Mock modularity and a secondary invariant

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These slides: categorified.net/StringMath2020.pdf.

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Plan for the talk:

The space of sqfts

Old and new invariants

Topological modular forms

The space of sqfts

Given an sqft $\mathcal{F}$ (in this talk: (1+1)d, $\mathcal{N}=(0,1)$), might ask:

(1) Is supersymmetry spontaneously broken in $\mathcal{F}$? I.e. is $\mathcal{F}$ null?

(2) Can spontaneous susy breaking be triggered by a small susy-preserving deformation?

(3) Can $\mathcal{F}$ be connected by a path in space $\mathrm{SQFT} = \{\text{sqfts}\}$ to one with spont susy breaking? I.e. is $\mathcal{F}$ nullhomotopic?

Questions (2,3) depend on analytic decisions about the space SQFT. I will use compact sqfts: all Wick-rotated partition functions $\mathrm{tr}_{\mathcal{H}}(\exp(-t\hat{H} - x\hat{P}))$ converge absolutely for $t > 0$.

The topology on I want on SQFT is something like "strong convergence of the resolvent." In this topology, an eigenvalue can go to $+\infty$, in which case the corresponding eigenvector is deleted.

Conjecture: $\{$possibly-noncompact sqfts$\}$ is contractible.

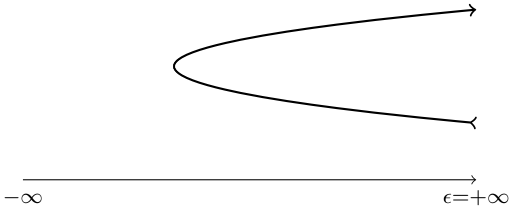
Example: Free $(0,1)$ scalar multiplet $(\phi, \bar{\phi}, \psi)$ is noncompact.
(ψ is right-moving fermion, the superpartner of full boson $(\phi, \bar{\phi})$.)
Add in a left-moving fermion λ. Turn on a superpotential $W = (\phi^2 - \epsilon)\lambda$, with $\epsilon \in \mathbb{R}$. This compactifies the sqft.

(1) $\epsilon < 0$: sqft is null.

(2) $\epsilon = 0$: far IR is a $(1,1)$ minimal model.

(3) $\epsilon > 0$: two massive vacua (of opposite Arf invariants).

As ϵ runs from $-\infty$ to ∞, these sqfts trace out a cobordism from $\emptyset$ to two points (of opposite orientation).



The $(0,1)$ sigma model makes sense if the target manifold X is string: metric, spin structure, and 3-form $\frac{1}{2\pi}H$ with integral periods solving $\frac{1}{2\pi}\mathrm{d}H = -\frac{1}{16\pi^2}\operatorname{tr}(R \wedge R)$; classically, $H = \mathrm{d}B$.

Gaiotto–JF–Witten: String cobordisms $\rightsquigarrow$ homotopies in SQFT.
Proof: Add a left-moving fermion λ. Turn on a superpotential. λ acts as a Lagrange multipler.

In particular, if X is string nullcobordant ($X = \partial Y$ for a string manifold Y), then sigma model for X is nullhomotopic.

Example: $X = S^3_k :=$ round S^3 with $\frac{1}{2\pi}\int_{S^3} H = k$. Far IR behaviour: $(0,1)$ WZW model with bosonic WZW levels $(|k|-1, |k|+1)$. (We believe susy spont breaks when $k = 0$.)

String nullcobordant iff $k \in 24\mathbb{Z}$ (via connect sum of $K3$ surfaces).

Question: If $k \notin 24\mathbb{Z}$, is S^3_k sigma model nullhomotopic?

Old and new invariants

Question: If $k \notin 24\mathbb{Z}$, is S^3_k sigma model nullhomotopic?

How to show $\mathcal{Y}$ is not nullhomotopic? Find a deformation invariant which is nonzero for $\mathcal{Y}$, but zero for null sqfts.

Famous example:

The Witten index aka elliptic genus is (up to a normalization convention) the Wick-rotated partition function of $\mathcal{Y}$ on flat tori with nonbounding spin structure (Ramond in both space and time).

A priori, it is an area-dependent real-analytic modular form $Z_{RR}(\mathcal{Y})(\tau, \bar{\tau}, \text{area})$, meromorphic at $\tau \to i\infty$.

Famous example (con't): However,

(i) $\dfrac{\partial}{\partial \bar\tau} Z_{RR} \propto \langle T_{\bar z,\bar z}\rangle \propto \langle \bar Q[\bar G_{\bar z}]\rangle, \quad \dfrac{\partial}{\partial \text{area}} Z_{RR} \propto \langle T_{\bar z,z}\rangle \propto \langle \bar Q[\bar G_{z}]\rangle$

If $\mathcal{Y}$ is compact, then $\langle \bar Q[O]\rangle = 0$ for any observable O.
($\bar Q$ is the $(0,1)$ susy, and $(\bar G_z, \bar G_{\bar z})$ is its supercurrent.)

So $Z_{RR}(\mathcal{Y})(\tau)$ is a (weakly) holomorphic modular form.

(ii) Break manifest modularity by choosing a small A-cycle and large B-cycle. Then recognize the q-expansion of $Z_{RR}(\mathcal{Y})$ as supersymmetric index of an S^1-equivariant $\mathcal{N}$=1 SQM model, i.e. a count of susy ground states. So $Z_{RR}(\mathcal{Y}) \in \mathbb{Z}_{RR}(\!(q)\!)$.

Since integers cannot deform, $Z_{RR}(-) : \mathrm{SQFT} \to \mathrm{MF}_{\mathbb{Z}}$ is a deformation invariant.

Sadness: $Z_{RR}(S^3_k) = 0$.

What if $\mathcal{Y}$ is noncompact?

If it is badly noncompact, then $Z_{RR}(\mathcal{Y})$ simply isn't defined.

Mild noncompactness: $\mathcal{Y}$ has cylindrical ends $\mathcal{X}$, parameterized by observable Φ if you can turn on a Lagrange multiplier λ and superpotential $W = (\Phi - \epsilon)\lambda$ so that when $\epsilon \ll 0$, theory is null, whereas when $\epsilon \gg 0$, theory $\to \mathcal{X}$.

I will write this as $\partial\mathcal{Y} = \mathcal{X}$.

If $Z_{RR}(\mathcal{X}) = 0$, then $Z_{RR}(\mathcal{Y})$ converges conditionally. In lagrangian formalism, it is again manifestly a real-analytic modular form (area-dependent).

Example: $\partial\big(\text{cigar } \mathrm{SL}(2, \mathbb{R})/\mathrm{SU}(2)\big) = S^1$.

What if $\mathcal{Y}$ is noncompact? Suppose $\partial\mathcal{Y} = \mathcal{X}$.

(i') Still true that

$$\frac{\partial}{\partial\bar{\tau}} Z_{RR} \propto \langle T_{\bar{z},\bar{z}} \rangle \propto \langle \bar{Q}[\bar{G}_{\bar{z}}] \rangle, \quad \frac{\partial}{\partial \text{area}} Z_{RR} \propto \langle T_{\bar{z},z} \rangle \propto \langle \bar{Q}[\bar{G}_z] \rangle.$$

Why $\langle \bar{Q}[O] \rangle_{\mathcal{Y}} = 0$ if $\mathcal{Y}$ is compact? Because it is the (path) integral of a total derivative. If $\partial\mathcal{Y} = \mathcal{X}$, then have Stokes' theorem: $\langle \bar{Q}[O] \rangle_{\mathcal{Y}} \propto \langle O \rangle_{\mathcal{X}}$. After checking normalizations,

Claim (Gaiotto–JF): Holomorphic anomaly equation

$$\sqrt{-8\tau_2}\,\eta(\tau) \frac{\partial}{\partial\bar{\tau}} Z_{RR}(\mathcal{Y}) = \langle \bar{G}_{\bar{z}} \rangle_{\mathcal{X}}$$

(up to convention-dependent power of $\sqrt[4]{-1}$.)

Similarly, $\frac{\partial}{\partial \text{area}} Z_{RR}(\mathcal{Y}) \propto \langle \bar{G}_z \rangle_{\mathcal{X}} = 0$ if $\mathcal{X}$ is superconformal.

What if $\mathcal{Y}$ is noncompact? Suppose $\partial\mathcal{Y} = \mathcal{X}$.

(ii') Any (nice enough) real-analytic modular form $\hat{f}(\tau, \bar{\tau})$ has a *q*-expansion, defined as the *q*-expansion of

$$f(\tau) := \lim_{\bar{\tau} \to -i\infty} \hat{f}(\tau, \bar{\tau}).$$

The limit breaks modularity. The *q*-expansion of $Z_{RR}(\mathcal{Y})$ is still an S^1-equivariant supersymmetric index:

$$\lim_{\bar{\tau} \to -i\infty} Z_{RR}(\mathcal{Y}) \in \mathbb{Z}(\!(q)\!).$$

(This is correct up to an $\mathcal{X}$-dependent shift related to APS invariants and mod-2 indexes, and for most $\mathcal{X}$ it is zero.)

Conclusion (Gaiotto–JF): If $\partial\mathcal{Y} = \mathcal{X}$ (and $\mathcal{X}$ is superconformal), then $\lim_{\bar{\tau} \to -i\infty} Z_{RR}(\mathcal{Y})$ is an integral (up to shift) (generalized) mock modular form with shadow $\langle \bar{G}_{\bar{z}} \rangle_{\mathcal{X}}$.

Conclusion (Gaiotto–JF): If $\partial\mathcal{Y} = \mathcal{X}$ (and $\mathcal{X}$ is superconformal), then $\lim_{\bar\tau\to -i\infty} Z_{RR}(\mathcal{Y})$ is an integral (up to shift) (generalized) mock modular form with shadow $\langle \bar{G}_{\bar z}\rangle_{\mathcal{X}}$.

Contrapositively, if we only know $\mathcal{X}$, can compute $g(\tau,\bar\tau) = \langle \bar{G}_{\bar z}\rangle_{\mathcal{X}}$ (& shift of integrality). The obstruction to g being the shadow of an integral (generalized) mock modular form lives in

$$\frac{\mathbb{C}(\!(q)\!)}{\mathbb{Z}(\!(q)\!) + \mathrm{MF}_{\mathbb{C}}}.$$

To compute this obstruction, solve $\sqrt{-8\tau_2}\frac{\partial}{\partial\bar\tau}\hat{f} = g$ among real-analytic modular forms (this can always be done). Then take the class of the q-expansion of $f = \lim_{\bar\tau\to -i\infty}\hat{f}$.

"Theorem" (Gaiotto–JF): This obstruction is a deformation invariant of the sqft $\mathcal{X}$. We call it the secondary elliptic genus.

Motivating example: Take $\mathcal{X} = S^3_k$, or rather its far-IR limit, the $\mathcal{N}{=}(0,1)$ WZW model with bosonic levels $(|k|-1, |k|+1)$.

$$\bar{G}_{\bar{z}} = \sqrt{\tfrac{-2}{|k|+1}}\, \bar{\psi}_1 \bar{\psi}_2 \bar{\psi}_3 + \text{proportional to } \bar{\psi}_a \bar{J}_a$$

where $\bar{J}_a$ are the right-moving currents in the bosonic WZW model, and $\bar{\psi}_a$ are their superpartners. Since $\langle \bar{\psi}_a \bar{J}_a \rangle = 0$, find:

$$\langle \bar{G}_{\bar{z}} \rangle = \sqrt{\tfrac{-2}{|k|+1}}\, \eta(\bar{\tau})^3\, Z(\text{bosonic } \mathrm{SU}(2)_{|k|-1}).$$

Harvey–Murthy–Nazaroglu: This is the shadow of an explicit (mixed) mock modular form equal to

$$kE_2(q) + q\mathbb{Z}[\![q]\!] = -\frac{k}{24} \mod \mathbb{Z}(\!(q)\!) + \mathrm{MF}_{\mathbb{C}}.$$

Corollary: S^3_k sigma model is not nullhomotopic if $k \notin 24\mathbb{Z}$.

Topological modular forms

Why did we look for our secondary invariant?

Conjecture (Stolz–Teichner, building on Witten, Segal, Hopkins, . . .): The Witten index $Z_{RR} : \mathrm{SQFT} \to \mathrm{MF}_{\mathbb{Z}}$ lifts to a topological Witten index $Z_{RR}^{\mathrm{top}} : \mathrm{SQFT} \to \mathrm{TMF}$, where TMF is the spectrum, aka generalized cohomology theory, of (weakly holomorphic) topological modular forms. Furthermore, Z_{RR}^{top} is a complete invariant: $\mathrm{SQFT} \simeq \mathrm{TMF}$ are homotopy-equivalent.

Definition: $\mathrm{MF}_{\mathbb{Z}}$ is the space of global sections of a graded vector bundle V on the stack $\mathcal{M}_{e\ell\ell}$ of smooth elliptic curves; fibre V_E at $E \in \mathcal{M}_{e\ell\ell}$ is $\bigoplus \mathrm{Lie}(E)^{\otimes n}$. This V_E is the coefficients of E-elliptic cohomology h_E. **Goerss–Hopkins–Miller–Lurie:** There is a derived stack $\mathcal{M}_{e\ell\ell}^{\mathrm{top}}$ of "derived elliptic curves" which carries a bundle of spectra $\mathcal{O}^{\mathrm{top}}$ whose fibre at E is h_E. TMF has an algebraic model as the space of derived global sections of $\mathcal{O}^{\mathrm{top}}$.

The conjecture offers an analytic model of TMF.

Why did we look for our secondary invariant?

The primary Witten index $\mathrm{TMF} \to \mathrm{MF}$ sees all of the non-torsion in TMF: it is an isomorphism after tensoring with $\mathbb{C}$.

Bunke–Naumann had already provided an algebrotopological description of a secondary Witten genus for classes in TMF, and proved that it was nonzero for the TMF class of S^3_k.

Their description makes no reference to mock modularity. We believe that our secondary invariant agrees with theirs (work in progress with Berwick-Evans).

There is further torsion in TMF which is not seen by the secondary Witten genus.

Open question: The group manifolds $\mathrm{SU}(3)$, $\mathrm{Spin}(5)$, and G_2 are known to be nonzero in TMF. Are the corresponding $(0,1)$ sigma models nullhomotopic?

Moonshine connection

Any scft $\mathcal{X}$ which is nullhomotopic in SQFT will provide an integral mock modular form. If $\mathcal{X}$ is nullhomotopic equivariantly for a finite group G of flavour symmetries, then the mock modular form will be valued in characters of G. I think that this is the (a?) physical explanation of umbral moonshine.

A priori, these mock modular forms are only weakly holomorphic: they can be badly meromorphic at the cusp $\tau = i\infty$. Important in "moonshine" is a genus zero / optimal growth condition. I think that this condition is best expressed in terms of G-equivariant topological cusp forms. (A cusp form is a modular form that vanishes at $\tau = i\infty$.)

Open question: What is the physics of strongly holomorphic topological modular forms (bounded at the cusp)? What is the physics of topological cusp forms?

Thank you!

Further details:

[arXiv:1811.00589] Holomorphic SCFTs with small index

[arXiv:1902.10249] A note on some minimally supersymmetric models in two dimensions

[arXiv:1904.05788] Mock modularity and a secondary elliptic genus

[arXiv:2006.02922] Topological Mathieu moonshine

[these slides] http://categorified.net/StringMath2020.pdf