i\eta^2}$$

Definition which we cannot explain logically.

$$\mathbf{b}^*(\zeta,\alpha)(X_{[k,l]}) := \left(\mathbf{f}(q\zeta,\alpha)+\mathbf{f}(q^{-1}\zeta,\alpha)-\mathbf{t}^*(\zeta,\alpha)\mathbf{f}(\zeta,\alpha)\right)(X_{[k,l]})\,.$$

Apart of usual trivial left reduction we have

$$\mathbf{b}^*(\zeta,\alpha)(X_{[k,m]}\otimes I_{[m+1,l]}) = \mathrm{Tr}_c\left\{\mathbb{T}_{c,[m+1,l]}(\zeta)\left(\mathbf{g}_c(\zeta,\alpha)(X_{[k,m]})\otimes I_{[m+1,l]}\right)\right\}\ .$$

This reduction relation allows to define $\mathbf{b}^*(\zeta)$ as operator acting from $\mathcal{W}_{\alpha-1,s+1}$ to $\mathcal{W}_{\alpha,s}$. Similarly we define

$$\mathbf{c}^*(\zeta,\alpha) := -\phi(\mathbf{b}^*)(\zeta,\alpha)\,.$$

The proof of commutativity with $\mathbf{t}^*$ is not very hard. The proof of anti-commutation relations with annihilation operators is very hard. The proof of anti-commutation relations of creation operators between themselves can be obtained indirectly.

All that was essentially based on HGSII.

Switching on the Matsubara direction. In HGSIII we prove the relations

$$Z^{\kappa}\Big\{\mathbf{t}^*(\zeta)(X)\Big\} = 2\rho(\zeta)Z^{\kappa}\{X\}\,,$$

$$Z^{\kappa}\Big\{\mathbf{b}^*(\zeta)(X)\Big\} = \frac{1}{2\pi i}\oint\limits_{\Gamma}\omega(\zeta,\xi)Z^{\kappa}\Big\{\mathbf{c}(\xi)(X)\Big\}\frac{d\xi^2}{\xi^2}\,,$$

$$Z^{\kappa}\Big\{\mathbf{c}^*(\zeta)(X)\Big\} = -\frac{1}{2\pi i}\oint\limits_{\Gamma}\omega(\xi,\zeta)Z^{\kappa}\Big\{\mathbf{b}(\xi)(X)\Big\}\frac{d\xi^2}{\xi^2}\,.$$

The function $\rho(\zeta)$ is simple:

$$\rho(\zeta) = \frac{T(\zeta,\kappa+\alpha)}{T(\zeta,\kappa)}\,.$$

The function $\omega(\zeta,\xi)$ was characterised as a quantum deformation of normalised second kind differential on hyperelliptic Riemann surface. For our goals another definition is more useful which is due to Boos and Göhmann.

Destri-Devega equations. We shall denote $Q = Q^-$. Consider the Baxter equation

$$T(\zeta,\kappa)Q(\zeta,\kappa) = (\zeta^2 q - 1)^{\mathbf{n}} Q(\zeta q^{-1},\kappa) + (\zeta^2 q^{-1} - 1)^{\mathbf{n}} Q(\zeta q,\kappa)\,.$$

Define

$$\mathfrak{a}(\zeta,\kappa) = \left(\frac{1-\zeta^2 q^{-1}}{1-\zeta^2 q}\right)^{\mathbf{n}} \frac{Q(\zeta q,\kappa)}{Q(\zeta q^{-1},\kappa)}\,.$$

Then it is easy to derive DDV equation

$$\log\mathfrak{a}(\zeta,\kappa) = -2\pi i\nu\kappa + \mathbf{n}\log\left(\frac{1-\zeta^2 q^{-1}}{1-\zeta^2 q}\right) - \int_\gamma K(\zeta/\xi)\log\left(1+\mathfrak{a}(\xi,\kappa)\right)\frac{d\xi^2}{\xi^2}\,,$$

where

$$K(\zeta,\alpha) = \frac{1}{2\pi i}\Delta_\zeta \psi(\zeta,\alpha), \quad K(\zeta) = K(\zeta,0)\,,$$

and γ goes *clockwise* around zeros of $Q(\zeta,\kappa)$.

Introduce

$$\delta_\zeta^- f(\zeta) = f(\zeta q) - \rho(\zeta) f(\zeta),$$

Set

$$f \star g = \int\limits_\gamma f(\eta) g(\eta) dm(\eta)\,,$$

where the measure is given by

$$dm(\eta) = \frac{d\eta^2}{\eta^2 \rho(\eta)\left(1 + \mathfrak{a}(\eta,\kappa)\right)}\,,$$

and the resolvent

$$R_{\mathrm{dress}} - R_{\mathrm{dress}} \star K_\alpha = K_\alpha\,.$$

Let

$$f_{\mathrm{left}}(\zeta,\xi) = \tfrac{1}{2\pi i}\,\delta_\zeta^- \psi(\zeta/\xi,\alpha), \quad f_{\mathrm{right}}(\zeta,\xi) = \delta_\xi^- \psi(\zeta/\xi,\alpha)\,.$$

The function ω is given by

$$\omega(\zeta,\xi) = \left(f_{\rm left}\star f_{\rm right} + f_{\rm left}\star R_{\rm dress}\star f_{\rm right}\right)(\zeta,\xi) + \omega_0(\zeta,\xi)\,,$$

where

$$\omega_0(\zeta,\xi) = \delta^-_\zeta\,\delta^-_\xi\,\Delta^{-1}_\zeta\psi(\zeta/\xi,\alpha)\,.$$

Important generalization.

First, introduce operators without transcendental part:

$$\mathbf{b}^*_{\rm rat}(\zeta) = e^{-\Omega_0}\ \mathbf{b}^*(\zeta)\ e^{\Omega_0},\quad \mathbf{c}^*_{\rm rat}(\zeta) = e^{-\Omega_0}\ \mathbf{c}^*(\zeta)\ e^{\Omega_0}\,,$$

where

$$\Omega_0 = \frac{1}{(2\pi i)^2}\oint\limits_\Gamma\oint\limits_\Gamma \omega_0(\zeta,\xi)\mathbf{c}(\xi)\mathbf{b}(\zeta)\ \frac{d\zeta^2}{\zeta^2}\frac{d\xi^2}{\xi^2}\,.$$

When $\mathbf{c}^*_{\rm rat}(\zeta)$ act to $\mathcal{W}_{\alpha,0}$ it behaves at $\zeta^2=0$ as

$$\mathbf{c}^*_{\rm rat}(\zeta) = \sum_{j=1}^{\infty}\zeta^{-\alpha+2j}\mathbf{c}^*_{{\rm screen},j}$$

Changing boundary conditions. Let us try

$$\mathrm{Tr}_{\mathrm{S}}\mathrm{Tr}_{\mathbf{M}}\Bigl(Y^{(s)}_{\mathbf{M}}T_{\mathrm{S},\mathbf{M}}q^{2\kappa S}\ X\Bigr),\quad X\in\mathcal{W}_{\alpha+s,-s}\,.$$

We could try

$$X=\mathbf{c}^*_{j_1}\cdots\mathbf{c}^*_{j_{m+s}}\mathbf{b}^*_{i_1}\cdots\mathbf{b}^*_{i_m}q^{2\alpha S(0)}\,,$$

but there would be a trouble with the main theorem because

$$\mathbf{c}^*_{\mathrm{rat}}(\zeta)=\sum_{j=-s+1}^{\infty}\zeta^{-\alpha+2j}\mathbf{c}^*_{\mathrm{screen},j}\,,$$

when the target is $\mathcal{W}_{\alpha+s,-s}$. The trick is to define

$$Z^{\kappa,-s}_{\mathbf{n}}\Bigl\{q^{2\alpha S(0)}\mathcal{O}\Bigr\}=\frac{\mathrm{Tr}_{\mathrm{S}}\mathrm{Tr}_{\mathbf{M}}\Bigl(Y^{(s)}_{\mathbf{M}}T_{\mathrm{S},\mathbf{M}}q^{2\kappa S}\ \mathbf{c}^*_{\mathrm{screen},-0}\cdots\mathbf{c}^*_{\mathrm{screen},-s+1}\bigl(q^{2\alpha S(0)}\mathcal{O}\bigr)\Bigr)}{\mathrm{Tr}_{\mathrm{S}}\mathrm{Tr}_{\mathbf{M}}\Bigl(Y^{(s)}_{\mathbf{M}}T_{\mathrm{S},\mathbf{M}}q^{2\kappa S}\ \mathbf{c}^*_{\mathrm{screen},-0}\cdots\mathbf{c}^*_{\mathrm{screen},-s+1}\bigl(q^{2\alpha S(0)}\bigr)\Bigr)}\,.$$

The determinant formula remains valid with the only change:

$$\rho(\zeta) = \frac{T(\zeta|\alpha+\kappa+s,-s)}{T(\zeta,\kappa)}\,.$$

Recall that

$$Q_{\mathbf{M}}(\zeta,\kappa) = \zeta^{-\kappa+\mathbf{S}} A_{\mathbf{M}}(\zeta^2)\,.$$

The Bethe equations for spin $-s$ and twist $\alpha+\kappa+s$ are formally the same as for spin 0 and twist

$$\kappa' = \alpha+\kappa-2s\tfrac{1-\nu}{\nu}\,.$$

The only difference is that the degree of $A(\zeta^2)$ is not $\mathbf{n}/2$, but $\mathbf{n}/2-s$. This will be irrelevant in the scaling limit when the number of roots is infinite anyway.

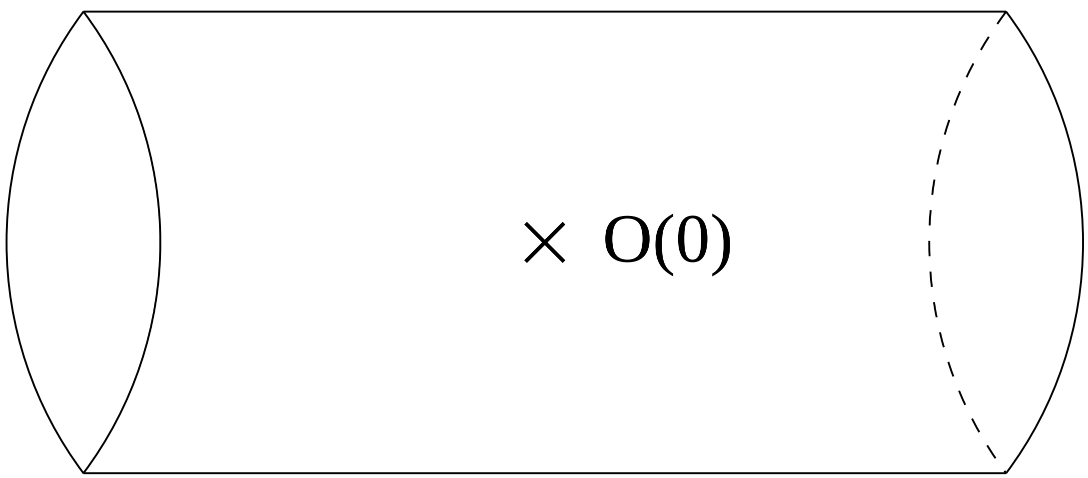
Back to CFT.

We believe that in proper scaling limit $Z_{\mathbf{n}}^{\kappa,-s}$ must give three-point functions for CFT with

$$c = 1 - 6\frac{\nu^2}{1-\nu}\,.$$

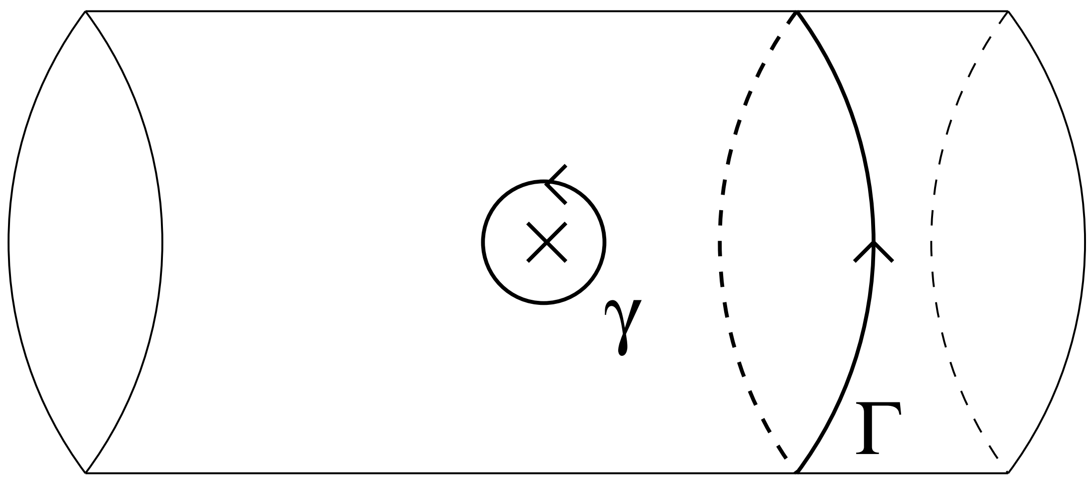
Consider the partition function with $O(0)$ inserted at $z=0$ on the cylinder

$$-\infty < \text{Re}(z) < \infty, \quad -\pi R < \text{Im}(x) < \pi R, \quad \mathbb{R} - \pi i R = \mathbb{R} + \pi i R\,,$$



Along with the local coordinate x, we shall also use the global coordinate $z = e^{-\frac{x}{R}}$. We have two Virasoro algebras. One is generated by

$$(\mathbf{l}_n O)(y) = \int\limits_{\gamma} \frac{dx}{2\pi i}(x-y)^{n+1} T(x) O(y)\,.$$



The OPE provide

$$[\mathbf{l}_m, \mathbf{l}_n] = (m-n)\mathbf{l}_{m+n} + \frac{c}{12} n(n^2-1)\delta_{m+n,0}\,.$$

In global coordinate $\tilde{T}(z) = \sum_{n=-\infty}^{\infty} L_n z^{-n-2}$, From $\tilde{T}(z)(dz)^2 = (T(x) - (c/12)\{z;x\})(dx)^2$, $\{z;x\} = \frac{z'''}{z'} - \frac{3}{2}\left(\frac{z''}{z'}\right)^2 = \frac{1}{2R^2}$ we find

$$T(x) = \frac{1}{R^2}\Big(\sum_{n=-\infty}^{\infty} L_n e^{\frac{nx}{R}} - \frac{c}{24}\Big)\,.$$

Fix a primary field $\phi_\Delta(y)$ with the scaling dimension Δ:

$$(\mathbf{l}_0\phi_\Delta)(y) = \Delta\phi_\Delta(y), \quad (\mathbf{l}_n\phi_\Delta)(y) = 0 \quad (n > 0)\,.$$

The boundary conditions are imposed

$$\lim_{x\to\pm\infty} T(x) = \frac{1}{R^2}\left(\Delta_\pm - \frac{c}{24}\right)\,.$$

To compute

$$Z_{\Delta_+,\Delta_-}\{\mathcal{O}_\alpha\} = \frac{\langle \mathcal{O}_\alpha(0)\rangle}{\langle \phi_\alpha(0)\rangle}\ ,$$

we use the OPE

$$T(x)T(y) = -\frac{c}{12R}\chi'''(x-y) - \frac{2T(y)}{R}\chi'(x-y) + \frac{T'(y)}{R}\chi(x-y) + O(1)\,,$$
$$T(x)\phi_\Delta(y) = -\frac{\Delta\phi_\Delta(y)}{R}\chi'(x-y) + \frac{\phi'_\Delta(y)}{R}\chi(x-y) + O(1)\,,$$

where

$$\chi(x) = \frac{1}{2}\coth\left(\frac{x}{2R}\right) = \sum_{n=0}^{\infty}\frac{B_{2n}}{(2n)!}\left(\frac{x}{R}\right)^{2n-1}\ .$$

Here $B_0 = 1, B_2 = 1/6, B_4 = -1/30, \cdots$ are the Bernoulli numbers.

The result is

$$
\begin{aligned}
&Z_{\Delta_+,\Delta_-}\{T(x_k)\cdots T(x_1)\phi_\Delta(y)\} \\
&\quad = -\frac{c}{12R}\sum_{j=2}^{k}\chi'''(x_1-x_j)\langle T(x_k)\cdots \overset{j}{\wedge} \cdots T(x_2)\phi_\Delta(y)\rangle_{\Delta_+,\Delta_-} \\
&\quad + \Big\{\sum_{j=2}^{k}\big(-\frac{2}{R}\chi'(x_1-x_j)+\frac{1}{R}(\chi(x_1-x_j)-\chi(x_1-y))\frac{\partial}{\partial x_j}\big)-\frac{\Delta}{R}\chi'(x_1-y) \\
&\quad + (\Delta_+-\Delta_-)\frac{1}{R^2}\chi(x_1-y)+\frac{1}{2R^2}(\Delta_++\Delta_-)-\frac{c}{24R^2}\Big\} \\
&\quad \times Z_{\Delta_+,\Delta_-}\{T(x_k)\cdots T(x_2)\phi_\Delta(y)\}\,.
\end{aligned}
$$

In particular,

$$
\begin{aligned}
Z_{\Delta_+,\Delta_-}\{\mathbf{l}_{-1}\phi_\alpha\} &= \frac{1}{R}(\Delta_+-\Delta_-)\,, \\
Z_{\Delta_+,\Delta_-}\{\mathbf{l}_{-2}\phi_\alpha\} &= \frac{1}{2R^2}\left(\Delta_++\Delta_--\frac{c}{12}-\frac{1}{2}\Delta\right)\,.
\end{aligned}
$$

Integrable structure of CFT.

There is an infinite set of local densities

$$h_2(x) = T(x), \quad h_4(x) = (\mathbf{l}_{-2}T)(x)\,, \cdots$$

such that

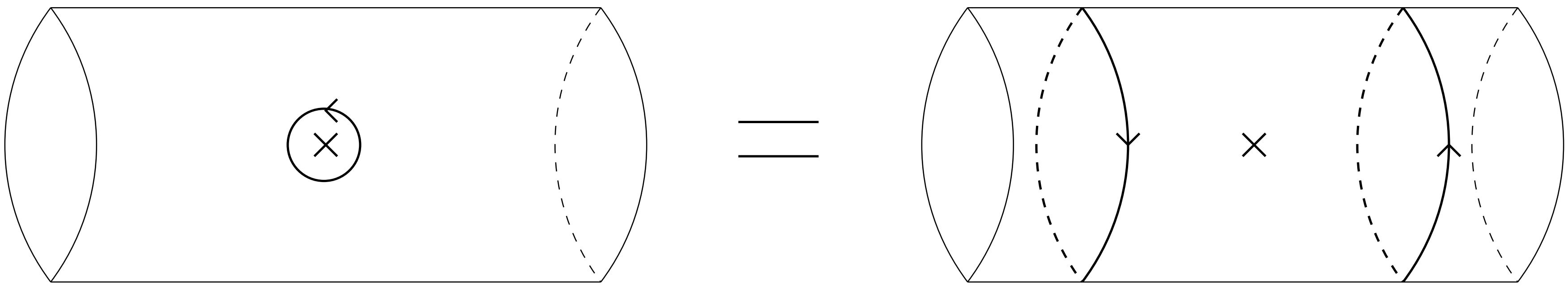
$$(\mathbf{i}_{2n-1}O)(y) = \int_\gamma \frac{dx}{2\pi i} h_{2n}(x)O(y) \quad (n \geq 1).$$

commute. In particular, $\mathbf{i}_1 = \mathbf{l}_{-1}$. We define also the global

$$I_{2k-1}(u) = \int\limits_\Gamma h_{2k}(z)\frac{dz}{2\pi i}\,.$$

In particular, $I_1 = R^{-2}\left(L_0 - \frac{c}{24}\right)$.

The primary fields are eigenstates of I_{2j-1} with eigenvalues I^{-}_{2j-1} and I^{-}_{2j-1}. Moving the contour as



we conclude that

$$Z_{\Delta_+,\Delta_-}\{\mathbf{i}_{2k-1}O(0)\} = (I^{+}_{2k-1} - I^{-}_{2k-1})Z_{\Delta_+,\Delta_-}\{O(0)\}\,.$$

Not very surprisingly this reminds out lattice operator $\mathfrak{t}^*(\zeta)$.
Basis in the Verma module

$$\mathbf{i}_{2k_1-1}\cdots\mathbf{i}_{2k_p-1}\mathbf{l}_{-2m_1}\cdots\mathbf{l}_{-2m_q}\phi_\alpha(0)\,.$$

Scaling limit in Matsubara direction.

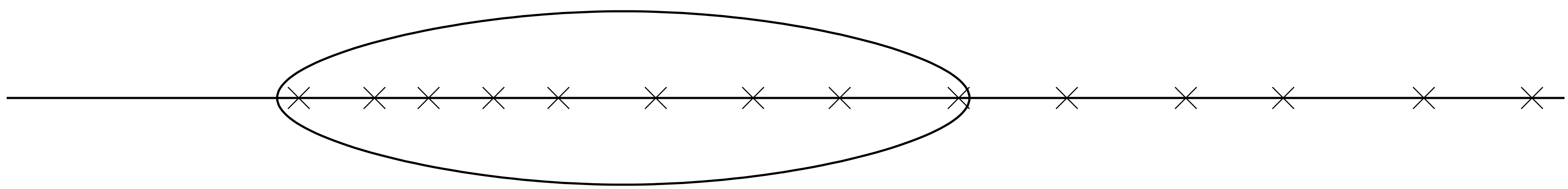
We consider $1/2 < \nu < 1$.
In the limit $\mathbf{n} \to \infty$ Bethe roots scale for $1 << j << \mathbf{n}$ as

$$\zeta_j = Const \cdot \left(\frac{j}{\mathbf{n}}\right)^{\nu} .$$

The scaling limit consists in

$$\mathbf{n} \to \infty, \quad a \to 0, \quad \mathbf{n}a = 2\pi CR \text{ fixed}$$

.



This scaling limit is chiral.

The following limit exists:

$$T^{\mathrm{sc}}(\lambda,\kappa)=\lim_{\substack{\mathbf{n}\to\infty,\ a\to 0,\\ 2\pi R=\mathbf{n}a}} T(\lambda a^{\nu}|\kappa)\,,\quad Q^{\mathrm{sc}}(\lambda,\kappa)=\lim_{\substack{\mathbf{n}\to\infty,\ a\to 0,\\ 2\pi R=\mathbf{n}a}} Q(\lambda a^{\nu}|\kappa)\,,$$

They satisfy

$$T(\lambda,\kappa)Q(\lambda,\kappa)=Q(\lambda q,\kappa)+Q(\lambda q^{-1},\kappa)\,.$$

since $(1-a^{2\nu}\lambda)^{\mathbf{n}}\to 0$.

Important:

$$\lim_{\substack{\mathbf{n}\to\infty,\ a\to 0,\\ 2\pi R=\mathbf{n}a}} T(\lambda a^{\nu}|\alpha+\kappa+s,-s)=T^{\mathrm{sc}}(\lambda,\kappa'),\quad \kappa'=\kappa+\alpha+2\tfrac{1-\nu}{\nu}s\,.$$

which implies

$$\rho^{\mathrm{sc}}(\lambda)=\frac{T^{\mathrm{sc}}(\lambda,\kappa')}{T^{\mathrm{sc}}(\lambda,\kappa)}\,.$$

Our favourite point $\zeta^2=1$ becomes $\lambda^2=\infty$.

Similarly the limit exists

$$\omega^{\rm sc}(\lambda,\mu)=\lim_{\substack{\mathbf{n}\to\infty,\ a\to 0,\\ 2\pi R=\mathbf{n}a}}\frac{1}{\sqrt{\rho(\lambda a^{\nu})\rho(\mu a^{\nu})}}\omega(\lambda a^{\nu},\mu a^{\nu})\,.$$

What can be said on general footing?
Bazhanov, Lukyanov, Zamolodchikov

$$\log T^{\rm sc}(\lambda,\kappa)\simeq_{\lambda^2\to\infty}\sum_{n=1}^{\infty}\lambda^{-\frac{2n-1}{\nu}}C_{2n-1}I_{2n-1}(\kappa)\,,$$

where $I_{2n-1}(\kappa)$ are eigenvalues of Hamiltonians corresponding to $\Delta_{1+\kappa}$, generally

$$\Delta_{\sigma}=\frac{\sigma(\sigma-2)\nu^2}{4(1-\nu)}\,.$$

Parameter λ is dimensional $\lambda^{-\frac{1}{\nu}}=[{\rm Length}]$.

$$\omega^{\mathrm{sc}}(\lambda,\mu)\simeq_{\lambda^2,\mu^2\to\infty}\sum_{l,m=1}^{\infty}\omega_{2l-1,2m-1}(\kappa,\kappa',\alpha)\lambda^{-\frac{2l-1}{\nu}}\mu^{-\frac{2m-1}{\nu}}\,.$$

Scaling limit in space direction.

In a weak sense we claim existence of the limits

$$\boldsymbol{\tau}^*(\lambda)=\lim_{a\to 0}\mathbf{t}^*(\lambda\bar{a}^{\nu}),$$

$$\boldsymbol{\beta}^*(\lambda)=\lim_{a\to 0}\mathbf{t}^*(\lambda\bar{a}^{\nu})^{-1/2}\mathbf{b}^*(\lambda\bar{a}^{\nu}),\quad \boldsymbol{\gamma}^*(\lambda)=\lim_{a\to 0}\mathbf{t}^*(\lambda\bar{a}^{\nu})^{-1/2}\mathbf{c}^*(\lambda\bar{a}^{\nu})\,,$$

such that

$$\boldsymbol{\tau}^*(\lambda)\simeq_{\lambda^2\to\infty}\exp\left(\sum_{n=1}^{\infty}\lambda^{-\frac{2n-1}{\nu}}C_{2n-1}\mathbf{i}_{2n-1}\right)\,,$$

$$\boldsymbol{\beta}^*(\lambda)\simeq_{\lambda^2\to\infty}\sum_{n=1}^{\infty}\lambda^{-\frac{2n-1}{\nu}}\boldsymbol{\beta}^*_{2n-1}\,,\qquad \boldsymbol{\gamma}^*(\lambda)\simeq_{\lambda^2\to\infty}\sum_{n=1}^{\infty}\lambda^{-\frac{2n-1}{\nu}}\boldsymbol{\gamma}^*_{2n-1}\,,$$

The Verma module allows the fermionic basis

$$\mathbf{i}_{2i_1-1}\cdots\mathbf{i}_{2i_p-1}\boldsymbol{\beta}^*_{2j_1-1}\cdots\boldsymbol{\beta}^*_{2j_q-1}\boldsymbol{\gamma}^*_{2k_1-1}\cdots\boldsymbol{\gamma}^*_{2k_q-1}\phi_\alpha(0)\,.$$

How to check all that? By comparing $\omega_{2l-1,2m-1}(\kappa,\kappa',\alpha)$ with CFT three-point functions.

Investigation of $T^{\mathrm{sc}}(\lambda,\kappa)$ and $\omega^{\mathrm{sc}}(\lambda,\mu)$.

We have

$$\mathfrak{a}^{\mathrm{sc}}(\lambda,\kappa)=\frac{Q^{\mathrm{sc}}(\lambda q,\kappa)}{Q^{\mathrm{sc}}(\lambda q^{-1},\kappa)}$$

which satisfies DDV equation

$$\log\mathfrak{a}^{\mathrm{sc}}(\lambda,\kappa)=-2\pi i\nu\kappa-\int\limits_\gamma K(\lambda/\mu)\log\left(1+\mathfrak{a}^{\mathrm{sc}}(\mu,\kappa)\right)\frac{d\mu^2}{\mu^2}\,,$$

Consider $\kappa \to \infty$. The smallest Bethe root behaves as $\lambda_1^2 \sim c(\nu)\kappa^{2\nu}$, where

$$c(\nu) = \Gamma(\nu)^{-2} e^{\delta} \left(\frac{\nu}{2R}\right)^{2\nu}, \quad \delta = -\nu \log \nu - (1-\nu)\log(1-\nu)\,.$$

We take

$$\lambda^2,\ \kappa \to \infty\,, \qquad \text{keeping}\ \ t = c(\nu)^{-1}\frac{\lambda^2}{\kappa^{2\nu}}\ \ \text{fixed.}$$

write

$$F(t,\kappa) = \log \mathfrak{a}^{\mathrm{sc}}(\lambda,\kappa)\,.$$

It can be shown that

$$F(t,\kappa) = -\kappa F_+(\arg t, \kappa^{-1})|t|^{\frac{1}{2\nu}} + O(|t|^{-\frac{1}{2\nu}}), \quad 0 < \arg t < \pi,$$

$$F(t,\kappa) = \quad \kappa F_-(\arg t, \kappa^{-1})|t|^{\frac{1}{2\nu}} + O(|t|^{-\frac{1}{2\nu}}), \quad -\pi < \arg t < 0,$$

Rewrite DDV equation as

$$F(t,\kappa) - \int_1^{\infty} K(t/u) F(u,\kappa) \frac{du}{u} = -2\pi i \nu \kappa$$

$$- \Big(\int_1^{e^{i\epsilon}\cdot\infty} K(t/u) \log(1+e^{F(u,\kappa)}) \frac{du}{u} - \int_1^{e^{-i\epsilon}\cdot\infty} K(t/u) \log(1+e^{-F(u,\kappa)}) \frac{du}{u} \Big) \,,$$

where

$$K(t) = \frac{1}{2\pi i} \cdot \frac{1}{2} \left(\frac{tq^2+1}{tq^2-1} - \frac{tq^{-2}+1}{tq^{-2}-1} \right) \,.$$

Set

$$F(t,\kappa) \simeq \sum_{n=0}^{\infty} \kappa^{-2n+1} F_n(t) \,.$$

For F_0 we have Wiener-Hopf equation

$$((I-K)F_0)(t) = -2\pi i\nu\,.$$

Mellin transforms

$$\hat{f}(k) = \int\limits_0^\infty f(t)t^{-ik}\frac{dt}{t}, \qquad f(t) = \int\limits_{-\infty}^\infty \hat{f}(k)t^{ik}\frac{dk}{2\pi}$$

We have

$$\hat{K}(k) = \frac{\sinh(2\nu-1)\pi k}{\sinh \pi k}\,.$$

Riemann-Hilbert factorisation

$$1-\hat{K}(k) = S(k)^{-1}S(-k)^{-1}\,,$$

$$S(k) = \frac{\Gamma(1+(1-\nu)ik)\Gamma(1/2+i\nu k)}{\Gamma(1+ik)\sqrt{2\pi(1-\nu)}}e^{i\delta k}\,,$$

If we demand that

$$F_0(t) = const.\, t^{\frac{1}{2\nu}} + O\left(t^{-\frac{1}{2\nu}}\right) \qquad (t \to \infty)\,,$$

then W-H equation admits a unique solution given by

$$F_0(t) = \int\limits_{-\frac{i}{2\nu}-i0} dl\, t^{il} S(l) \frac{-if}{l(l+\frac{i}{2\nu})}\,, \qquad (t > 1)$$

where $f = \frac{1}{2\sqrt{2(1-\nu)}}$. Similarly for the resolvent:

$$R(t,u) = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \frac{dl}{2\pi}\frac{dm}{2\pi} t^{il} u^{im} \hat{K}(l) S(l) S(m) \frac{-i}{l+m-i0} = \int\limits_{-\infty}^{\infty} \frac{dl}{2\pi} t^{il} S(l) \hat{K}(l) \hat{R}(l,u)\,.$$

Set

$$F(t,\kappa) = \kappa F_0(t) + \int\limits_{-\infty}^{\infty} dl\, t^{il} S(l)\hat{K}(l)(\Psi(l,\kappa) - \kappa\Psi_0(l)),$$

where

$$\Psi(l,\kappa) \simeq \sum_{n=0}^{\infty} \kappa^{-2n+1}\Psi_n(l), \quad \Psi_0(l) = \frac{-if}{l(l+\frac{i}{2\nu})}.$$

Making the substitution $u = e^{\frac{ix}{f\kappa}}$ we arrive at

$$\Psi(l,\kappa) - \kappa\Psi_0(l) = -\frac{i}{f\kappa}\Big\{ \int\limits_0^{-i\infty+\epsilon} \frac{dx}{2\pi}\hat{R}(l, e^{ix/f\kappa}) \log(1 + e^{F(e^{ix/f\kappa},\kappa)})$$

$$+ \int\limits_0^{i\infty+\epsilon} \frac{dx}{2\pi}\hat{R}(l, e^{-ix/f\kappa}) \log(1 + e^{-F(e^{-ix/f\kappa},\kappa)})\Big\}.$$

Previously unknown to us lemma.

Consider a Fourier integral

$$G(x) = \int_{-\infty}^{\infty} e^{ikx} g(k) dk \,,$$

$g(k)$ is holomorphic on the lower half plane, it satisfies asymptotic expansion

$$g(k) \simeq \sum_{n=-n_0}^{\infty} g_n (ik)^{-n} \quad (k \to \infty,\ \mathrm{Im}\, k < 0).$$

Suppose we know *a priori* that $G(x)$ allows analytical continuation around $x = 0$. Then the Taylor series is

$$G(x) = 2\pi i \, \mathrm{res}_k [e^{ikx} g(k)] \,.$$

Corollaries.

$$\hat{R}(l, e^{ix/f\kappa}) = \mathrm{res}_h \Big[\frac{e^{-hx/f\kappa}}{l+h} S(h) \Big] \, .$$

$$F(e^{ix/f\kappa}, \kappa) = -2\pi \left(x - \bar{F}(x,\kappa) \right) \, , \quad \bar{F}(x,\kappa) = x + \mathrm{res}_h \Big[e^{-hx/f\kappa} S(h) i\Psi(h,\kappa) \Big] \, .$$

Final equation

$$i\Psi(l,\kappa) - i\kappa\Psi_0(l)$$

$$\simeq \frac{2}{f\kappa} \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^{\infty} \frac{dx}{2\pi} \left\{ \mathrm{res}_h \Big[\frac{e^{-hx/f\kappa}}{l+h} S(h) \Big] \bar{F}(x,\kappa)^n \Big(-\frac{\partial}{\partial x} \Big)^n \right\}_{\text{even}} \log(1 + e^{-2\pi x}) \, .$$

can be iterated using

$$\int_0^{\infty} \frac{dx}{2\pi} x^m \Big(-\frac{\partial}{\partial x} \Big)^n \log(1 + e^{-2\pi x}) = m! (1 - 2^{-m-1+n}) \frac{\zeta(m-n+2)}{(2\pi)^{m-n+2}} \, .$$

The first few terms of the expansion read

$$i\Psi(l,\kappa) = \frac{1}{l(l+\frac{i}{2\nu})}f\kappa + \frac{1}{24}\frac{1}{f\kappa} + \frac{7}{2^6\cdot 90}\left(l-\frac{i}{2\nu}\right)\left(l-i\frac{2\nu^2-6\nu+6}{7\nu(1-\nu)}\right)\frac{1}{(f\kappa)^3}+\cdots$$

Finally,

$$\log T^{\rm sc}(\lambda,\kappa) \simeq \pi i\nu\kappa + \frac{\sqrt{1-\nu}}{\sqrt{2\pi}}\int_{-\frac{i}{2\nu}-i0} dl\ \frac{\Gamma(1-il)\Gamma(\frac{1}{2}+i\nu l)}{\Gamma(1-i(1-\nu)l)}\Psi(l,\kappa)\Big(\frac{e^{\delta-\pi i\nu}\lambda^2}{\kappa^{2\nu}c(\nu)}\Big)^{il}$$

$$\simeq \sum_{n=0}^{\infty} C_n I_{2n-1}(\kappa)\lambda^{-\frac{2n-1}{\nu}}$$

where

$$C_n = -\frac{\sqrt{\pi}}{\nu}\frac{1}{n!}\frac{\Gamma(\frac{2n-1}{2\nu})}{\Gamma(1+\frac{1-\nu}{2\nu}(2n-1))}(1-\nu)^n\ \Gamma(\nu)^{-\frac{2n-1}{\nu}}\ ,$$

This gives

$$I_1(\kappa) = \frac{1}{R}\Big(\Delta_{\kappa+1} - \frac{c}{24}\Big),$$
$$I_3(\kappa) = \frac{1}{R} I_1(\kappa)^2 - \frac{1}{6R^2} I_1(\kappa) + \frac{c}{1440 R^3}\,,$$

in perfect agreement with CFT computations.
Similar computations give

$$\omega^{\mathrm{sc}}(\lambda, \mu|\kappa, \kappa, \alpha) \simeq \frac{1}{2\pi i}\iint dl dm \tilde{S}(l,\alpha)\tilde{S}(m, 2-\alpha)\Theta(l + i0, m|\kappa, \alpha)$$
$$\times \Big(\frac{e^{\delta+\pi i\nu}\lambda^2}{\kappa^{2\nu}c(\nu)}\Big)^{il}\Big(\frac{e^{\delta+\pi i\nu}\mu^2}{\kappa^{2\nu}c(\nu)}\Big)^{im},$$

$$\tilde{S}(k,\alpha) = \frac{\Gamma\big(-ik + \frac{\alpha}{2}\big)\Gamma\big(\frac{1}{2} + i\nu k\big)}{\Gamma\big(-i(1-\nu)k + \frac{\alpha}{2}\big)\sqrt{2\pi}(1-\nu)^{(1-\alpha)/2}}\,.$$

The first non-trivial term reads

$$i\Theta(l,m|\kappa,\alpha)\simeq\frac{1}{l+m}+\frac{i}{24\nu}\frac{1}{(f\kappa)^2}\left(-i\nu(l+m)-\frac{1}{2}+\Delta_\alpha\right)+O\left(\frac{1}{\kappa^4}\right).$$

Important property of the remainder is that it vanishes when $l,m=\frac{2i}{2\nu}$. All together we get

$$\omega^{\mathrm{sc}}(\lambda,\mu|\kappa,\kappa,\alpha)=\lambda^{-\frac{1}{\nu}}\mu^{-\frac{1}{\mu}}D_1(\alpha)D_1(2-\alpha)\frac{1}{2R^2}\left(2\Delta_{\kappa+1}-\frac{c}{12}-\frac{1}{2}\Delta\right),$$

generally

$$D_{2m-1}(\alpha)=\sqrt{\frac{i}{\nu}}\,\frac{\Gamma\left(\frac{\alpha}{2}+\frac{2m-1}{2\nu}\right)}{(m-1)!\,\Gamma\left(\frac{\alpha}{2}+\frac{(2m-1)(1-\nu)}{2\nu}\right)}\left(\Gamma(\nu)^{-1/\nu}\sqrt{1-\nu}\right)^{-2m-1}..$$

Hence

$$\boldsymbol{\beta}_1^*\boldsymbol{\gamma}_1^*\phi_\alpha(0)\equiv D_1(\alpha)D_1(2-\alpha)\mathbf{l}_{-2}\phi_\alpha(0),\quad (\mathrm{mod}\ \ \mathbf{i}).$$

Generally,

$$\boldsymbol{\beta}^*_{I^+}\boldsymbol{\gamma}^*_{I^-}\Phi_\alpha \equiv \prod_{2j-1\in I^+}\prod_{2k-1\in I^-} D_{2j-1}(\alpha)D_{2k-1}(2-\alpha)$$
$$\times\left[P^{\mathrm{even}}_{I^+,I^-}(\{\mathbf{l}_{-2k}\}|\Delta_\alpha,c)+d_\alpha P^{\mathrm{odd}}_{I^+,I^-}(\{\mathbf{l}_{-2k}\}|\Delta_\alpha,c)\right]\Phi_\alpha \pmod{\mathbf{i}},$$

where

$$d_\alpha=\tfrac{1}{6}\sqrt{(25-c)(24\Delta_\alpha+1-c)}\,.$$

This has been checked up to level 8.
Second chirality: $\lambda\to\lambda^{-1}$ and $\alpha\to 2-\alpha$.

$$\bar{\boldsymbol{\beta}}^*(\lambda)\simeq_{\lambda^2\to 0}\sum_{j=1}^{\infty}\lambda^{\frac{2j-1}{\nu}}\bar{\boldsymbol{\beta}}^*_{2j-1}\,,\quad \bar{\boldsymbol{\gamma}}^*(\lambda)\simeq_{\lambda^2\to 0}\sum_{j=1}^{\infty}\lambda^{\frac{2j-1}{\nu}}\bar{\boldsymbol{\gamma}}^*_{2j-1}\,.$$

Returning to sine-Gordon model.

OPE:

$$\Phi_{\alpha_1}(z,\bar{z})\Phi_{\alpha_2}(0) = \sum_{m=-\infty}^{\infty} \sum_{N,\bar{N}} \left(\boldsymbol{\mu}^2 r^{2\nu}\right)^{|m|} C_{\alpha_1,\alpha_2}^{m,N,\bar{N}} \left(\boldsymbol{\mu}^4 r^{4\nu}\right)$$

$$\times\, r^{\frac{\nu^2}{1-\nu}\alpha_1\alpha_2+2m^2(1-\nu)+2\alpha m\nu}\; z^{|N|}\bar{z}^{|\bar{N}|}\; \mathbf{l}_{-N}\bar{\mathbf{l}}_{-\bar{N}}\Phi_{\alpha+2m\frac{1-\nu}{\nu}}(0)\,,$$

where $\alpha = \alpha_1 + \alpha + 2$.
We need to find

$$\frac{\langle \boldsymbol{\beta}^*_{I^+}\bar{\boldsymbol{\beta}}^*_{\bar{I}^+}\bar{\boldsymbol{\gamma}}^*_{\bar{I}^-}\boldsymbol{\gamma}^*_{I^-}\,\Phi_{\alpha+2m\frac{1-\nu}{\nu}}(0)\rangle_R^{\mathrm{sG}}}{\langle\Phi_\alpha(0)\rangle_R^{\mathrm{sG}}}\,.$$

I did not explain how to shift α, so, the nearest goal is $m = 0$.

Mimicking sG model on the lattice.

Let us try the same construction as before, but inhomogeneous:

$$T_{\mathbf{S},\mathbf{M}} = \overset{\curvearrowright}{\prod_{j=-\infty}^{\infty}} T_{j,\mathbf{M}}(\zeta_0^{(-1)^j})\,, \quad T_{j,\mathbf{M}}(\zeta) = \overset{\curvearrowleft}{\prod_{\mathbf{m}=\mathbf{1}}^{\mathbf{n}}} R_{j,\mathbf{m}}(\zeta\zeta_0^{-(-1)^{\mathbf{m}}} q^{-\frac{1}{2}})\,.$$

Fermions can be introduced in that case as before. Annihilation operators

$$\mathbf{b}(\zeta) = \mathbf{b}^{+}(\zeta) + \mathbf{b}^{-}(\zeta), \qquad \mathbf{c}(\zeta) = \mathbf{c}^{+}(\zeta) + \mathbf{c}^{-}(\zeta)\,,$$

$$\mathbf{b}^{\pm}(\zeta) = \sum_{p=1}^{\infty} \left(\zeta^2\zeta_0^{\mp 2} - 1\right)^{-p} \mathbf{b}_p^{\pm}, \qquad \mathbf{c}^{\pm}(\zeta) = \sum_{p=1}^{\infty} \left(\zeta^2\zeta_0^{\mp 2} - 1\right)^{-p} \mathbf{c}_p^{\pm}\,.$$

Local operators are created coefficients of by $\mathbf{b}^*(\zeta)$ at $\zeta^2 \to \zeta^{\pm 2}$. It is important to make the Bogolubov transformation defining

$$e^{-\Omega_0^{+-}-\Omega_0^{-+}}\mathbf{b}^*(\zeta)e^{\Omega_0^{+-}+\Omega_0^{-+}} \underset{\zeta^2\to\zeta_0^{\pm 2}}{\simeq} \mathbf{b}^{\pm *}(\zeta) = \sum_{p=1}^{\infty}(\zeta^2\zeta_0^{\mp 2}-1)^{p-1}\mathbf{b}^{\pm *}_{0,p}\,.$$

where

$$\Omega_0^{\epsilon,\epsilon'} = \frac{1}{(2\pi i)^2}\oint_\Gamma\oint_\Gamma \omega_0(\zeta,\xi)\mathbf{c}^{\epsilon}(\xi)\mathbf{b}^{\epsilon'}(\zeta)\,\frac{d\zeta^2}{\zeta^2}\frac{d\xi^2}{\xi^2}\,.$$

the reason for that $\zeta\to\infty$.

$$\mathbf{b}_k^{+*}\,\mathbf{c}_l^{+*}\,\mathbf{b}_r^{-*}\,\mathbf{c}_s^{-*}(q^{2\alpha S(0)}) = \left(\mathcal{O}_0 + \zeta_0^{-2}\mathcal{O}_1 + \zeta_0^{-4}\mathcal{O}_2 + \cdots\right)q^{2\alpha S(0)}\,,$$

and we exactly want to take the limit

$$a\to 0,\quad \zeta_0\to\infty,\qquad \text{keeping } R \text{ and } M = 4a^{-1}\zeta_0^{-\frac{1}{\nu}} \text{ fixed}\,.$$

We get rid of our old enemy $\rho(\zeta)$ by considering $Z_{\mathbf{n}}^{-s}$, and then continuing analytically.
We conjecture the existence of

$$\begin{aligned}
\mathbf{b}^{+*}(\zeta) \underset{\text{scaling}}{\longrightarrow} \boldsymbol{\beta}^{+*}(\zeta) &\simeq \boldsymbol{\beta}^{*}(\boldsymbol{\mu}\zeta) + \bar{\boldsymbol{\beta}}^{*}_{\text{screen}}(\zeta/\boldsymbol{\mu}), \quad \zeta \to \infty\,, \\
\mathbf{c}^{+*}(\zeta) \underset{\text{scaling}}{\longrightarrow} \boldsymbol{\gamma}^{+*}(\zeta) &\simeq \boldsymbol{\gamma}^{*}(\boldsymbol{\mu}\zeta) + \bar{\boldsymbol{\gamma}}^{*}_{\text{screen}}(\zeta/\boldsymbol{\mu}), \quad \zeta \to \infty\,, \\
\mathbf{b}^{-*}(\zeta) \underset{\text{scaling}}{\longrightarrow} \boldsymbol{\beta}^{-*}(\zeta) &\simeq \bar{\boldsymbol{\beta}}^{*}(\zeta/\boldsymbol{\mu}) + \boldsymbol{\beta}^{*}_{\text{screen}}(\boldsymbol{\mu}\zeta), \quad \zeta \to 0\,, \\
\mathbf{c}^{-*}(\zeta) \underset{\text{scaling}}{\longrightarrow} \boldsymbol{\gamma}^{-*}(\zeta) &\simeq \bar{\boldsymbol{\gamma}}^{*}(\zeta/\boldsymbol{\mu}) + \boldsymbol{\gamma}^{*}_{\text{screen}}(\boldsymbol{\mu}\zeta), \quad \zeta \to 0\,,
\end{aligned}$$

where

$$\boldsymbol{\mu} = \left[M \frac{\sqrt{\pi}\;\Gamma(\frac{1}{2\nu})}{2\Gamma(\frac{1-\nu}{2\nu})} \Gamma(\nu)^{-\frac{1}{\nu}} \right]^{\nu}\,.$$

Thermodynamical functions. DDV equation:

$$\frac{1}{i}\log\mathfrak{a}(\zeta) = \pi M R(\zeta^{1/\nu} - \zeta^{-1/\nu}) - 2\mathrm{Im}\int\limits_0^\infty R(\zeta/\xi)\log(1+\mathfrak{a}(\xi e^{+i0}))\frac{d\xi^2}{\xi^2}\,,$$

where

$$R(\zeta,\alpha) = \int\limits_{-\infty}^{\infty}\zeta^{2ik}\widehat{R}(k,\alpha)\frac{dk}{2\pi}, \quad \widehat{R}(k,\alpha) = \frac{\sinh\pi((2\nu-1)k - i\alpha/2)}{2\sinh\pi((1-\nu)k + i\alpha/2)\cosh(\pi\nu k)}\,,$$

$$R(\zeta) = R(\zeta,0)\,.$$

Introducing

$$G(k) = 2\int_0^\infty \zeta^{-2ik}\mathrm{Re}\left(\frac{1}{1+\mathfrak{a}(\zeta e^{-i0})}\right)\frac{d\zeta^2}{\zeta^2}\,,$$

we write the equation

$$\Theta_R^{\mathrm{sG}}(l,m|\alpha) + G(l+m) + \int\limits_{-\infty}^{\infty} G(l-k)\widehat{R}(k,\alpha)\Theta_R^{\mathrm{sG}}(k,m|\alpha)\frac{dk}{2\pi} = 0\,,$$

$\Theta_R^{\mathrm{sG}}(l,m|\alpha)$ is related to $\omega_R^{\mathrm{sG}}(\zeta,\xi|\alpha)$ by Mellin transform:

$$\omega_R^{\mathrm{sG}}(\zeta,\xi|\alpha) = -\frac{\pi i}{2}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\frac{dl}{2\pi}\frac{dm}{2\pi}\zeta^{2il}\xi^{2im}\frac{e^{-\pi\nu l}}{\cosh(\pi\nu l)}\Theta_R^{\mathrm{sG}}(l,m|\alpha)\frac{e^{-\pi\nu m}}{\cosh(\pi\nu m)}\,.$$

Final result.

$$\frac{\langle \boldsymbol{\beta}^*_{I^+}\bar{\boldsymbol{\beta}}^*_{\bar{I}^+}\bar{\boldsymbol{\gamma}}^*_{\bar{I}^-}\boldsymbol{\gamma}^*_{I^-}\,\Phi_\alpha(0)\rangle_R^{\mathrm{sG}}}{\langle\Phi_\alpha(0)\rangle_R^{\mathrm{sG}}}$$

$$=\boldsymbol{\mu}^{2m\alpha-2m^2+\frac{1}{\nu}\left(|I^+|+|I^-|+|\bar{I}^+|+|\bar{I}^-|\right)}\,\mathcal{D}_R^{\mathrm{sG}}\left(I^+\cup(-\bar{I}^+)\mid I^-\cup(-\bar{I}^-)|\alpha\right)\ ,$$

where

$$\mathcal{D}_R^{\mathrm{sG}}(A|B|\alpha)=\prod_{j=1}^{n}\mathrm{sgn}(a_j)\mathrm{sgn}(b_j)\left(\frac{i}{2\pi\nu^2}\right)^n\det\left(D_{a_n,b_k}(\alpha)\right)|_{j,k=1,\cdots,n},$$

$$D_{a,b}(\alpha)=\Theta_R^{sG}\left(\tfrac{ia}{2\nu},\tfrac{ib}{2\nu}|\alpha\right)-\delta_{a,-b}\mathrm{sgn}(a)2\pi\nu\cot\tfrac{\pi}{2\nu}(\nu\alpha+a)\,.$$

In principle we have to consider the case

$$\#(I^+)=\#(I^-),\quad \#(\bar{I}^+)=\#(\bar{I}^-)\,.$$

which corresponds to descendants of $\Phi_\alpha(0)$.

However, the only real limitation in our formula is

$$\#(I^{+}) + \#(\bar{I}^{+}) = \#(I^{-}) + \#(\bar{I}^{-}) \,.$$

More profound study of the conformal case shows that

$$\begin{aligned} &\boldsymbol{\beta}^*_{I^+}\bar{\boldsymbol{\beta}}^*_{\bar{I}^+}\bar{\boldsymbol{\gamma}}^*_{\bar{I}^-}\boldsymbol{\gamma}^*_{I^-}\,\Phi_{\alpha+2m\frac{1-\nu}{\nu}}(0) \\ &\cong C_m(\alpha)\boldsymbol{\beta}^*_{I^++2m}\bar{\boldsymbol{\beta}}^*_{\bar{I}^+-2m}\bar{\boldsymbol{\gamma}}^*_{\bar{I}^-+2m}\boldsymbol{\gamma}^*_{I^--2m}\boldsymbol{\beta}^*_{I_{\mathrm{odd}}(m)}\bar{\boldsymbol{\gamma}}^*_{I_{\mathrm{odd}}(m)}\Phi_{\alpha}(0)\,, \end{aligned}$$

where for negative indices

$$\boldsymbol{\gamma}^*_{-a} = -t_a(\alpha)\boldsymbol{\beta}_a, \quad \bar{\boldsymbol{\beta}}^*_{-a} = -t_a(2-\alpha)\bar{\boldsymbol{\gamma}}_a, \quad t_a(\alpha) = \tfrac{i}{\nu}\cot\tfrac{\pi}{2\nu}(\alpha\nu + a)\,,$$

and

$$C_m(\alpha) = \boldsymbol{\mu}^{2m\alpha-2m^2}\prod_{j=0}^{m-1} C_1(\alpha + 2j\,\tfrac{1-\nu}{\nu})\,,$$

$$C_1(\alpha) = -\nu\Gamma(\nu)^{4x}\,\frac{\Gamma(-2\nu x)}{\Gamma(2\nu x)}\cdot\frac{\Gamma(x)}{\Gamma(x+1/2)}\cdot\frac{\Gamma(-x+1/2)}{\Gamma(-x)}i\cot\pi x\,, \quad x = \frac{\alpha}{2} + \frac{1-\nu}{2\nu}.$$