SUPER DUPER VECTOR SPACES II: THE HIGHER-CATEGORICAL GALOIS GROUP OF $\mathbb{R}$

THEO JOHNSON-FREYD
"HIGHER STRUCTURES IN FUNCTORIAL FIELD THEORY,"
UNIVERSITY OF REGENSBURG, 18 AUGUST 2023

Abstract. A theorem of Deligne suggests that the complex numbers are not algebraically closed in a 1-categorical sense but that their 1-categorical algebraic closure is the category $\mathbf{sVec}_{\mathbb{C}}$ of complex super vector spaces. In fact, this property uniquely (up to non-unique isomorphism) characterizes $\mathbf{sVec}_{\mathbb{C}}$ amongst complex-linear symmetric monoidal categories.

In these talks, we will outline work in progress on constructing complex-linear symmetric n-categories which are higher categorical analogues of $\mathbf{sVec}_{\mathbb{C}}$ in that they are uniquely characterized by being the n-categorical separable closure of the complex numbers. We will explore the resulting higher-categorical absolute Galois group of the complex numbers, and outline a construction of that group very much akin to the surgery-theoretic description of the stable piecewise linear group PL.

This is the second half of a 2-part lecture. The first part is given by David Reutter, with whom this work is joint in progress. The slides from Part I are available at https://homepages.uni-regensburg.de/~lum63364/ConferenceFFT/Reutter.pdf.

Contents

The conference "Higher Structures in Functorial Field Theory" is part of the SFB 1085 on Higher Invariants at the Faculty of Mathematics Regensburg, and is supported by the Deutsche Forschungsgemeinschaft. The speaker's research is supported by an NSERC Discovery Grant and by the Simons Collaboration on Global Categorical Symmetries, and the Perimeter Institute is by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Economic Development and Innovation. We thank Claudia Scheimbauer and Matthias Ludewig for their excellent organization of the conference. We had many valuable conversations about this project during the conference, and particularly thank Chris Schommer-Pries for many long and clarifying conversations. We also thank Tomer Schlank and Mike Hopkins for their ongoing influence on this project.

Perimeter Institute for Theoretical Physics, Waterloo, Ontario, theojf@pitp.ca.
Department of Mathematics, Dalhousie University, Halifax, Nova Scotia, theojf@dal.ca.

0. Caveat lector

These notes are unedited and verbose. They accomplish exactly the worst type of talk preparation: too little practice and organization for this to be a good "well prepared" type of lecture; too much practice and material for this to a good "off the cuff" lecture.

After the talk, we spent more time carefully calculating the low-degree homotopy groups. The calculations are as reported in these notes except for $\pi^4\mathcal{W}$, or dually $\pi_5\mathrm{B}\,\mathrm{Gal}$. The non-torsion in the Witt group does not simply go away when we decide to only work with torsion things. It gets adjusted. Indeed, working spectrally, $\mathbb{Z}[\mathrm{tor}]$ is a degree-shifted $\mathbb{Q}/\mathbb{Z}$, and *not* just trivial.

The end result is that $\mathrm{fibre}(\mathrm{G}/\,\mathrm{Gal} \to \ell^\vee)$ has two nontrivial homotopy groups, and not just one as reported. They are:

$$\pi_4\,\mathrm{fibre}(\mathrm{G}/\,\mathrm{Gal} \to \ell^\vee) = \mathrm{TrueQuantWitt}[\mathrm{tor}]^\vee = \mathrm{Ext}^1(\mathrm{TrueQuantWitt}, \widehat{\mathbb{Z}}) \cong (\mathbb{Z}/2)^{\times\infty}$$
$$\pi_5\,\mathrm{fibre}(\mathrm{G}/\,\mathrm{Gal} \to \ell^\vee) = \mathrm{Ext}^0(\mathrm{TrueQuantWitt}, \widehat{\mathbb{Z}}) \cong (\widehat{\mathbb{Z}})^{\times\infty}$$

In other words, the fibre looks like the $I_{\widehat{\mathbb{Z}}}$-dual of the full truly quantum Witt group, and not just a dual of the (non-spectral) torsion subgroup. It is also the profinite completion of the $I_{\mathbb{C}^\times}$-dual. Profinite completion of spaces/spectra *does not* simply profinitely complete their homotopy groups. That is all it does for finitely generated homotopy groups, but the spectral profinite completion of $\mathbb{C}^\times$ is a shifted $\widehat{\mathbb{Z}}$ and *not* just trivial.

The Postnikov extension from π_4 to π_5 is noncannonically zero just because in TrueQuantWitt the torsion subgroup is noncannonically a direct summand. I do not have a conjecture for the full map from $K(\mathbb{Z}/2, 4) = \mathrm{fibre}(\mathrm{G}/\mathrm{PL} \to L) \to \mathrm{fibre}(\mathrm{G}/\,\mathrm{Gal} \to \ell^\vee)$. On π_4, it should be the map reported in Section 9 below. But it can also involve a degree-1 map to π_5. Together, these maps should arise from an "organic" map TrueQuantWitt $\to \mathbb{Z}/2$ which on the pseudounitary torsion subgroup records whether the central charge is 0 or $\frac{1}{4}$ mod $\frac{1}{2}$. The existence of such an "organic" map is one of the corollaries of our conjectured that $\mathrm{PL} \to \mathrm{Gal}$.

1. Starting point

In the previous lecture, David explained how to construct $\mathcal{W}^\bullet$, the *higher separable closure* of $\mathbb{R}$. My goal in this lecture is to actually compute as much as I can about the result of this construction. In particular, I will try to analyze the *higher absolute Galois group* $\mathrm{Gal}(\mathcal{W}/\mathbb{R})$.

Let me pick up where David ended. Suppose you have built $\mathcal{W}$ up to height $n-1$. Then we have a map

$$(\Sigma^\bullet \mathcal{W}^{n-1})^\times[\mathrm{tor}] \to I_{\mathbb{Q}/\mathbb{Z}}$$

Aside on conventions: Towers are naturally cohomologically-graded. So I'll use upper indices, and they just mean the negativized lower indices. Later we'll write down some long exact sequences. The pneumonic for the degree of the connecting map: it goes up with respect to upper indices, and down with respect to lower indices. I'm always going to index towers so that the top-morphisms are in degree 0. This includes when I write something like "$\Sigma^\bullet\mathcal{W}^{n-1}$": the notation is slightly abusive, but what I mean is the tower which in degree $N > n$ looks like $\Sigma^{N-(n-1)}\mathcal{W}^{n-1}$.

The fibre F^n of the above map are the "missing roots of unity." The idea is to add some of the missing roots to get up to $\mathcal{W}^n$, and then repeat. The procedure works because F^n is *n-coconnective*: it has

$$\pi^k F^n := \pi_{-k} F^n = 0 \text{ if } k < n.$$

To get to $\mathcal{W}^n$, we add exactly the missing roots in degree n. In other words, $\mathcal{W}^n$ is an extension of $\Sigma\mathcal{W}^{n-1}$ by the group $\pi^n F^n$. Example. When $n = 1$, I'm trying to build $\mathcal{W}^1 = \mathbf{sVec}_{\mathbb{C}}$ from $\mathcal{W}^0 = \mathbb{C}$. So I'm looking at a map of spectra which is an iso on π^0 and the injection $1 \to \mathbb{Z}/2$ on π^1, so $\pi^0(F^0) = \mathbb{Z}/2$ and $\pi^{<0}(F^0)$ are trivial. To build $\mathbf{sVec}_{\mathbb{C}}$, I extend $\mathbf{Vec}_{\mathbb{C}}$ by this $\mathbb{Z}/2$ (in degree 1).

Note that the defining property of $\Sigma\mathcal{W}^{n-1}$ is that (it deloops $\mathcal{W}^{n-1}$ and) every object is connected to the unit object by a nonzero morphism. Consider the set of indecomposable objects in $\mathcal{W}^n$ modulo connection by a nonzero morphism. This relation is symmetric because $\mathcal{W}$ has adjoints, but it isn't obviously transitive. It turns out — the argument is essentially due to Deligne — that it is transitive when restricted to simple objects. But $\mathcal{W}$ that we've constructed is semisimple, so "indecomposable" and "simple" are the same. The equivalence classes under this relation (on $\mathcal{W}^n$) are, naturally, called $\pi_0\mathcal{W}^n$. I'll also write it as $\pi^n\mathcal{W}$.

The statement that $\mathcal{W}^n$ is an extension of shape $\Sigma\mathcal{W}^{n-1}.\pi^n F^n$ then implies that:

$$\pi^n\mathcal{W} = \pi^n F^n.$$

From the long exact sequence for the homotopy groups of a fibre sequence, we find that a certain exact sequence

$$\dots \to \underbrace{\pi^{n-1}F^n}_{=0} \to \pi^n(\Sigma^\bullet\mathcal{W}^{n-1})^\times[\mathrm{tor}] \to \pi^n I_{\mathbb{Q}/\mathbb{Z}} \to \underbrace{\pi^n F^n}_{=\pi^n\mathcal{W}} \to \pi^{n+1}(\Sigma^\bullet\mathcal{W}^{n-1})^\times[\mathrm{tor}] \to \pi^{n+1}I_{\mathbb{Q}/\mathbb{Z}} \to \dots$$

2. TQFT interpretation

Let me interpret these groups in terms of TQFTs. Specifically, I mean framed TQFTs valued in $\mathcal{W}$, so this interpretation is purely linguistic: I'm invoking the cobordism hypothesis to give myself language and pictures for objects and morphisms. I won't ever actually need to compile the TQFTs.

Then of course $\pi^n I_{\mathbb{Q}/\mathbb{Z}}$ are the iso classes of nD invertible TQFTs. The injection

$$\pi^n(\Sigma^\bullet\mathcal{W}^{n-1})^\times[\mathrm{tor}] = \pi_0(\Sigma\mathcal{W}^{n-1})^\times[\mathrm{tor}] \hookrightarrow \pi^n I_{\mathbb{Q}/\mathbb{Z}}$$

selects the invertible TQFTs which admit a nonzero boundary condition, and so the cokernel are the invertible TQFTs which do not admit a boundary condition. Indeed, an object of $\Sigma\mathcal{W}^{n-1}$ is a TQFT which can be connected to the vacuum by a boundary condition. Taking $(-)^\times$ asks the bulk TQFT, but not the boundary, to be invertible. Then we take iso classes of TQFTs with these two properties of invertibility and boundability.

The more novel map is

$$\pi^{n+1}(\Sigma^\bullet\mathcal{W}^{n-1})^\times[\mathrm{tor}] = \pi_0(\Sigma^2\mathcal{W}^{n-1})^\times[\mathrm{tor}] \to \pi^{n+1}I_{\mathbb{Q}/\mathbb{Z}}.$$

Let's try to understand the domain of this map. There is a map $\Sigma^2\mathcal{W}^{n-1} \to \Sigma\mathcal{W}^n$, and we understand the latter: it is $(n+1)$D TQFTs Q which admit a boundary condition. More precisely, it is the TQFTs which admit a sufficiently-generating boundary condition. If the bulk Q is simple (and invertible implies simple!), then "sufficiently generating" just means nonzero: any choice of nonzero $X : 1 \to Q$ will write the bulk Q as the condensate of $\mathrm{End}_{\hom(1,Q)}(X)$. Suppose I've chosen X; then these data will lift to $\Sigma^2\mathcal{W}^{n-1}$ exactly when $\mathrm{End}_{\hom(1,Q)}(X) \in \mathcal{W}^n$ lifts to $\Sigma\mathcal{W}^{n-1}$.

I claim that this latter lift is automatic if X was simple [over $\mathbb{R}$, I would mean "absolutely simple": simple with endomorphisms $\mathbb{R}$ and not some other division ring]. Since I might as well have chosen X simple [warning: this requires that we have at least built $\mathbb{C}$, and not just $\mathbb{R}$], what I conclude is that the map $\Sigma^2\mathcal{W}^{n-1} \to \Sigma\mathcal{W}^n$ is an essential surjection (when restricted to simples, but then it is for all objects). This is actually really useful, since it means that in our above LES, for the purposes of computing $\pi^n\mathcal{W}$, I could replace $\pi^n(\Sigma^\bullet\mathcal{W}^{n-1})^\times[\mathrm{tor}]$ with $\pi^n(\Sigma^\bullet\mathcal{W}^{n-2})^\times[\mathrm{tor}]$, since the maps from these groups to $\pi^n I_{\mathbb{Q}/\mathbb{Z}}$ have the same image. In other words, we get a nice systematic LES:

$$\dots \to \pi^n(\Sigma^2\mathcal{W}^{n-2})^\times[\mathrm{tor}] \to \pi^n I_{\mathbb{Q}/\mathbb{Z}} \to \pi^n\mathcal{W} \to \pi^{n+1}(\Sigma^2\mathcal{W}^{n-1})^\times[\mathrm{tor}] \to \dots \qquad (*)$$

Suppose you know all of pure homotopy theory, for example the homotopy groups of spheres. Then you will feel like you understand $\pi^n\mathcal{W}$ if you can understand the groups $\pi^n(\Sigma^2\mathcal{W}^{n-2})^\times[\mathrm{tor}]$.

Look again at this LES (*). There is an obvious map of (non-Karoubian) towers

$$I_{\mathbb{Q}/\mathbb{Z}} \to \mathcal{W}$$

that includes the invertible TQFTs among all the TQFTs. Up to a degree convention, the fibre of this map is the tower of *anomalous TQFTs*: pairs $(\mathcal{Q}, \alpha)$, where α is invertible, $\mathcal{Q}$ is a boundary condition of α, and I think of α as the anomaly of $\mathcal{Q}$. A morphism $(\mathcal{Q}, \alpha) \to (\mathcal{Q}', \alpha')$ in this tower consists of: an isomorphism $\alpha \cong \alpha'$, and an interface $\mathcal{Q} \to \mathcal{Q}'$ of TQFTs with identified anomaly. In terms of the formula "$\pi^n(\Sigma^2\mathcal{W}^{n-2})^\times[\mathrm{tor}]$", the $(-)^\times$ is saying that Q should be invertible, and π^n is saying that Q should be nD and that I take some equivalence classes. If I just had $\Sigma\mathcal{W}^{n-1}$, then I would just be asking for the existence of X, but the morphisms would ignore it. The fact that I'm asking for $\Sigma^2\mathcal{W}^{n-2}$ is what makes the morphisms involve interfaces in the X variable:

- $\pi^n(\mathcal{W}^n)^\times$: Invertible nD TQFTs.
- $\pi^n(\Sigma\mathcal{W}^{n-1})^\times$: Invertible nD TQFTs which have the property of admitting a boundary condition.
- $\pi^n(\Sigma^2\mathcal{W}^{n-2})^\times$: Invertible nD TQFTs with a choice of boundary condition, modulo interfaces between boundary conditions.

We will feel like we understand $\pi^n\mathcal{W}$ if we understand the third of these groups.

3. Surgery for Mfr (motivational recitation)

An analogy to motivate the next step of the construction. A TQFT is like a manifold. A boundary condition is like a filling — a nullbordism. An interface is a bordism. Morse theory: bordisms are built from handle attachments. If you scan a sequence of handle attachments, it looks like a sequence of surgeries.

An invertible TQFT is like a (stably) framed sphere. The trivial TQFT is the sphere with boundary (stable) framing, but for example S^1 has another framing, S^3 has $\mathbb{Z}$ many framings, ... An anomalous TQFT is like a filling of a framed sphere. (Fillings of the boundary-framed sphere correspond to closed manifolds, up to some convention about marked points.)

So in this analogy, $\pi^n(\Sigma^2\mathcal{W}^{n-2})^\times$ is like the space of framed spheres equipped with a filling, modulo surgering the filling. Foreshadowing: by "sphere" I mean that it might have an exotic smooth structure.

Now, if you are a manifold theorist, you might could ask: surger your filling to simplify it as much as possible. For example, can you simplify it all the way to a disk, thereby trivializing the boundary? This is a relative version of the question for closed manifolds of surgering them until they become spheres. In the problem we actually want, I could ask: can you surger your boundary condition X all the way until it is invertible, thereby trivializing the anomaly Q? If not, how close to trivial can you get? This is basically a story of Sullivan; other major players include Wall and Ranicki.

Let me remind the closed manifold version. Let's try to surger it at least to being a homotopy sphere. We can worry about whether homotopy spheres are spheres some other time. The relative version that we actually want is the same, you just have to talk about relative homology relative to the boundary sphere, which is mild because it's relative to the homology of a sphere

To *surger* a manifold M^n, what you need to do is to select an embedded S^k; the surgery cuts out a tubular neighbourhood $S^k \times D^{n-k}$ of that selected sphere, leaving a manifold with boundary $S^k \times S^{n-k-1}$, and then glues in an $D^{k+1} \times S^{n-k-1}$. To actually do this requires a little more information than the embedded sphere: you need to choose a trivialization of the normal bundle in order to identify the boundary with $S^k \times S^{n-k-1}$. I want framed manifolds, so I also need to make a few more choices in order for the surgery to produce a framed answer. A priori, this cutting and gluing introduces a piecewise-smooth crease into the manifold. The piecewise-smooth category is also called the piecewise-linear (PL) category. It is a nontrivial statement that, in the presence of a framing, any PL (or PDiff) framed manifold has a unique (up to a contractible space) smoothing compatible with the framing. The manifolds before and after surgery are (rather obviously) cobordant: you take $M \times I$ and then attach a handle $D^{k+1} \times D^{n-k}$ along the original $S^k \times D^{n-k}$, and the new boundary is the result of the surgery. Going the other way, pick a cobordism $M_0 \to M_1$, and select parameterizing function to $[0, 1]$ (i.e. the function is i on M_i), and wiggle it to being normal at the boundary and Morse in the bulk. The function selects a handle decomposition of the bordism, and these handles, in order, are a sequence of surgeries to get from M_0 to M_1.

Ok, so now what you can do is: take some S^1s that generate π_1. Assuming n is high enough, these 1D spheres can fully unlink from each other. So you can surger all of them at the same time: the surgeries won't interfere. This kills H_1, but might introduce some higher homology. Now repeat with π_2 ($= \mathrm{H}_2$, since we've killed π_1) and so on. You keep killing low homology. But manifolds have Poincaré duality. So you also kill high homology.

This works as long as the dimensions of the surgered spheres are low enough that you can unlink them. You get stuck when you get to middle dimension (and it becomes technically hard as you approach middle dimension). The end result is that if n is odd, then you can surger M^n to a homology sphere. If n is even, then you can surger it until it only has homology in degree $n/2$.

Now again recall Poincaré duality. The homology in degree $n/2$ has an intersection pairing, which is symmetric if $n/2$ is even and antisymmetric if $n/2$ is odd. Slightly nontrivial fact: if the manifold is stably framed, the framing selects a quadratic refinement of the intersection pairing.

The *quadratic Witt group of* $\mathbb{Z}$ is the commutative monoid of (isomorphism classes of) finitely-generated free abelian groups $A \cong \mathbb{Z}^r$ equipped with a nondegenerate quadratic form $q : A \to \mathbb{Z}$, modulo a relation I'm about to explain. (The group law is orthogonal direct sum.) By *quadratic form*, I mean that $q(0) = 0$, that $q(-x) = q(x)$, and that $\langle x, y\rangle := q(x + y) - (q(x) + q(y))$ is bilinear. This $\langle,\rangle$ is the *second derivative* of q, and it is a symmetric bilinear form. It recovers q by polarization: $q(x) = \frac{1}{2}\langle x, x\rangle$. In other words, "quadratic refinement" in this example just means that $\langle,\rangle$ is even. The form q is called *nondegenerate* if the map $A \to \hom(A, \mathbb{Z})$ induced by $\langle,\rangle$ is an iso.

By definition, a subgroup $Y \subset (A, q)$ is called *isotropic* if $q|_L$ vanishes identically. Suppose you pick an isotropic subgroup. Then the *isotropic reduction* $A /\!\!/ Y$ is by definition $Y^\perp/Y$, where $Y^\perp \subset A$ is the space of vectors with trivial $\langle,\rangle$ against Y. Isotropy for $\langle,\rangle$ is what makes $Y \subset Y^\perp$; isotropy for q is what makes q descend to $L /\!\!/ Y$. The relation that defines the Witt group is generated by declaring that $A \sim A /\!\!/ Y$ for all isotropics Y.

A theorem about lattices says that this Witt group is isomorphic to $\mathbb{Z}$. Specifically, the *signature* of L is $p-q$, where $L\otimes\mathbb{R} = \mathbb{R}^{p,q}$ with p positive-definite directions and q negative-definite directions. For even (quadratic) nondegenerate lattices, the signature is always in $8\mathbb{Z}$, and the theorem is that the signature is the only invariant of a class in the Witt group. (There is a stronger theorem: if L and L' have the same signature, then you can find m, n so that $A \oplus H^m \cong A' \oplus H^n$, where H is $\mathbb{Z}^2$ with pairing $\langle(x, y),(x', y')\rangle = xy' + x'y$.)

Since I'm about to repeat this story anyway, let me finish the sketch of the surgery story in the case $n \in 4\mathbb{Z}$. I'll be sketchy. Then what you can show is that any two cobordant manifolds have intersection forms which are equivalent in the Witt group. On the other hand, the surgery shows that you can surger your manifold to get any representative of the desired Witt class. So for the question of achieving a homology sphere, the only obstruction is the signature.

If you work a while, you can do this in families. What we're really doing is analyzing the stably-framed cobordism spectrum Mfr, isomorphic to the sphere spectrum $. You've probably heard of the *j-homomorphism* $j : \pi_n\mathrm{O} \to \pi_n\$$, given by the action of the orthogonal group on the sphere. [The more precise thing, if you want to work in families, is to say that you have a map $\mathrm{O} \to \mathrm{G} := \mathrm{GL}_1(\$)$; the non-group map $\Omega^\infty\mathrm{G} \to \Omega^\infty\$$ is an iso on positive homotopy groups.] In terms of $\$ \cong \mathrm{Mfr}$, what this map does is to select the stable framings on the standard smooth sphere. This map actually factors through the *piecewise linear group* PL, and $j : \pi_n\mathrm{PL} \to \pi_n\$$ selects the framed possibly-exotic spheres. The fancy families version of the surgery story is that the PL j-homomorphism is part of a LES

$$\pi_n \mathrm{PL} \xrightarrow{j} \pi_n \mathrm{G} \to \pi_n \mathrm{L} \to$$

where L is a fancy spectral version of the Witt group.

Actually, this is not quite true. The one part I left out: can you actually engineer every Witt class as coming from a manifold? The answer is yes *except* in dimension 4, where Rokhlin's theorem says that for smooth spin 4-manifolds, the signature is divisible not just by 8 but by 16 (and you can realize signature 16). The final statement is that there is a map

$$\mathrm{G}/\mathrm{PL} \to \mathrm{L}$$

and it is almost an isomorphism: its fibre is just a $K(\mathbb{Z}/2, 3)$, which just encodes this ratio $16\mathbb{Z}/8\mathbb{Z}$ of the actually-realizable signatures modulo the algebraically-possible signatures.

4. Surgery for TQFTs

With this construction in mind, let me now actually explain the surgery for TQFTs. Recall the problem. We want to understand the set

$$\begin{aligned}\pi^n(\Sigma^2 \mathcal{W}^{n-1})^\times &= \pi^n \operatorname{fibre}(I_{\mathbb{Q}/\mathbb{Z}} \to \mathcal{W}) \\ &= \frac{\{\text{invertible } n\text{D TQFTs with a choice of boundary condition}\}}{\{\text{boundary interfaces}\}} \\ &= \frac{\{\text{anomalous } (n-1)\text{D TQFTs }\}}{\{\text{interfaces}\}}\end{aligned}$$

We will do this by a version of surgery: we will pick up an $(n-1)$D anomalous TQFT, and try to systematically modify it, always in ways that are separated by topological interfaces, aiming to simplify it all the way to invertible.

Any $(n-1)$D TQFT has an monoidal $(n-2)$-category of *extended operators*. More generally, if you only want extended operators of dimension $\leq k < n-1$, then you get a k-category. The *state-operator correspondence* asserts that if your $(n-1)$D TQFT is $\mathcal{Q}$, then the k-category of $\leq k$D operators is $\hom(1, \mathcal{Q}(S^{n-2-k}))$. The sphere should be given the framing that bounds a disk. This is actually just a statement about fully dualizable objects in any category, and how adjunctibility relates to endomorphism algebras: it generalizes the statement that $\operatorname{End}(\mathcal{Q}) = \hom(1, \mathcal{Q} \otimes \mathcal{Q}^*) = \hom(1, \mathcal{Q}(S^0))$. This is the category of "external" operators. If you allow "enriched" operators, you just remove the $\hom(1, -)$: you get an object of your target tower, and not plain category. If your $\mathcal{Q}$ is anomalous, or more generally relative, then instead of the sphere S^{n-2-k}, you evaluate on the closed disk D^{n-1-k} in which you put the anomaly theory α in the bulk and you put $\mathcal{Q}$ on the boundary.

Note that if $\mathcal{Q}$ is invertible, then its dual $\mathcal{Q}^*$ is its inverse $\mathcal{Q}^{-1}$, so you can detect invertibility from looking at operators: $\mathcal{Q}$ is invertible iff there are no operators at all. A stronger statement is true: if $\mathcal{Q}$ is $(n-1)$D, then it is invertible as soon as it has no operators of dimension $\leq \frac{n-2}{2}$. This follows from a more general *Poincaré duality* statement that I don't have time to explain, which says that if you have no operators of dimension k, then all there are no "new" operators of dimension $n-2-k$: they are all built by condensing operators of lower dimension.

The algebra $\mathcal{Q}(S^{n-2-k}) \in \mathcal{W}^{k+1}$ of operators of dimension $\leq k$ is automatically E_{n-1-k}-monoidal, and of category number k. (Counting: you get one unit of monoidality for each direction transverse to the operator.)

Now I really use strongly that our $\mathcal{W}$ is coconnective, so that an object of $\mathcal{W}^{k+1}$ is like a (k,k)-category and not like an (∞,k)-category. In this case, the *stability hypothesis* says that E_{n-1-k}-monoidality canonically and automatically becomes E_∞-monoidality as soon as $n-1-k \geq k+2$. I.e. this happens when $k \leq \frac{n-3}{2}$.

In the analogy, this E_∞-monoidality of the top approximately-half of the operators corresponds to the unlinking of the bottom approximately-half of the homology.

But now I am in business, because I am working over a separably-closed tower $\mathcal{W}$! Let $k := \lfloor \frac{n-3}{2} \rfloor$, and $\mathcal{A} = \mathcal{Q}(S^{n-2-k})$. Then there is a natural "$\Sigma^\bullet \mathcal{A}$" which is a $\mathcal{W}$-linear tower. It is fully finite over $\mathcal{W}$ because of the full dualizability of our TQFT. So it maps ($\mathcal{W}$-linearly) back to $\mathcal{W}$.

Let $X = \hom_{\mathcal{W}}(\mathcal{A}, \mathcal{W})$. Part of the separable closure story that David didn't have time to tell you is that (X has finite homotopy groups and) the canonical map

$$\mathcal{A} \to \mathcal{W}^X$$

built from currying is an equivalence — you always have this map, but it is separable-closure of $\mathcal{W}$ that makes it an equivalence.

Pick a map $x \in X$. If $\mathcal{Q}$ was simple, which I assumed it was anyway, then this turns out not to be much of a choice: X is connected. What I'm going to do is to "condense" or "ungauge" the subsector $\mathcal{A}$ of $\mathcal{Q}$. It's called "ungauging" because $X = \mathrm{B}G$ for $G = \Omega_x X$, and if you built a gauge theory with gauge group G, it would have Wilson operators parameterized by $\mathcal{A} = \mathcal{W}^X = \mathbf{Rep}_{\mathcal{W}}(G)$. It's called "condensing" because it has the effect of freezing out these operators down to just being in $\mathcal{W}$. The actual procedure is the following. The composition $\mathcal{W} \to \mathcal{A} \xrightarrow{x} \mathcal{W}$ is the identity. So the composition $\mathcal{A} \xrightarrow{x} \mathcal{W} \to \mathcal{A}$ is an idempotent. But $\mathcal{A}$ was realized as (internal, aka $\mathcal{W}$-enriched) operators on object $\mathcal{Q} \in \mathcal{W}$. Using also the algebra structure on $\mathcal{A}$, and the full-dualizability of $\mathcal{Q}$, all together you can build a higher idempotent $\mathcal{Q} \to \mathcal{Q}$. By definition, the *condensation* $\mathcal{Q} /\!\!/ \mathcal{A}$ (at the map x) is the splitting of this higher idempotent.

Since it arises from splitting an idempotent, $\mathcal{Q} /\!\!/ \mathcal{A}$ comes with a morphism to $\mathcal{Q}$: they are "cobordant". On the other hand, because we've condensed them away, $\mathcal{Q} /\!\!/ \mathcal{A}$ has no operators of dimension $\leq k$!

I want you to think of this like surgering away for low-dimensional homology. By the way, I did this once and for all for $k := \lfloor \frac{n-3}{2} \rfloor$. But I could have done it step by step, just like in the manifold case.

Now look. We've killed all operators of dimension $\leq \frac{n-3}{2}$ in our $(n-1)$D TQFT. We are invertible if we have killed all operators of dimension $\leq \frac{n-2}{2}$. Half the time these are the same bound: when n is odd, $\lfloor \frac{n-2}{2} \rfloor = \lfloor \frac{n-3}{2} \rfloor = \frac{n-3}{2}$.

Every (odd $-\,1$)D TQFT, possibly anomalous, is cobordant to an invertible TQFT (which in turn trivializes the anomaly).

The anomaly itself is an nD invertible theory, i.e. an element $\pi^n I_{\mathbb{Q}/\mathbb{Z}}$. **So what I'm saying is: a nontrivial odd-dimensional invertible theory never has topological boundary conditions.** Remember why we cared about this: we wanted to understand $\pi^n \mathcal{W}$. So what I'm saying is: when n is odd, $\pi^n I_{\mathbb{Q}/\mathbb{Z}} \to \pi^n \mathcal{W}$ is injective. The cokernel is precisely the bordism classes of non-anomalous nD theories that cannot be surgered to an invertible theory, which is part of the "$n+1$" case of the above surgery.

When n is even, then $\lfloor \frac{n-2}{2} \rfloor = \frac{n-2}{2} > \lfloor \frac{n-3}{2} \rfloor = \frac{n-2}{2} - 1$. So we are potentially left with some operators in "middle" dimension $\frac{n-2}{2}$.

5. Middle-dimensional operators

Let's continue to analyze the case where $n = 2m$ is even. We have surgered our $n-1$D anomalous TQFT to one whose operators are only in dimension $m-1$ (in the sense that all other operators, of all other dimensions, are condensates). $(m-1)$-dimensional operators compile into a $\mathcal{W}$-enriched $(m-1)$-category; it is E_m-monoidal since the total theory if $2m-1$-dimensional.

Moreover, since $\mathcal{W}$ is (ind-finite) semisimple and $\mathcal{Q}$ is very dualizable, this E_m-monoidal $(m-1)$-category is a (ind-finite) fusion.

Proposition (generalization of JF-Yu): Suppose $m-1 \geq 2$. Then any (possibly ind-finite) fusion $m-1$-category $\mathcal{C}$ with operators only in dimension $m-1$ is automatically *groupal*: there is a (ind-finite) group A so that $\mathcal{C}$ is a twisted group algebra of A. *This really uses semisimplicity!*

Because I want my group algebra to be $\mathcal{W}$-linear, the twist is a cohomology class valued in $I_{\mathbb{C}^\times}$; it factors through $I_{\mathbb{Q}/\mathbb{Z}}$ because A is finite. Because I want my algebra to be E_m-monoidal, the cohomology is of the space $K(A,m)$. And because I want an $(m-1)$-category, the twist has m more units of cohomological degree than there were units of monoidality. In other words:

$$\{E_m\text{-monoidal } \mathcal{W}\text{-linear } (m-1)\text{-category group algebras of } A\} = I^{2m}_{\mathbb{Q}/\mathbb{Z}}(K(A,m)).$$

First nontrivial calculation: For any finite abelian group A and $m > 0$,

$$I^{2m}_{\mathbb{Q}/\mathbb{Z}}(K(A,m)) = \begin{cases} \{\mathbb{Q}/\mathbb{Z}\text{-valued symmetric forms on } A\}, & m \text{ even}, \\ \{\mathbb{Q}/\mathbb{Z}\text{-valued skew-symmetric forms on } A\}, & m \text{ odd}. \end{cases} \qquad \square$$

There is an obvious map from the LHS to the RHS — the nontrivial calculation is that this map is an iso for these specific degrees. The map is called the *S-matrix*. It is also called the *commutator*. Here's what you do. Take $a, b \in A$; they correspond to invertible objects, well-defined up to invertible scalar, in our group algebra $\mathcal{C}$. Now take a pair of Hopf-linked S^{m-1}s; this makes sense because my ambient space is $2m-1$-dimensional. (To be precise: use invertibility to know that you only need stable framings, not framings, on your sphere, and then choose the bounding framings.) Place a on the first S^{m-1} and b on the second and evaluate. The choice of overall scalar cancels when I wrap around the sphere. This is all valid because I'm in an E_m-monoidal $(m-1)$-category with lots of duality. The result will evaluate to a $\mathbb{C}^\times$-number, which will be a root of unity because A is finite. Indeed, it is rather obviously linear in a, b, and so we have defined a pairing $\langle,\rangle : A \otimes A \to \mathbb{Q}/\mathbb{Z}$. Finally, it is a fun exercise that this pairing is $(\text{skew})^m$-symmetric.

The "Poincaré duality" that applies to any TQFT in this case says that this pairing is nondegenerate: the induced map $A \to A^\vee := \hom(A, \mathbb{Q}/\mathbb{Z})$ is an isomorphism. Conversely, any A with a nondegenerate class in $I^{2m}_{\mathbb{Q}/\mathbb{Z}}(K(A,m))$ does arise. The way you should think is: a class in $I^{2m}_{\mathbb{Q}/\mathbb{Z}}(K(A,m))$ is an $I_{\mathbb{Q}/\mathbb{Z}}$-valued function on $K(A,m)$. The pairing $\langle,\rangle$ is the *second derivative* of the function, evaluated at the basepoint (the only point!) in $K(A,m)$. Nondegeneracy is the request that the function be Morse.

Finally, for any choice of isotropic $Y \subset A$, we can further condense to a theory with operators $A /\!\!/ Y := Y^\perp / Y$.

The *Witt group of* $(\textit{skew})^m$*-symmetric finite abelian groups* is the commutative monoid of $(\text{skew})^m$-symmetric finite abelian groups, modulo isotropic reduction.

Cobordism classes of anomalous $(\text{even}-1)$-dimensional TQFTs, in dimension even $>$ 4, are classified by a Witt group of finite abelian groups.

Finally, what are these Witt groups? Every finite abelian group factors canonically into a product of abelian p-groups for different primes p. (A *p-group* is *not* a group of categorical height p, in spite of the language that my friends use. For a hundred years "p-group" has meant "group of order a power of p". Let's keep that language.) This means that the Witt groups also factor over primes.

If A has odd order, then every nondegenerate skew-symmetric form on A admits a Lagrangian Y (I think this is a lemma of Davydov), i.e. $Y = Y^\perp$, so $A /\!\!/ Y = *$. If A is a 2-group, then there is either a Lagrangian or there is a maximal isotropic for which $A /\!\!/ Y = \mathbb{Z}/2$ with the unique skew-symmetric pairing.

In the symmetric case, up to Witt equivalence, every 2-group is either trivial or $\mathbb{Z}/2$. If p is an odd prime, then $\mathbb{Z}/p$ has two isomorphism classes of symmetric forms. The group $(\mathbb{Z}/p)^2$ also has two isomorphism classes of symmetric forms, one of which is Witt trivial and the other isn't. Specifically, the forms on $\mathbb{Z}/p$ are 2-torsion — their direct sum is Witt trivial — exactly when -1 is a square mod p; otherwise they are each other's inverses. The nontrivial form on $(\mathbb{Z}/p)^2$ is then either the sum of two copies of the same form on $\mathbb{Z}/p$ or the sum of the two different ones. This is all the Witt-equivalence classes.

Summarizing, we find that for $n > 4$:

$$\frac{\{\text{anomalous } (n-1)\text{D TQFTs valued in } \mathcal{W}\}}{\text{interfaces}} = \begin{cases} 0, & n \equiv 1,3 \pmod 4 \\ \mathbb{Z}/2, & n \equiv 2 \pmod 4 \\ \text{Witt}, & n \equiv 0 \pmod 4 \end{cases}$$

where

$$\text{Witt} = \bigoplus_{p \text{ prime}} \begin{cases} \mathbb{Z}/2, & p = 2, \\ (\mathbb{Z}/2)^2, & p \equiv 1 \pmod 4, \\ \mathbb{Z}/4, & p \equiv 3 \pmod 4 \end{cases}$$

6. Gravitational anomalies usually enforce gaplessness

Let me introduce a name: ℓ^n is the 4-periodic sequence of groups above. To remind, the computation that produced ℓ^n was in serve to computing $\pi^n F^n = \pi^n \mathcal{W}$. Namely, we had a long exact sequence

$$\dots \to \ell^n \to \pi^n I_{\mathbb{Q}/\mathbb{Z}} \to \pi^n \mathcal{W} \to \ell^{n+1} \to \dots.$$

Now pick a class $[A, \langle,\rangle] \in \ell^n$. What can we say about its image in $\pi^n I_{\mathbb{Q}/\mathbb{Z}}$? The answer is: a lot. If n is odd, there's nothing to say. Suppose that $n = 2m$ is even. Then, by retracing our steps, what we have is the bulk TQFT defined by a groupal E_m-monoidal $(m-1)$-category. Since it is groupal, the physics of this TQFT is just a (generalized) Dijkgraaf–Witten theory. So in principle we can compute it exactly.

Suppose that m is even. I'm going to quote three facts. First fact: every $\mathbb{Q}/\mathbb{Z}$-valued symmetric form on a finite abelian group admits a quadratic refinement. Second fact: a choice of quadratic refinement selects an orientation structure on this a-priori-framed invertible theory. Third fact: in dimension n divisible by 4, the map $\Omega^n_{\mathrm{SO}} \to \pi^n I_{\mathbb{Q}/\mathbb{Z}}$ vanishes, because the elements of Ω^n_{SO} are all characteristic classes, which vanish on framed manifolds.

Suppose, on the other hand, that m is odd (so $n = 2 \pmod 4$). If $[A, \langle,\rangle] \neq 0$, then A might as well be $\mathbb{Z}/2$, with the form $\langle a, b\rangle = (-1)^{ab}$, which is skew-symmetric because we're in characteristic 2. The corresponding Dijkgraaf–Witten theory is called the *Arf–Kervaire invariant*.

A highly nontrivial computation of Hill, Hopkins, and Ravenel says that the Arf–Kervaire invariant vanishes on framed n-manifolds unless $n = 2$, 6, 14, 30, 32, and possibly 126.

Except for five or six choices (these Arf–Kervaire invariants), a nontrivial invertible framed TQFT does not admit a topological boundary condition, in *any* semisimple category. Any physical boundary condition cannot be topological in the deep IR, and so must have massless modes.

(Topological=gapped theories are insulation. Gapless theories are conducting, because it easy to excite a current to run through them. If you could engineer a bulk material to be nontrivially-invertible even as a framed theory, then it will automatically be a topological insulator with conductance on the boundary.)

7. Coherent surgery and finite abelian Chern–Simons theories

What's really going on coherently?

Define the *higher absolute Galois group of* $\mathbb{R}$ to be the group of automorphisms $\mathrm{Gal} := \mathrm{Gal}(\mathcal{W}/\mathbb{R}) = \mathrm{Aut}_{\mathbb{R}}(\mathcal{W})$. Using the uniqueness of $\mathcal{W}$, together with what is essentially a classical Galois theory argument, what you find is that in low homotopy types:

$$\pi_{\leq n} \mathrm{Gal} = \mathrm{Gal}(\mathcal{W}^n/\mathbb{R}).$$

This lets you compute $\pi_n \operatorname{Gal}$: the only thing a degree-n automorphism of $\mathcal{W}^n$ could do is to assign a $\mathbb{C}^\times$-number to each object, and the only conditions are that connected objects have the same number and the products of objects get the product of numbers. In other words,

$$\pi_n \operatorname{Gal} = (\pi^n \mathcal{W})^\vee$$

where $(-)^\vee$ means $\hom(-, \mathbb{C}^\times)$, or equivalently (since the groups we care about are all torsion) $\hom(-, \mathbb{Q}/\mathbb{Z})$. So we get a dual surgery long exact sequence when $n > 4$:

$$\cdots \to \ell^\vee_{n+1} \to \pi_n \operatorname{Gal} \to \pi_n \$ \to \ell^\vee_n \to \dots$$

This is extremely close to the surgery sequence for G/PL.

The group Gal manifestly acts on $I_{\mathbb{Q}/\mathbb{Z}}$. The automorphism group of $I_{\mathbb{Q}/\mathbb{Z}}$ is essentially the automorphism group G of the sphere. It would be if I used S^1 instead of $\mathbb{Q}/\mathbb{Z}$. That I'm just using torsion parts means only that G is profinitized to $\widehat{\mathrm{G}}$, and the map $\mathrm{G} \to \widehat{\mathrm{G}}$ is an iso in positive degrees and the inclusion $\mathbb{Z}^\times \to \widehat{\mathbb{Z}}^\times$ in degree 0.

In general, the action of a Galois group on the roots of unity is called the *cyclotomic character*. Every absolute Galois group will act on $I_{\mathbb{Q}/\mathbb{Z}}$.

So our LES looks like it is a description of $\widehat{\mathrm{G}}/\operatorname{Gal}$. Is it?

A point in $\mathrm{B}\operatorname{Gal}$ is a tower noncanonically isomorphic to $\mathcal{W}$: $\mathrm{B}\operatorname{Gal}$ is "the space of abstract $\mathcal{W}$s." Similarly, BG is "the space of abstract $I_{\mathbb{Q}/\mathbb{Z}}$s." Pick some reference standard $I_{\mathbb{Q}/\mathbb{Z}}$. Then G/G is the (trivial!) space of abstract $I_{\mathbb{Q}/\mathbb{Z}}$s equipped with an isomorphism to the standard reference one.

So a point in $\widehat{\mathrm{G}}/\operatorname{Gal}$ is a pair $(\mathcal{W}, \gamma)$ where $\mathcal{W}$ is some (non-canonical) separable closure of $\mathbb{R}$ in towers, and γ is an isomorphism of your chosen $\mathcal{W}^\times[\mathrm{tor}]$ with my standard reference $I_{\mathbb{Q}/\mathbb{Z}}$.

The LES means we should expect that there is some "L-theory spectrum" $\ell^\bullet$, and that $\widehat{\mathrm{G}}/\operatorname{Gal}$ looks like $\Omega^\infty \ell^\vee_\bullet$ except in low degree, where for π-(ind- or pro-)finite spectra $(-)^\vee$ means the $I_{\mathbb{Q}/\mathbb{Z}}$-dual. By "L-theory" here I mean a rather general sense, first sketched by Lurie in his surgery lectures and fully constructed in a beautiful nine-author paper.

To get this to work, I need to build a coherent version of ℓ. An obvious guess: there is an L-theory of π-finite spectra equipped with a symmetric form. This will produce exactly the right groups. But is it the right guess?

Remember that what ℓ wants to do is to map to the space of pairs (invertible theory, boundary condition). It's specifically supposed to do this via some rather simple Dijkgraaf–Witten type construction about abelian groups and symmetric forms. Suppose that A is a π-finite spectrum, and $\langle,\rangle : A \otimes A \to I_{\mathbb{Q}/\mathbb{Z}}$ is a symmetric form of degree n. Can you build from this a Dijkgraaf–Witten theory? Yes, but it is the wrong one. To build a Dijkgraaf–Witten theory, what you need is a degree-n *function* $\Omega^\infty A \to I_{\mathbb{Q}/\mathbb{Z}}$. A symmetric pairing determines the quadratic function $\langle a, a\rangle$, but its second derivative is $2\langle,\rangle$, and in characteristic 2 that is obviously the wrong choice.

The next guess is that we should use quadratic forms. There is also an L-theory of those. But it has the wrong L-groups, and not every symmetric bilinear form on a spectrum admits a quadratic refinement (not even up to Witt equivalence). For example, the skew-symmetric form on $\mathbb{Z}/2$, when converted into a symmetric form on $B\mathbb{Z}/2$, does not.

Here are the criteria I want. I want to do something L-theoretical, which is to say something about quadratic functions. And I want to be able to engineer second derivatives of quadratic functions. The trick is: no one forces me to use a *homogeneous* quadratic function (aka quadratic form).

The spectrum ℓ is defined as the L-theory spectrum of π-finite spectra equipped with a *possibly-inhomogeneous* quadratic function q. Heuristically, "possibly inhomogeneous" means that $q(0) = 0$ (no constant term) but $q(a)$ is not necessarily equal to $q(-a)$, i.e. there can be a linear term.

I've been talking about L-theory spectra. What are they? Heuristically, a k-cocycle is a *nondegenerate* thing of the type that you care about, where the quadratic information is of degree k. A co-homology between cocycles is a Lagrangian correspondence. In other words, you build a "symplectic category" that goes in both directions, where a k-endomorphism of the identity is (A, q) with q of degree k and nondegenerate (induced map $A \to A^\vee$ is an iso), and a map $(A, q) \to (A', q')$ is a span $A \leftarrow B \to A'$, and isomorphism $q|_B \cong q'|_B$, and the nondegeneracy condition is that the induced commutative square $B \to A \oplus A' \cong A^\vee \oplus A'^\vee \to B^\vee$ is exact (pushout equiv pullback).

Calculation: This ℓ-theory has the correct homotopy groups. Indeed, the forgetful map to the ℓ-theory of symmetric forms is an iso in positive cohomological degrees.

Now we're in business. Suppose that A, q is a connective π-finite spectrum with a degree-n inhomogeneous quadratic function $q : A \to I_{\mathbb{Q}/\mathbb{Z}}$. It selects an object in $\mathcal{W}$ as follows:

$$\mathrm{Gauss}(A, q) = \mathrm{colim}_{\Omega^\infty A}\, q.$$

More precisely, I use some chosen isomorphism $\gamma : \mathcal{W}^\times \xrightarrow{\sim} I_{\mathbb{Q}/\mathbb{Z}}$, so I should write this as

$$\mathrm{Gauss}(A, q) = \mathrm{colim}_{\Omega^\infty A}\, \gamma^{-1}(\Omega^\infty q).$$

I call this function "Gauss" because the corresponding TQFT, when evaluated on n-manifolds M^n, compiles a (generalized) Gauss sum, where you sum over the A-valued cohomlogy of M with the function $\Omega^\infty q$. A theorem of Gauss: if (A, q) is nondegenerate, then $\mathrm{Gauss}(A, q)$ is invertible, and a root of unity up to a factor of $\sqrt{|A|}$. Note that $|A| = \prod_i |\pi_i A|^{(-1)^i}$ is a positive $\mathbb{Q}$-number. In the higher version, this factor "$\sqrt{|A|}$" should be interpreted as an euler characteristic theory, so it trivializes on framed manifolds of positive dimension. For the purposes of narration, I'm going to fudge the degree-0 part.

But the invertibles in $\mathcal{W}$ are in $I_{\mathbb{Q}/\mathbb{Z}}$, at least up to a factor of γ.

The end result is that we just in the process constructed a map

$$\widehat{\mathrm{G}}/\,\mathrm{Gal} \to \Omega^\infty \ell_\bullet^\vee$$

that sends $(\mathcal{W}, \gamma : \mathcal{W}^\vee[\mathrm{tor}] \xrightarrow{\sim} I_{\mathbb{Q}/\mathbb{Z}})$ to the (linear!) function

$$(A, q) \mapsto \gamma^{-1}(\mathrm{Gauss}(A, q)).$$

up to a fudged factor of $\sqrt{|A|}$.

This is the map that our LES is really about. Our LES computation shows that this map is an isomorphism in degrees > 4.

8. 4-dimensional exceptions

Relatively easy by-hand computations show that $\pi_i \widehat{\mathrm{G}}/\,\mathrm{Gal} \to \pi_i \ell^\vee$ is an iso when $0 < i < 4$. Running the LES, you find that $\pi_1\,\mathrm{Gal} = \mathbb{Z}/2$ and $\pi_2\,\mathrm{Gal} = *$. The $\mathbb{Z}/2$ in degree 1 acts by fermion parity flip, and arises from the extension $\mathbf{Vec} \to \mathbf{sVec} = \mathcal{W}^1$. That $\pi_2\,\mathrm{Gal} = *$ means that there is no extension in degree 2: $\mathcal{W}^2 = \Sigma\mathcal{W}^1$. I fudged the degree-0 piece already, so let me just briefly say: if I were working over $\mathbb{Q}$ rather than $\mathbb{R}$, then $\pi_0 \widehat{\mathrm{G}}/\,\mathrm{Gal}$ would be trivial. Over $\mathbb{R}$, this group is nontrivial, but without the hats $\pi_0 \mathrm{G}/\,\mathrm{Gal}$ is trivial.

The interesting case is in dimension 4 (affecting $\pi_3\,\mathrm{Gal}$). That's where both classical and quantum surgery has trouble. The fundamental origin of this trouble is the existence of nonabelian knots.

The question in dimension 4 is to classify possibly-anomalous 3D TQFTs modulo interfaces. This is a "classical" question in quantum algebra, where a piece of mathematics is called "classical" if it is from before when the speaker started graduate school; the complete description of the group in question "well-known", defined to mean the speaker learned it during graduate school, and so assumed everyone else already knew it.

Suppose you have a 3D TQFT, perhaps with 4D invertible anomaly. You try to start applying surgery: you condense out all the operators of dimension $\leq \frac{4-3}{2}$. Well, that is something: you condense out the 0D operators. But that's all you can do. The remaining middle-dimensional operators are in 1D. The operator content of your TQFT is a braided fusion category $\mathcal{B}$ — er, a braided *super* fusion category because we're working over $\mathcal{W}^2 = \Sigma\mathbf{sVec}$. The Poincaré duality enjoyed by all TQFTs asserts that $\mathcal{B}$ is super-nondegenerate, also called *slightly degenerate*. Two super-nondegenerate braided fusion categories $\mathcal{B}, \mathcal{B}'$ are separated by an interface precisely when $\mathcal{B} \boxtimes \mathcal{B}'^{\mathrm{rev}}$ is a Drinfel'd centre of a super fusion category. The group with these generators and relations is called, not surprisingly, the *Witt group of slightly degenerate braided fusion categories*. I'll call it QuantWitt, where I'm emphasizing the quantum nature of the construction, and I'll not record the super-ness in the notation.

If $\mathcal{B}$ happens to be groupal, then the analysis from earlier of how $\mathcal{B}$ must look proceeds: such a $\mathcal{B}$ is classified by a finite abelian group A together with a nondegenerate symmetric form. This supplies a map Witt $\to$ QuantWitt, which we knew there must be (since $\widehat{\mathrm{G}}/\operatorname{Gal}$ maps to $\ell^\vee$ in all degrees). It is an amusing exercise to show that a QuantWitt-equivalence between groupal BFCs is itself groupal, and so a Witt-equivalence. So Witt $\to$ QuantWitt is an injection.

The problem is that in this dimension, my result with Yu does not apply: there are non-groupal BFCs. The essential reason for this is the existence of nonabelian knotting and linking: there are just a lot more ways that an S^1 can knot in $\mathbb{R}^3$ than there are ways an S^m can knot in $\mathbb{R}^{2m+1}$. This is also what interferes with manifold surgery from working: there are linkings that homology cannot see, and those linkings interfere with surgery.

An impressive paper of Davydov, Nikshych, and Ostrik gives the complete isomorphism type of QuantWitt. Their result is: the inclusion Witt $\to$ QuantWitt is non-canonically split; the quotient is noncanonically isomorphic to $(\mathbb{Z}/2)^{\oplus\infty} \oplus \mathbb{Z}^{\oplus\infty}$. We call this quotient the *truly-quantum Witt group* TrueQuantWitt.

The construction David explained only adjoins torsion things — it doesn't use the $\mathbb{Z}^\infty$ part, but it does use the $(\mathbb{Z}/2)^\infty$ part. The end result is that (in positive degrees):

$$\operatorname{fibre}(\mathrm{G}/\operatorname{Gal} \to \Omega^\infty \ell^\vee) = K(\mathrm{TrueQuantWitt}[\mathrm{tor}]^\vee, 4) \cong K((\mathbb{Z}/2)^{\times\infty}, 4)$$

$$\pi_3 \operatorname{Gal} = (\mathrm{QuantWitt}[\mathrm{tor}].\mathbb{Z}/24)^\vee$$

$$\pi^3 \mathcal{W} = \mathbb{Z}/24.\mathrm{QuantWitt}[\mathrm{tor}]$$

The periods mean "some extension of abelian groups TBD."

By the way, because of the $\mathbb{Z}^\infty$ term, the map $\mathcal{W}^\times \to I_{\mathbb{C}^\times}$ is not an isomorphism. It is an isomorphism on torsion, and so the deficit is a $\mathbb{Q}$-vector space. It is specifically

$$\operatorname{fibre}(\mathcal{W}^\times \to I_{\mathbb{C}^\times}) = K(\mathbb{Q}^{\oplus\infty}, 4).$$

You could have run David's construction but not restrict to torsion things. The results will not be finite separable, but since the only difference starts in high degree, it is still semisimple, and still a "Galois" extension in the same way that $\mathbb{Q} \to \mathbb{C}$ is a Galois extension: the map $\mathbb{Q} \to \mathbb{C}^{\mathrm{Aut}(\mathbb{C}/\mathbb{Q})}$ is an iso. (The $\mathbb{C}$ story requires the axiom of choice. Ours does not.) The larger thing is a $K(\mathbb{Q}^{\times\infty}, 4)$-Galois extension of the smaller, (ind-)fully finite thing we've talked about. Anyway, I'll continue to focus on the separable version David constructs.

Let's compare with the manifold theory. There is also a fibre of $\mathrm{G}/\mathrm{PL} \to L$, which is also only in degree 4. Namely, Rokhlin's Theorem says that for a smooth, or equivalently PL, manifold in dimension 4, the signature is divisible by 16 (and 16 is achievable), whereas L-theory would predict that 8 was achievable. 8 is achievable for topological manifolds, so the surgery exact sequence does exactly compute BTop. A bit more work shows that this factor of 2 from Rokhlin's Theorem is the only failure of the surgery exact sequence for PL:

$$\operatorname{fibre}(\mathrm{G}/\mathrm{PL} \to \Omega^\infty L) = K(\mathbb{Z}/2, 4).$$

The term in degree 4 is the *Kirby–Siemenmann invariant*.

All together we find that Gal and PL look extremely similar. They are built from different L-theories: by its very nature, Gal is a much more profinite object than PL, and its L-theory is the profinite version of the L-theory for PL (in the sense that ℓ is built from profinite abelian groups and

L is built from finite-rank free abelian groups). And Gal has *infinitely many Kirby–Siemenmann invariants*.

We strongly believe, but have not proved, that there is a canonical map $\mathrm{PL} \to \mathrm{Gal}$. Specifically:

Conjecture: The PL j-homomorphism factors through the cyclotomic character.

9. Postscript on the Kirby–Siemenmann invariant and $\pi_3\,\mathrm{Gal}$

To end, maybe I'll mention that I do think I know where the factorization of the cyclotomic character should send the manifold KS invariant, and I definitely know the precise group $\pi_3\,\mathrm{Gal}$.

There is a version of QuantWitt built from *pseudounitary* fusion categories and braided fusion categories. Pseudounitarity is a property, and there is a natural map $c : \mathrm{PseudounitaryQuantWitt} \to \mathrm{QuantWitt}$ that includes the nonunitary stuff. I don't know if this map is an injection, and I think it's not expected to be a surjection. But the advantage of PseudounitaryQuantWitt is that it comes with a canonical map to $\mathbb{Q}/(\frac{1}{2}\mathbb{Z})$, called the *central charge*. Restrict to the torsion subgroup; then this map factors through $\mathbb{Z}/2 = \{0, \frac{1}{4}\} \subset \mathbb{Q}/(\frac{1}{2}\mathbb{Z})$. This is not the trivial map: the super modular tensor category $\mathbf{Rep}(\mathrm{SO}(n)_n)$ represents a nontrivial element under this map when n is odd. I claim that this central charge map c extends canonically to QuantWitt[tor] (and trivializes on the classical subgroup Witt). It probably does not extend canonically to all of QuantWitt, although from abstract characterization you know it extends noncanonically.

Our conjectured map $\mathrm{PL} \to \mathrm{Gal}$ comes with two pieces. It predicts a map $L \to \ell^\vee$. And the remaining piece is a map $\mathrm{fibre}(\mathrm{G}/\mathrm{PL} \to \Omega^\infty L) \to \mathrm{fibre}(\mathrm{G}/\,\mathrm{Gal} \to \Omega^\infty \ell^\vee)$, which I predict sends the KS invariant to this "central charge" function $\mathrm{QuantWitt[tor]} \to \mathbb{Z}/2$.

Finally, let me write $\mathrm{QuantWitt[tor]}_{c=0}$ for the kernel of this "central charge" map, so that $\mathrm{QuantWitt[tor]} = \mathrm{QuantWitt[tor]}_{c=0}.\mathbb{Z}/2$. There is an extension $\mathbb{Z}/24 \to \mathbb{Z}/24 \to \mathbb{Z}/2$. I claim that there is a not-quite-canonical isomorphism

$$\pi_3\,\mathrm{Gal} = (\mathrm{QuantWitt[tor]} \times_{\mathbb{Z}/2} \mathbb{Z}/24)^\vee = \mathbb{Z}/2.(\mathbb{Z}/24 \times \mathrm{QuantWitt[tor]}_{c=0}^\vee)$$

where the latter extension is the diagonal one extending 24 to 48 and $\mathrm{QuantWitt[tor]}_{c=0}^\vee$ to QuantWitt[tor].