Factorization Algebras

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1 Lecture 1: 2023 May 1

Three things we want from the talks:

1. What is a factorization algebra?
2. Relate factorization algebras to field theories?
3. Be curious and learn more!

Rough schedule for now:

1. Physical (field theories) motivation
2. Topological FT
3. Functorial things

4. Holomorphic FT

5. Applications/Current Work

Goal: What are factorization algebras and how do we find them in nature?

1.1 Classical Story

Let's assume that we have some container X (a box, manifold, etc.) and that we have a particle running around in X. Mathematically, this is a path $[0,1] \to X$. The particle can't just do *anything*; it has to be constrained by at least the laws of motion. Intiuitively, we may have:

- The particle is lazy, i.e., it only takes least energy paths;
- $X = \mathbb{R}^2$ maybe the particle only takes straight lines
- $X = \mathbb{S}^2$ may only move in great circles
- If X is a Riemannian manifold, maybe the particle only moves in geodesics.

Physical constraints are called "equations of motion" or Euler-Lagrange (EL) equations and presented as a PDE. But PDEs are scary! So instead we equivalently encode the PDE as a map

$$S : \mathrm{Map}(I, X) \to \mathbb{R}$$

called the "action functional."

The allowable paths are a subset

$$\mathrm{EL} \subseteq \mathrm{Map}(I, X)$$

where

$$\mathrm{EL} := \{f : I \to X \mid (\mathrm{d}S)(f) = 0\}.$$

Note that the set EL is also called the *critical locus of S*.

More generally, instead of looking for maps $I \to X$, we might want to think of paths as parametrized by a space N. In particular, we may look for maps

$$N \times I \to X$$

which we call spacetime. Note we also define $M := N \times I$.

Definition 1.1.1. A *classical field theory* is a:

1. Spacetime M;
2. A space of fields $\mathrm{Map}(M, X)$;
3. An action functional $S : \mathrm{Map}(M, X) \to \mathbb{R}$.

Remark 1.1.2. We'll be revisiting and amending this definition as we go!

Definition 1.1.3. The *dimension* of the field theory is the dimension of M.

Example 1.1.4 (Classical Mechanics, Massless Free Theory)**.** Let $I = [a, b]$ be a time interval and let our fields be $\mathrm{Map}([a, b], \mathbb{R}^n)$ for some $n \in \mathbb{N}, n \geq 1$. Our action function is then

$$S(f) := \int_a^b \left\langle f(t), \frac{\partial^2}{\partial t^2} f(t) \right\rangle \mathrm{d}t.$$

The critical locus is then

$$\mathrm{EL} = \{\text{straight lines}\}.$$

This example can be generalized to Riemannian manifolds.

Remark 1.1.5. We want to think of our spacetime M as a compact manifold and we want X to be something like a manifold. The action functional S needs to be something local. What do we mean? Well: S is *local* when it can be written as an integral over M of some polynomial in the fields and their derivatives.

Example 1.1.6 (Gauge Theory)**.** Let G be a Lie group and let M be a spacetime. Denote this theory by $\mathrm{Gauge}_G(M)$. The fields are $\mathrm{Bun}_G^{\nabla}(M)$ (principal G-bundles with connection). If we drop the connection condition we get that

$$\mathrm{Map}(M, BG) \cong \mathrm{Bun}_G(M)$$

(note: BG is not necessarily a manifold but it's close enough). There's a way to modify this to get a connection classifying space $B_{\nabla}G$ and produce

$$\mathrm{Map}(M, B_{\nabla}G) = \mathrm{Bun}_G^{\nabla} M.$$

Our action functional (actional — suggestion from Geoff) is given by the Yang Mills equation

$$S(A) = \frac{-1}{2} \int_M \mathrm{tr}(\mathrm{d}A \wedge *\mathrm{d}A).$$

Remark 1.1.7. Our definition of saying that fields are mapping spaces $\mathrm{Map}(M, X)$ might not always be true. When it is true, the field theory is called a sigma model.

Question from Hank: How important is it for the actional to be positive definite? Araminta will get back to us!

1.2 Quantization

Answer the important motivating questions. Say we have a box with a spider in it. We can ask:

- Can the spider get out?
- How fast is the spider?
- Where is the spider?
- Is the box open?
- Are there holes?

The first question is a questio nthat we need to know everything about the field theory to know; the second and third questions are local geometric questions'; and the fourth and fifth questions are global questionas about the topology of X. These are *measurements* we can make on a field theory. In quantum field theory we cannot answer these all. This is the *Heisenberg Uncertainty Principle.* This says the following:

Definition 1.2.1 (Heisenbewrg Uncertainty Principle)**.** In Quantum Field Theory (QFT) we cannot precisely know the position and momentum of the particle at the same time.

Goal: mathematicatize this principle!

Definition 1.2.2. Given a classical field theory

$$S : \mathrm{Map}(M, X) \to \mathbb{R}$$

the *classical observables* are the real functions[1] on the critical locus of S, i.e., when $\mathrm{EL} \subseteq \mathrm{Map}(M, X)$,

$$\mathrm{Obs}^{\mathrm{cl}} = \mathcal{O}_{\mathrm{EL}}$$

[1] This is purposefully vague at the moment.

Remark 1.2.3. The classical observables $\mathrm{Obs}^{\mathrm{cl}}$ is a commutative algebra via pointwise multiplication, for $f, g : \mathrm{EL} \to \mathbb{R}$,

$$(fg)(\lambda) = f(\lambda)g(\lambda).$$

This fails in QFT!

In QFT let's say we have two observables $f, g \in \mathrm{Obs}$. We want to get an observable fg which is the product of fg. The trick is to do the measurements at distinct times. So: If we do f and g on different disjoint time intervals we get a product of observables dependent on disjoint intervals. More generally, if $M = N \times I$ then we look to do the observations f and g on disjoint open discs in M.

In particular: the set of quantum observables Obs^q should have this weird alg structure.

1.3 Factorization Algebra

Definition 1.3.1 (Factorization Algebra)**.** Let M be a manifold. A *factorization algebra on M* valued in the category of chain complexes is a functor

$$\mathcal{F} : \mathbf{Open}(M) \to \mathbf{Ch}$$

together with isomorphisms

$$\mathcal{F}\left(U \coprod V\right) \cong \mathcal{F}(U) \otimes \mathcal{F}(V)$$

if U, V are disjoint opens which satisfying a Weiss topology cosheaf condition[2].

What data do we get from a factorization algebra? Well:

- Given a disjoint family of open discs $\coprod_{i=1}^k D_i \to M$ we get an isomorphism

$$\mathcal{F}\left(\coprod_{i=1}^k D_i\right) \cong \bigotimes_{i=1}^k \mathcal{F}(D_i)$$

- If we have a n open disc D in M and disjoint open discs $\coprod_{i=1}^k D_i$ in D, then we have a map

$$\bigotimes_{i=1}^k \mathcal{F}(D_i) \cong \mathcal{F}\left(\coprod_{i=1}^k D_i\right) \to \mathcal{F}(D)$$

  which is our "multiplication" in the factorization algebra.

Theorem 1.3.2 (Costello = Gwilliam)**.** *The quantum observables* Obs^q *of a FT has the structure of a factorization algebra on the spacetime.*

The main take-away is that we should think of QFTs *as* factorization algebras.

Given a factorization algebra $\mathrm{Obs}^q : \mathbf{Open}(M) \to \mathbf{Ch}$ and an open $U \subseteq M$ we get a chain complex

$$\mathrm{Obs}^q(U) = \text{measurements we can take on } U$$

which are local observables on U.

Question: Are there *global* observables?

Since Obs^q is a cosheaf we can take its global sections.

Definition 1.3.3. Let $\mathcal{F}$ be a factorization algebra on M. The *factorization homology* of M with coefficients in $\mathcal{F}$ is the global sections

$$\int_M \mathcal{F} := \mathcal{F}(M)$$

[2] This is a local-to-global condition.

Remark 1.3.4 (ACHTUNG)**.** The integral symbol above is **NOT AN END OR COEND OR INTEGRAL IN THE USUAL SENSE.** It *just* means factorization algebra.

A question about what the dimension of a FT means from Pedro: The dimension tells us how many directions in which the particle can moove around.

A question: How do we quantize a classical algebra into a quantum factorization algebra. This is something we'll discuss in the week, but the trick is that we have to take commutative algebras (the classical commutative algebras) and turn them into functors. Factorization algebras on $\mathbb{R}$ are like associative algebras.

2 Lecture 2: 2023 May 2

How to do Question 1.6 from the problem session. Recall that a classical field theory is

$$\mathrm{Obs}^{\mathrm{cl}} = \mathcal{O}_{\mathrm{EL}} \,.$$

Question 1.6 asked us to show that $\mathrm{Obs}^{\mathrm{cl}}$ form a factorization algbera on M. More generally, we should show that if A is a commutative algebra then A induces a factorization algebra on M.

Definition 2.0.1. A factorization algebra on M is a Weiss cosheaf on M, $\mathcal{F} : \mathbf{Open}(M) \to \mathbf{Ch}$, and has equivalences

$$\mathcal{F}\left(U \coprod V\right) \cong \mathcal{F}(U) \otimes \mathcal{F}(V).$$

Assume that A is a commutative algebra in $\mathbf{Ch}$. Define the constant copresheaf $\mathcal{F}_1 : \mathbf{Open}(M) \to \mathbf{Ch}$ by

$$U \mapsto A, \mathrm{incl} : U \to V \mapsto \mathrm{id}_A \,.$$

This then induces a functor $\mathcal{F}_2 \to \mathbf{Open}(M) \to \mathbf{Com}(\mathbf{Ch})$. Coseafify $\mathcal{F}_2$ in the Weiss (Grothendieck) topology to get a cosheaf $\mathcal{F}$ factoring as

$$\mathcal{F} : \mathbf{Open}(M) \to \mathbf{Com}(\mathbf{Ch}) \to \mathbf{Ch} \,.$$

To show that this gives a factorization algebra, because in $\mathbf{Com}(\mathbf{Ch})$ the coproduct is tensor products, we get that

$$\mathcal{F}\left(U \coprod V\right) \cong \mathcal{F}(U) \coprod \mathcal{F}(V) = \mathcal{F}(U) \otimes \mathcal{F}(V).$$

A question: does this work because something about the Weiss topology allow us to know that our colimits and limits work out properly.

2.1 Lecture 2

Recall that quantum field theory to have quantum observables as a factorization algebra on M. Today we will talk about topological field theories.

Remark 2.1.1. Intuition: A field theory is topological if it desn't depend on metrics.Recall that the actional $S : \mathrm{Map}(M, X) \to \mathbb{R}$ *can* use metrics. If if it doesn't use metrics, it is essentially topological in nature.

Question: How does being topological affect Obs. Well: Consider that we have to describe $\mathrm{Obs}(B_r(0))$ for all $r > 0$ for $r \in \mathbb{R}$. Then since we have a chain complex $\mathrm{Obs}(B_r(0)) \in \mathbf{Ch}$ for any $r \in \mathbb{R}$, so if $r < s$ we get an inclusion

$$\mathrm{Obs}(B_r(0)) \to \mathrm{Obs}(B_s(0))$$

Because our field theory is topological, this must be an equivalence. Similarly, if we have any other ball $B_s(1)$ for $s, r > 0$ and $s, r \in \mathbb{R}$ then there is also an equivalence

$$\mathrm{Obs}(B_r(0)) \to \mathrm{Obs}(B_s(1))$$

because the difference between the balls is in this case a metric issue. This warrants a definition which allows us to capture locally constant cosheaves.

Definition 2.1.2. A factorization algebra $\mathcal{F}$ is *locally constant* if there is an isomorphism of complexes

$$\mathcal{F}(D_1 \subseteq D_2) : \mathcal{F}(D_1) \xrightarrow{\cong} \mathcal{F}(D_2)$$

whenever $D_1 \subseteq D_2$ for D_1, D_2 open discs in M.

Definition 2.1.3. A field theory is topological if the quantum observables are locally constant.

Example 2.1.4. Let $M = \mathbb{R}$ and consider a topological QFT with observables Obs^q. Then the functor Obs^q sends open sets to the category of chain complexes. We claim that Obs^q is determined by $\mathrm{Obs}^q(\mathbb{R})$. Define

$$\mathrm{Obs}^q(\mathbb{R}) := A.$$

Consider two disjoint opens balls $U, V \subseteq \mathbb{R}$ and assume that $U \coprod V \subseteq W$ for another open ball $W \subseteq \mathbb{R}$. Then we get a commuting diagram

$$\begin{tikzcd}
\mathrm{Obs}^q\left(U \coprod V\right) \arrow[r, "\cong"] \arrow[d, "\cong"'] & \mathrm{Obs}^q(U) \otimes \mathrm{Obs}^q(V) \arrow[r] \arrow[d, "\cong"] & \mathrm{Obs}^q(W) \arrow[d, "\cong"] \\
A \otimes A \arrow[r] & A \otimes A \arrow[r] & A
\end{tikzcd}$$

and because we cannot slide U past V while keeping the opens disjoint, the multiplication is noncommutative. Thus locally constant factorization algebras are noncommutative algebras.

Example 2.1.5. Consider $M = \mathbb{R}^\infty$ and let $\mathcal{F}$ be a factorization algebra on $\mathbb{R}^\infty$ which is locally constant. By playing the exact same game as in the prior example and setting $A := \mathcal{F}(\mathbb{R}^\infty)$ we get a multiplication map

$$\mu : A \otimes A \to A.$$

Because we can slide open balls around because of the infinite dimensions of movement and freedom in the literal space $\mathbb{R}^\infty$, the multiplication law is (coherently!) commutative.

We showed that:

Spacetime	Locally constant factorization algebras
$\mathbb{R}$	Algebras
$\mathbb{R}^2$	?
$\vdots$	$\vdots$
$\mathbb{R}^\infty$	Commutative algebras

Example 2.1.6. Consider in $\mathbb{R}^2$ that we have once again we have for a locally constant factorization algebra we cannot move around stuff. We get $\mathbb{E}_2$-algebras for locally constat factorization algebras because our algebras are only *homotopy* commutative — the trick is that the multiplication map

$$A \otimes A \cong \mathrm{Obs}^q(U) \otimes \mathrm{Obs}^q(V) \to A = \mathrm{Obs}^q(W)$$

holds for any two disjoint opens and any open containing the disjoints. Basically you can pick up monodromy of $\mathbb{S}^1$ as you run around and don't get strict commutativity.

Now:

Spacetime	Locally constant factorization algebras
$\mathbb{R}$	Algebras
$\mathbb{R}^2$	$\mathbb{E}_2$-algebras
$\vdots$	$\vdots$
$\mathbb{R}^\infty$	Commutative algebras

We will extrapolate and get to $\mathbb{E}_n$-algebras and describe them as locally constant factorization algebras on $\mathbb{R}^n$. To do this we will need operads!

2.2 Operads

Let $\mathbf{Fin}^{\mathrm{bij}}$ be the category of finite sets and bijections.

Definition 2.2.1. The category of *symmetric sequences* is the category

$$\mathbf{SSeq} := \left[\mathbf{Fin}^{\mathrm{bij}}, \mathbf{Spaces}\right].$$

The category **SSeq** is a monoidal category. How? Well assume that $R, S \in \mathbf{SSeq}$. Then define the monoidal product $R \circ S$ by, if $[n]$ is a fixed set of n-elements

We will copy this from the notes later.

$$(R \circ S)[n] :=$$

Note that the argument involving $[n]$ is a skeletal argument involving $\mathbf{Fin}^{\mathrm{bij}}$, as Fin has one isomorphism class of objects for each $n \in \mathbb{N}$.

The unit of the monoidal structure is $\mathcal{O}_{\mathrm{triv}}$ sends a finite set B to the unit $\mathbb{1}_{\mathbf{Spaces}}$ if $|B| = 1$ and $*$ otherwise.

Definition 2.2.2. An *operad* is a monoidal object in **SSeq**.

If $\mathcal{O}$ is an operad for every $k \in \mathbb{N}$ we get

$$\mathcal{O}(k) \in \mathbf{Spaces}$$

as well as an operation

$$(\mathcal{O} \circ \mathcal{O})(k) \to \mathcal{O}(k)$$

Definition 2.2.3 (Also an Example)**.** The $\mathbb{E}_n$-operad is the SSeq

$$\mathbb{E}_n(k) := \mathrm{Conf}_k(\mathbb{R}^n)$$

where

$$\mathrm{Conf}_k(\mathbb{R}^n) := \{X \subseteq (\mathbb{R}^n)^k \mid X \,\text{has}\, k \,\text{distinct points in}\, \mathbb{R}^n\}.$$

Note that there is a map

$$\mathbb{E}_n \circ \mathbb{E}_n \to \mathbb{E}_n$$

built as follows. View a point in $\mathbb{R}^n$ as a little ε-small open disc with centre at the given point. Then from embedding disjoint unions of discs into bigger discs we can manipulate our operations.

Definition 2.2.4. If $\mathcal{O}$ is an operad then an *$\mathcal{O}$-algebra* in a symmetric monoidal closed category $\mathscr{C}$ is an object $V \in \mathscr{C}$ together with morphisms (note the adjoint transpose below)

$$\frac{\mathcal{O}(k) \to \mathscr{C}(V^{\otimes k}, V)}{\mathcal{O}(k) \otimes V^{\otimes k} \to V}$$

for all $k \in \mathbb{N}$.

Theorem 2.2.5 (Lurie; cf. *Higher Algebra*)**.** *There is an equivalence of categories*

$$\mathbb{E}_n\,\mathbf{Alg} \simeq \mathbf{FactAlg}(\mathbb{R}^n)_{\mathrm{l.c.}}$$

*where $\mathbb{E}_n$ **Alg** is the category of $\mathbb{E}_n$-algebras and $\mathbf{FactAlg}(\mathbb{R}^n)_{\mathrm{l.c.}}$ si the category of locally constant factorization algebras.*

The examples show that this works for $\mathbb{E}_1$ and $\mathbb{E}_\infty$. The idea is that ther eis an isomorphism

$$\mathrm{Conf}_k(\mathbb{R}^n) \simeq \mathrm{Emb}^{\mathrm{fr}}\left(\coprod_k \mathbb{R}^n, \mathbb{R}^n\right).$$

Tomorrow: Factorization Homology. If $\mathcal{F}$ is a factorization algebra on M, we should think of

$$\int_M \mathcal{F} = \mathcal{F}(M).$$

So if $\mathcal{F}$ is locally constant, how do we define factorization homology of $\mathbb{E}_n$-algebras? Basically, how can we pass through the equivalence of categories to define and make sense of statements like

$$\int_M \mathcal{F} = \int_M A_{\mathcal{F}}?$$

3 Lecture 3: 2023 May 3

3.1 Foactroization Homology in the $\mathbb{E}_n$-language

Recall:

Definition 3.1.1. A *factorization algebra* on M is a Weiss cosheaf on M, $\mathcal{F}$, together with isomorphisms

$$\mathcal{F}\left(U \coprod V\right) \cong \mathcal{F}(U) \otimes \mathcal{F}(V)$$

for disjoint opens $U, V \subseteq M$.

Definition 3.1.2. The *factorization homology* of a factorization algebra $\mathcal{F}$ on M is

$$\int_M \mathcal{F} := \mathcal{F}(M).$$

Yesterday we showed that for any TQFT the locally constant facorization algebras correspond to $\mathbb{E}_n$-algebras on $M = \mathbb{R}^n$. Today we want to unerstand $\int_M A$ for $A \in \mathbb{E}_n\,\mathbf{Alg}$.

Definition 3.1.3. Let $\mathbf{Mfld}_n$ be the ∞-category n-dmension manifolds (without boundary niceness) together with smooth embeddings.

Remark 3.1.4 (ACHTUNG)**.** Today our lecture is going to be in a derived setting, so be particularly careful. In particular the category $\mathbf{Ch}$ is now used to mean the derived ∞-category $\mathsf{D}(\mathbf{Ch})$.

Remark 3.1.5. For now, an ∞-category $\mathscr{C}$ is a category such that $\mathscr{C}(X, Y)$ is a topological space for all objects X, Y.

Example 3.1.6. Let $X, Y \in \mathbf{Mfld}_n$. Then

$$\mathbf{Mfld}_n(X, Y) = \mathrm{SEmb}(X, Y)$$

and the space of smooth embeddings carries a topology.

Remark 3.1.7. There is a framed version of the ∞-category $\mathbf{Mfld}_n$. We denote this as $\mathbf{Mfld}_n^{\mathrm{fr}}$.

Definition 3.1.8. Let $\mathbf{Disc}_n \hookrightarrow \mathbf{Mfld}_n$ be the full subcategory of n-manifolds isomorphic to disjoint unions of Euclidean spaces

$$\coprod_k \mathbb{R}^n$$

and similarly for the framed version.

This category of discs is symmetric monoidal under disjoint union; its unit is the manifold of every dimension: $\varnothing$.

Let $M \in \mathbf{Mfld}_n$. Take the overcategory (slice category) $\mathbf{Disc}_n/M$ of smooth embeddings

$$\coprod_k \mathbb{R}^n \to M.$$

Morphisms in this category are diagrams

$$\begin{tikzcd} U \arrow[rr] \arrow[dr] & \arrow[d, phantom, "\Downarrow\varphi"] & V \arrow[dl] \\ & M & \end{tikzcd}$$

where φ is some chosen isotopy. There is also a forgetful functor from the slice category to the total categorry which sends an object $U \to M$ to U.

Definition 3.1.9. An *n-disc algera* is a symmetric monoidal functor

$$A : \mathbf{Disc}_n \to \mathbf{Ch}$$

Let $\mathbf{Alg}_n(\mathbf{Ch})$ donote the category n-disc algebras.

We have diagrams of categories

$$\mathrm{FactAlg}(\mathbb{R}^n) \hookleftarrow \mathbb{E}_n\,\mathbf{Alg} \hookrightarrow \mathbf{Alg}_n(\mathbf{Ch})$$

where the left arrow includes locally constant factorization algebras and the right arrow picks out framed disc algebras. Now since

$$\mathrm{SEmb}^{\mathrm{fr}}\left(\coprod_k \mathbb{R}^n, \mathbb{R}^n\right) \cong \mathrm{Conf}_k(\mathbb{R}^n).$$

I missed some things :(

Definition 3.1.10. Let M be an n-manifold and let A be an n-disc algebra in $\mathbf{Ch}$. The factorization homology of M with coefficients in A is the diagram

$$\int_M A = \mathrm{colim}\left(\mathbf{Disc}_n/M \to \mathbf{Disc}_n \xrightarrow{A} \mathbf{Ch}\right).$$

We now can ask if factorization homology is in fact in some way a homology theory?

Well a usual homology theory $H_0(M; A)$ satisfies the Eilenberg-STeenrod axioms. In particular, if we write

$$M = U \coprod_W V$$

the Eiulenberg-Steenrod axioms tell us how to compute M via excision.

Definition 3.1.11. A symmetric monoidal functor

$$\mathbf{Mfld}_n \to \mathbf{Ch}$$

satisfies $\otimes$-*exicision* if given a cullor gluing

$$M \simeq U \coprod_{V \times \mathbb{R}} U'$$

there is ane equivalence

$$T(M) \simeq T(U) \overset{\mathbb{L}}{\otimes}_{T(V \times \mathbb{R})} T(U').$$

Example 3.1.12. We saw an example of excision for manifolds with boundary where a person god excised from their body by identifying the neck as a celinder $V \times \mathbb{R}$ separating/gluing the head to the body.

Definition 3.1.13. The category of homology theories of n-manifolds is the full subcategory

$$\mathbb{H}\left(\mathbf{Mfld}_n; \mathbf{Ch}\right) \hookrightarrow \mathbf{SMFunc}(\mathbf{Mfld}_n, \mathbf{Ch})$$

of symmetric monoidal functors satisfying $\otimes$-excision.

Theorem 3.1.14 (Ayala-Francis)**.** *There is an equivalence of categories*

$$\mathbb{H}\left(\mathbf{Mfld}_n; \mathbf{Ch}\right) \simeq \mathbf{Alg}_n(\mathbf{Ch})$$

Sketch. The functor from homology to algebras is evaluation on $\mathbb{R}^n$ while the functor from algebras to homologies is the global sections with coefficients in A functor. $\square$

Given M, is $H_\bullet(M;\mathbb{R}) = \int_M A$ for some factorization algebra? This answer is yes, but factorization homology is not a generic homology theory for technical reasons.

3.2 Compute Things!

Example 3.2.1. Let $M = \mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} A = \operatorname{colim}\left(\mathbf{Disc}_n/\,\mathbb{R}^n \to \mathbf{Disc}_n \xrightarrow{A} \mathbf{Ch}\right).$$

Now, since $\mathbf{Disc}_n/\,\mathbb{R}^n$ has a terminal object (which is the identity morphism $\mathbb{R}^n = \mathbb{R}^n$), the colimit has the value

$$\int_{\mathbb{R}^n} A = (A \circ \mathrm{Forget})(\mathbb{R}^n \xrightarrow{\mathrm{id}_{\mathbb{R}^n}} \mathbb{R}^n) = A(\mathbb{R}^n).$$

Remark 3.2.2. If $\mathcal{F}$ is a factorization algebra on M and $\mathbb{R}^n \subseteq M$ then

$$\int_M \mathcal{F} = \Gamma(M, \mathcal{F})$$

and

$$\int_{\mathbb{R}^n} \mathcal{F} = \Gamma(\mathbb{R}^n, \mathcal{F}) = \mathcal{F}(\mathbb{R}^n).$$

Example 3.2.3. Let $M = \coprod_k \mathbb{R}^n$. Then

$$\int_M A = \operatorname{colim}\left(\mathbf{Disc}_n/M \to \mathbf{Disc}_n \xrightarrow{A} \mathbf{Ch}\right)$$

and since the slice category has terminal object $\mathrm{id} : M \to M$, we have

$$\int_M (A) = (A \circ \mathrm{Forget})(\mathrm{id}_M : M \to M) = A(M).$$

Since $M = \coprod_k \mathbb{R}^n$ we further have that

$$A(M) \simeq \otimes_k A(\mathbb{R}^n).$$

Example 3.2.4. Let $M = \mathbb{S}^1$ and assume that A is a framed disc algebra so A is an $\mathbb{E}_1$-algebra, i.e., A is an associative algebra. Consider that

$$\int_M A = A(M).$$

However, we compute now weith excision. Consider that we can write $\mathbb{S}^1$ as the union of the oepn northern hemisphere U and the open lower hemisphere U' glued along the overlaps. If $V = \mathbb{S}^0$, we can write these overlaps as a space homeomorphic to $V \times \mathbb{R}$. Note also that everything in sight is framed. Now

$$\int_{\mathbb{S}^1} A = A(\mathbb{S}^1) \simeq \int_U A \otimes_{\int_{V\times\mathbb{R}} A} \int_{U'} A \simeq A(\mathbb{R}) \otimes_{\mathbb{S}^0 \times \mathbb{R}} A(\mathbb{R}).$$

Since we have to keep track of all orientations and one of the copies of $\mathbb{R}$ in $\mathbb{S}^0 \times \mathbb{R}$ points the other direction, we get

$$A(\mathbb{R}) \otimes_{\mathbb{S}^0 \times \mathbb{R}} A(\mathbb{R}) \simeq A(\mathbb{R}) \otimes_{A(\mathbb{R})^{\mathrm{op}} \otimes A(\mathbb{R})} A(\mathbb{R})$$

so

$$\int_{\mathbb{S}^1} A = A(\mathbb{S}^1) = A(\mathbb{R}) \otimes_{A(\mathbb{R})^{\mathrm{op}} \otimes A(\mathbb{R})} A(\mathbb{R})$$

which is the Hochshild homology of A.

Recall that we have a forgetful functor $\mathbf{alg}_n(\mathbf{Ch}) \to \mathbf{Ch}$ given by $A \mapsto A(\mathbb{R}^n)$.

Definition 3.2.5. The *free n-disc algebra* is the left adjoint ot the forgetchul functor and is a functor

$$\mathrm{Free}_n : \mathbf{Ch} \to \mathbf{Alg}_n(\mathbf{Ch}).$$

Example 3.2.6. Let $V \in \mathbf{Ch}$. Then $\mathrm{Free}_n(V)$ sends $\coprod_k \mathbb{R}^n$ to

$$\bigoplus_{m\geq 0} C_0 \left(\mathrm{SEmb} \left(\coprod_m \mathbb{R}^n, \coprod_k \mathbb{R}^n \right) \right) \otimes_{S_m} V^{\otimes m}$$

where S_m is the symmetric group on m letters. If we're framed we find that

$$\mathrm{Free}_n^{\mathrm{fr}}(V)(\mathbb{R}^n) = \bigoplus_{m\geq 0} C_0 \left(\mathrm{Conf}_m \mathbb{R}^n \right) \otimes_{S_m} V^{\otimes m}.$$

Proposition 3.2.7. *If M is a framed n-manifold,*

$$\int_M \mathrm{Free}_n^{\mathrm{fr}}(V) = \bigoplus_{m\geq 0} C_0(\mathrm{Conf}_m(M)) \otimes_{S_m} V^{\otimes m}.$$

Remark 3.2.8. Use this to compute *harder* $\mathbb{E}_n$-algebra A homology by finding a free $\mathbb{E}_n$-algebra resolution of A. This works well because of the Barr-Beck Theorem.

Start of the proof. Consider, if $U = \coprod_k \mathbb{R}^n$, that

$$\mathrm{Free}_n(V)(U) = \bigoplus_{m\geq 0} C_0 \left(\mathrm{Conf}_m(\mathbb{R}^n) \right) \otimes_{S_m} V^{\otimes m} \Big)^{\otimes k} \simeq \bigoplus_{m\geq 0} C_0 \left(\mathrm{Conf}_m(U) \right) \otimes_{S_m} V^{\otimes m}.$$

Thus $\int_U \mathrm{Free}_n^{\mathrm{fr}} V$ works. Now

$$\begin{aligned} \int_M \mathrm{Free}_n^{\mathrm{fr}}(V) &= \operatorname*{colim}_{U \in \mathbf{Disc}_n/M} \bigoplus_{m\geq 0} C_0(\mathrm{Conf}_m U) \otimes V^{\otimes m} \\ &\simeq \bigoplus_{m\geq 0} \operatorname*{colim}_{U \in \mathbf{Disc}_n/M} C_0(\mathrm{Conf_m U}) \otimes V^{\otimes m}. \end{aligned}$$

One shows that

$$\operatorname*{colim}_{U \in \mathbf{Disc}_n/M} C_0(\mathrm{Conf_m U}) \otimes V^{\otimes m} \simeq C_0(\mathrm{Conf}_m(M)).$$

Remark: have to show that can compute this colimit over ordinary cats and ∞-cats. $\square$

If $A = \mathrm{Obs}^q$ for a QFT on M then $\int_M \mathrm{Obs}^q$ tells you the global measurements. $\mathrm{Obs}^q(U)$ tells you measurements on U.

Will asked a question about where is the cosheaf condition in this story? Remember that a factorization algebra was a functor

$$\mathcal{F} : \mathbf{Open}(\mathbb{R}^n) \to \mathbf{Ch}$$

and an n-disc algebra is a functor

$$A : \mathbf{Disc}_n \to \mathbf{Ch}\,.$$

The Weiss cosheaf condition omes down to saying that $\mathcal{F}$ is determined by $\mathcal{F}(\coprod_k \mathbb{R}^n)$. In particular, $F(U)$ is determined by the Weiss cosheaf condition and computing

$$A(U) \simeq \int_U A$$

which by construction is determined by $A(\coprod_k \mathbb{R}^n)$.

4 Lecture 4: 2023 May 4

4.1 Functorial Day

Remark 4.1.1. Observables, in the way we've been thinking of them, Obs are sometimes called "pout observalbes."

Observables	Costello-Gwilliam	Topological ($\mathbb{R}^n$)
Point Observables	Factorization algebra in **Ch**	$Z(\mathbb{S}^{n-1})$: $\mathbb{E}_n$-algebra
Line Operators	Guess: Factorization alg in **Cat** Approx: Modules over $\int_{\mathbb{S}^{n-2}\times\mathbb{R}^2}\mathrm{Obs}$	$\mathbb{Z}(\mathbb{S}^{n-2})$: $\mathbb{E}_{n-2}$-algebra in **Cat**

Recall that if $\mathcal{F}$ is a locally constant factorization algebra on $\mathbb{R}^n$ then the $\mathbb{E}_n$-aglerba is $\mathcal{F}(D^n)$ for some D^n open. Now if we have an open in $\mathbb{R}^2$ with two disjoint discs then by pulling the two discs back and embedding the two discs back in $\mathbb{R}^3$ we arrive at a 2-manifold with boundary given by the two discs. This is our bordism.

Note that the multiplicative structure of $\mathcal{F}(D^n)$ is determined by the linking sphere $Z(\mathbb{S}^{n-1})$.

Example 4.1.2 (Wilson Loop Operator)**.** Let G be a Lie group G and consider the Gauge theory of rG and M. We have

$$\mathrm{Fields} = \mathbf{Bun}_G^\nabla M$$

Given a lop C in M we get a map

$$\mathbf{Bun}_G^\nabla M \to \mathbb{R}$$

by taking the trace of holonomy around C:

$$(P, A) \in \mathbf{Bun}_G^\nabla M, \quad \mathrm{Hol}_C(P, A) : \mathfrak{g} \to \mathfrak{g}\,.$$

Set

$$\gamma := \mathrm{trace}(\mathrm{Hol}_C(P, A)).$$

Note that γ is a function on fields. Now since

$$\mathrm{Obs} = \mathrm{Map}(\mathrm{EL}, \mathbb{R})$$

but Obs don't know about dependence on C.

In general, maps

$$\text{Fields} \to \mathbb{R}$$

which depend on lines/curves in spacetime are called *line operators.*

What structure should line operators have?

We can *stack* lines in the sense that if we have n line segments we can glue them along their endpoints to make one big line. The main insight is that there should be a category of line operators! Here is the category **LineOp**:

- Objects: Line operators, i.e., pairs (L, ρ) of a line L in M and the corresponding operator $\rho : \textbf{Fields} \to \mathbb{R}$.
- Morphisms: We define
$$\textbf{LineOp}\big((L, \rho), (L', \rho')\big)$$
to be the set of point observables that can be placed between L and L'. We set $\textbf{LineOp}\big((L, \rho), (L', \rho')\big) := \varnothing$ if the lines L and L' are nowhere near each other.
- Composition: Stacking three lines.
- Identities: Added formally.

We get an algebraic structure as follows: by collision of lines. If our field theory is topological of dimension $n \geq 2$, the linking spheres are now $\mathbb{S}^{n-2}$. This gives **LineOp** the structure of an $\mathbb{E}_{n-1}$-monoidal category.

We guess this pattern continues:

Point	$\mathbb{E}_n$ in **Ch**
Lines	$\mathbb{E}_{n-1}$ in **Cat**
Surfaces	$\mathbb{E}_{n-2}$ in ???
$\vdots$	$\vdots$
dim k manifolds	$\mathbb{E}_{n-k}$ in ???

If X is an oriented manifold, let $\bar{X}$ denote the manifold X with reveresed orientation.

Definition 4.1.3. Let $\textbf{Cob}(n)$ the category with:

- $(n-1)$ dimensional closed oriented manifolds;
- A morphism $M \to N$ is an n-dimensional oriented manifold W with $\partial W = \overline{M} \coprod N$.
- Composition: Gluing tubes along shared boundaries.

Definition 4.1.4 (Atiyah-Segal)**.** An n-dimensional TQFT is a symmetric monoidal functor

$$Z : \textbf{Cob}(n)^{\coprod} \to \textbf{Ch}^{\otimes} .$$

We think of $Z(\mathbb{S}^{n-1})$ as the point observables of the field theory.

Definition 4.1.5. An $\mathbb{E}_n$-algebra is the data of

$$\mathbb{E}_n(k) \otimes Z(\mathbb{S}^{n-1})^{\otimes k} \to \mathbb{Z}(\mathbb{S}^{n-1})$$

where

$$\mathbb{E}_n(k) = \text{Conf}_k(\mathbb{R}^n)$$

Example 4.1.6. If $k = 2$ and $n = 2$ then these come from bordisms of pairs of pants.

Remark 4.1.7. Atiyah-Segal's definition considers all n-manifolds at once rather than a single spacetime. If we want to specify a spacetime M, there is a version with cobordisms embedded inside M.

Definition 4.1.8. If V is a (real) vector space (or chain complex) we write

$$V^\vee := \mathscr{C}(V, \mathbb{R}).$$

Let Z be an n-dimensional TQFT and let K ne an $n-1$ dimensional oriented manifold. Then the cylinder $K \times [0,1]$ gives a morphism in $\mathbf{Cob}(n)$ in a few ways:

- As a map $K \to K$ (which is the "horizontal" cylinder with faces K pointing left and right).
- As a map $K \coprod \bar{K} \to \varnothing$ by bending the cylinder to have the two faces at the left (the domain) side. This is a left macaroni and describes
$$\mathrm{ev} : Z(K) \otimes Z(\bar{K})^\vee \to \mathbb{R}\,.$$
- By facing the macaroni to the right we get a map $\varnothing \to \bar{K} \coprod K$ and hence induce
$$\mathrm{coev} : \mathbb{R} \to \mathbb{Z}(K)^\vee \otimes Z(K).$$

Remark 4.1.9. Recall that a pairing $V \otimes W \to k$ for a field k is *perfect* if it induces an isomorphism $V \to W^\vee$.

Proposition 4.1.10. *The vector space $Z(M)$ is always finite dimensional and the pairing* ev *is perfect.*

4.2 Classification (Benefit of the Funcorial Setting)

Example 4.2.1. Dimension 1 TQFTs:
$$Z : \mathbf{Cob}(1) \to \mathbf{Ch}\,.$$
Let P be a single point ald net $Q = \bar{P}$. Then $Z(P)^\vee = Z(Q)$. In general, an object in $\mathbf{Cob}(1)$ is

$$M := \left(\coprod_{S_+} P\right) \coprod \left(\coprod_{S_-} Q\right)$$

of points of positive and negative orientation and

$$Z(M) = \left(\bigotimes_{S_+} V(P)\right) \otimes \left(\bigotimes_{S_-} Z(P)^\vee\right).$$

Get a perfect pairing
$$Z(P) \otimes Z(P)^\vee \to \mathbb{R}$$
whose coevaluation is a morphism

$$\mathbb{R} \to \mathbb{Z}(P) \otimes Z(P)^\vee \cong \mathrm{End}(Z(P)).$$

Now note that $\mathbb{S}^1$ in $\mathbf{Cob}(1)$ is a map $\varnothing \to \varnothing$ and so we get that $Z(\mathbb{S}^1)$ determines a number $\lambda \in \mathbb{R}$. This number λ coming from $Z(\mathbb{S}^1)$ is the dimension $Z(P)$ and the composition

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\mathrm{coev}"'] \arrow[rr, bend left] & \mathrm{End}(Z(P)) \arrow[r, "\mathrm{ev}"'] & \mathbb{R} \end{tikzcd}$$

gives the multiplication by λ map. The take-away is that $Z(\mathbb{S}^1)$ is the dimension invariant.

Remark 4.2.2. Every $1d$ TQFT determines a funite dimensional vector space $Z(P)$ and conversely every finite dimensional vector space V determines a $1d$ TQFT.

Example 4.2.3. Two dimensional TQFTS:

$$Z : \mathbf{Cob}(2) \to \mathbf{Ch}$$

and $Z(\mathbb{S}^1)$ is an $\mathbb{E}_2$-algebra. Because our morphisms are pairs of pants, these pants give us our $\mathbb{E}_2$-algebra structure. The left-facing pill cap allows us to realize $B_1(0,0) : \mathbb{S}^1 \to \varnothing$ in $\mathbf{Cob}(2)$ and the right-facing pill cap gives a map $\varnothing \to \mathbb{S}^1$; the map $\mathbb{R} \to Z(\mathbb{S}^1)$ gives the identity of the algebra.

Now consider the pair of paints with two $\mathbb{S}^1$-boundaries on the domain side and one $\mathbb{S}^1$ on the codomain side and place pill caps on on each part of the pants, we get a space homotopic to $\mathbb{S}^2$. Putting a cap on only one leg of the pants gives a cylinder and the macaronis give a perfect pairing $Z(\mathbb{S}^1) \otimes Z(\mathbb{S}^1) \to \mathbb{R}$.

Definition 4.2.4. A *commutative Frobenius* $\mathbb{R}$-algebra is an $\mathbb{R}$-algebra A with a linear map $A \to \mathbb{R}$ and a nondegenerate pairing $A \otimes A \to \mathbb{R}$.

Consequently we see that a $2d$ TQFT determines a commutative Frobeneius algebra through $Z(\mathbb{S}^1)$. Consequently we induce:

Theorem 4.2.5. *Three is an equivalence of categories*

$$2dTQFTs \simeq \mathbf{CFrobAlg}.$$

Tomorrow: Try higher dimensions! Classify $3d$ and bigger TQFTs *and* get version of TQFT that accounts for line operators, surface operators, and so on. Also: holomorphic things and some fun recent stuff!

5 Lecture 5: 2023 May 5, Revenge of the Fifth

5.1 Review

Yesterday we say that for a TQFT over $\mathbb{R}^n$, observables are $\mathbb{E}_n$-algebras and the functorial version are functors $Z : \mathrm{Cob}(3) \to \mathbf{Vect}$. We classified them in dimensions 1 and 2 and today we will study higher dimensions. In dimension 3, objects of Cob(3) are surfaces like many-holed torii. Given such a surface we want to think of such a surface as the gluing of torii along some closed disk. So here is what we want:

A version of $\mathbf{Cob}(n)$ which knows about $(n-2)$-manifilds and a version of

$$Z : \mathbf{Cob}(n) \to \mathscr{C}$$

that assigns an ($\mathbb{E}_n$-monoidal) $(\infty, 1)$-category to $\mathbb{S}^{n-2}$ (like line operators). Solution: Use higher categories!

Definition 5.1.1. A *strict n-category* is a category $\mathscr{K}$ enriched over $(n-1)$-categories.

Remark 5.1.2. The enrichement definition implies in particular that $\mathscr{K}(X, Y)$ is an $(n-1)$-category.

Example 5.1.3. Let's consider the category $\mathbf{Vect}_2^{\mathbb{R}}$:

- 0-cells: $\mathbb{R}$-linear categories.
- Morphism categories: $\mathbf{Vect}_2^{\mathbb{R}}(E, D) = \mathbf{Func}^{\text{nice}}(E, D)$.

Example 5.1.4. Define $\mathbf{Cob}_2(n)$ as:

- Objects: Closed oriented manifolds of dimension $n-2$;
- Morphism categories: $\mathbf{Cob}_2(n)(M, N)$ = equivalence classes of cobordisms (and maps between them).

Example 5.1.5. For any $n \in \mathbb{N}$ define the n-category, $\mathbf{Cob}_n(n)$ whose objects are closed oriented manifolds of dimension 0 ... points. But there's a problem.

Remark 5.1.6 (ACHTUNG)**.** There are some issues with the strictness we're asking for! Requiring that things be bang-on *equal* to each other is very, very unnatrual — things should only be equal up to a homotopy and not on the nose.

Our solution is to move to (∞, n)-categories instead of strict n-categories.

Proposition 5.1.7. *There is an* (∞, n)*-category* $\mathsf{Cob}_n(n)$ *which does what we want.*

Definition 5.1.8. Let $\mathscr{C}$ be a symmetric monoidal (∞, n)-category. An extended n-dimensional TQFT is a symmetric monoidal functor

$$Z : \mathsf{Cob}_n(n) \to \mathscr{C} .$$

Remark 5.1.9. Think of Z as an assignment

Object	Assignment
n-manifolds	Something
$(n-1)$-manifolds	Some other thing
$\vdots$	$\vdots$
pt	Some other things

Example 5.1.10. Consider

Object	Assignment
n-manifolds	$\mathbb{C}$
$(n-1)$-manifolds	$\mathbf{Vect}_{\mathbb{C}}$
$\vdots$	$\vdots$
pt	$(\infty, n-1)$-categories with a linear over $\mathbb{C}$ condition

In this case $Z(\mathbb{S}^{n-2})$ is a linear category over $\mathbb{C}$ and the $\mathbb{E}_{n-1}$-monoidal structure comes from pairs of pants.

Here we can also decompose manifolds into smaller and smaller pieces. For instance, we can take a many-holed torus, pick out a circle, and then further decompose the circle into points. So in particular we cna decompose n-manifolds down to the level points.

What does this buy us? Well yesterday we say that the value of a point completely determined a value on a point.

Theorem 5.1.11 (Cobordism Hypothesis; cf. Baez-Dolan, Hopkins-Lurie, Lurie, Grady-Pavlov)**.** *Let* $\mathscr{C}$ *be a symmetric monoidal* (∞, n)*-category with duals/ Tjem tjere os am eqiova;emce omdiced ny* $Z \mapsto Z(*)$ *between*

$$\mathsf{Fun}^{\otimes}(\mathsf{Cob}_n(n), \mathscr{C}) \xrightarrow{\simeq} \mathscr{C}^{\text{f.d.}}$$

where $\mathscr{C}^{\text{f.d.}}$ *is the full sub-*(∞, n)*-symmetric monoidal category of fully dualizable objects.*

Remark 5.1.12. Given $Z(*)$, what is $Z(M)$? It is expected that there is a version of factorization homology for (∞, n)-categories so that

$$\int_M Z(*) = Z(M)$$

for all manifolds that Z can see. This is due to Ayala-Francis and is called β-factorization homology.

Remark 5.1.13. Could take $\mathscr{C}$ to be an appropriate Morita category Morita_{n-1}.

Example 5.1.14. The category Morita_1 is an $(\infty, 2)$-category whose 0-cells are $\mathbb{E}_1$-algebras, 1-cells are bimodules for $\mathbb{E}_1$-algebras, and 2-cells and higher are morphisms and equivalences between bimodule maps.

If we have a symmetric monoidal ∞-functor $Z : \mathsf{Cob}_n(n) \to \mathsf{Morita}_{n-1}$ then $Z(*)$ is a (special type[3] of) $\mathbb{E}_{n-1}$-algebra. Thus by the Cobordism Hypothesis, an n-dimensional TQFT with target Morita_{n-1} is determined by a (special type of) $\mathbb{E}_{n-1}$-algebra.

Before now: field theories as determined by observables giv rise to $\mathbb{E}_n$-algebras, yet now we have $\mathbb{E}_{n-1}$-algebras. Two questions:

1. How to go from $\mathbb{E}_{n-1}$ to $\mathbb{E}_n$?
2. How to go from $Z(*)$ to $Z(\mathbb{S}^{n-1})$?

5.2 Drinfeld Centres

Yesterday: Drinfeld centres of monoidal categories were braided monoidal categories. In the $\mathbb{E}_n$-language,

$$Z\left(\mathbb{E}_1\,\mathbf{Cat}\right) \simeq \mathbb{E}_2\,\mathbf{Cat}\,.$$

Moreover, there is a notion of Drinfeld centre for $\mathbb{E}_k$-algebras in (∞, n)-categories.

Theorem 5.2.1 (Deligne Conjecture; Lurie)**.** *The $\mathbb{E}_n$-Drinfeld Centre of an $\mathbb{E}_n$-category is an $\mathbb{E}_{n+1}$-category.*

Remark 5.2.2. This answers our first question: to go from $\mathbb{E}_n$ to $\mathbb{E}_{n+1}$, use Drinfeld Centres.[4]

Remark 5.2.3. We have

$$Z_{\mathbb{E}_n}(Z(*)) = Z(\mathbb{S}^{n-1}).$$

Example 5.2.4 (Ben-Zvi, Francis, Nadler)**.** If X is a perfect derived stack then

$$Z_{\mathbb{E}_n}\left(\mathbf{QCoh}(X)\right) \simeq \mathbf{QCoh}\left(\mathscr{L}^n(X)\right).$$

5.3 Holomorphic Field Theories

Recalll that a continuous function $f : \mathbb{C} \to \mathbb{C}$ is *holomorphic* if it is complex differentiable (complex smooth).

We want a notion of what it means to be a holomorphic field theory over spacetime $\mathbb{C}$. Before we say tha topological field theories give equivalences

$$\mathrm{Obs}^q(D_1) \xrightarrow{\simeq} \mathrm{OPbs}^q(D_2).$$

Definition 5.3.1. A factorization algebra $\mathscr{F}$ on $\mathbb{C}$ is *holomorphic translation invariant* if we have an equivalence

$$p_x : \mathscr{F}(U) \xrightarrow{\simeq} \mathscr{F}(\tau_x U)$$

where τ_x is a translation of U by a vector $x \in \mathbb{C}$ such that p_x is holomorphic in x and commute with the factorization algebra maps.

Theorem 5.3.2 (Costello-Gwilliam)**.** *A holomorphic translation invariant field theory has factorization algebr aobservables taht are holomorphically translation invariant. Moreover, holomorphically translation invariant factorization algebras (with an $\mathbb{S}^1$-action) determines a vertex algebra.*

Definition 5.3.3 (Ben-Zvi, Frenkel; cf. also Borcherds)**.** A vertex algebra over $\mathbb{C}$ consists of the following:

- A vector space V over $\mathbb{C}$ (the state space);
- A nonzero vector $|0\rangle \in V$ (the vacuum vector);
- A linear map $T : V \to V$ (the shift operator);

[3] Related to dualizability.
[4] Is there a way to go from $\mathbb{E}_n$ to $\mathbb{E}_\infty$?

- A linear map $y(-,z) : V \to \mathrm{End}(V)[\![z, z^{-1}]\!]$ (the state-space correspondence);

which satisfy some conditions omitted here.

Remark 5.3.4. The reference [Ben-Zvi, Frenkel] is the standard reference, but the original is due to Borcherds.

Idea. The main step is

$$V \cong \bigoplus_k \varinjlim_{r\in\mathbb{N}} H^\bullet\left(\mathcal{F}_k\, B_r(0)\right)$$

where $\mathcal{F}_k$ is the k-th eigenspace of the $\mathbb{S}^1$-action. □

Example 5.3.5. If X is a manifold, then Diff_X is an associative algebra (which is a 1d FT). If we work with a 2D FT we instead get chiral differential operators (which are vertex algebras) and looks like an enveloping algbera $U^{\mathrm{fact}}(\mathfrak{h}_u)$ which is the enveloping algebra, turned into a factorization algebra, of the Heisenberg Lie algebra.

Theorem 5.3.6 (Malikov-Schectman-Vautrb[5])**.** *Chiral differential operators determin Witten genus (expected by Worle an Witten). The relation to chromatic homotopy theory via Stolz-Teichner.*

The process of understanding a field theory by its observables gives us a process for turning something physical in nature to something algebraic. This lets us:

- Give a prcise definition of topological field theories
- understand quantization as deformations.

But also, there is:

5.4 Duality

In manifold theory, one of the greatest results is Poincaré duality and we have seen in the exercise nonAbelian Poincaré duality.

Question: are there duality notions for field theories and their observables?

Cool type of duality for algebras is called *Koszul duality.*

Definition 5.4.1. If A is an associatiative algebra then the Koszul dual of A is

$$\mathbb{D}(A) = \left(\mathbb{1} \overset{L}{\otimes} \mathbb{1}\right)^\vee$$

Remark 5.4.2. Note that

$$\mathbb{D}(A) = \left(\int_{\mathbb{D}^1} A\right)^\vee$$

Definition 5.4.3 (Ayala-Francis)**.** Let A be an $\mathbb{E}_n$-algebra. The Kozul Dual of A is

$$\mathbb{D}(A) = \left(\int_{\mathbb{D}^n} A\right)^\vee$$

Theorem 5.4.4 (Getzler-Jones, Lurie/Francis)**.** *The Kozul dual of an $\mathbb{E}_n$-algebra is an $\mathbb{E}_n$-algebra.*

Theorem 5.4.5 (Poincaré/Koszul Duality; cf. Ayala-Francis)**.** *Given an n-manifold M,*

$$\int_M A \simeq \int_{M^+} \mathbb{D}(A).$$

[5] I am sure there is a typo here, and I apologize.

The Koszul dual of objservables for a TQFT on $\mathbb{R}^n$ has the structure of observables for a TQFT on $\mathbb{R}^n$. Consequently we can ask is there a dual field theory Y to a field theory X for which

$$\mathbb{D}(\mathrm{Obs}_X) \simeq \mathrm{Obs}_Y ???$$

Conjecture 5.4.6 (Costello=Paquette, Costello-Li)**.** *Koszul duality for observables corresponds to AdS/CFT duality.*