FACTORIZATION ALGEBRAS: QUANTIZATION

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1. Quantization

Previously we talked about how classical observables form a commutative algebra, and quantum observables form a factorization algebra. Recall the definition of a factorization algebra.

Definition 1.1. A factorization algebra on M is a Weiss cosheaf $\mathcal{F}$ on M with equivalence

$$\mathcal{F}(U \sqcup V) \simeq \mathcal{F}(U) \otimes \mathcal{F}(V).$$

Today I want to start by talking about how to *quantize* a field theory.

To get at this question, we need an easy, workable example. Let $\mathbb{R}$ be our spacetime, and take as fields

$$\mathsf{Map}(\mathbb{R}, T^*\mathbb{R}).$$

Lastly, our action functional is going to be $S = 0$. From the exercises, we know that

- the critical locus EL of S has functions
$$\mathsf{Obs}^{\mathrm{cl}} = C^\infty(T^*\mathbb{R}),$$
- and the quantum observables are a factorization algebra on $\mathbb{R}$. Assume that Obs^q comes from an associative algebra A.

On the level of observables, we can rephrase our quantization question as:

Given a commutative algebra, how do we get a factorization algebra out of it?

The process of going from a commutative algebra to an associative algebra is called *deformation.*

Definition 1.2. Let T be a commutative algebra. A *deformation* of T is an associative algebra structure on $T[[\hbar]]$ such that

$$T[[\hbar]]/\hbar \simeq T$$

as algebras.

So this cannot be it. If this was all we asked for, we could always take

$$\mathsf{Obs}^q = \mathsf{Obs}^{\mathrm{cl}}[[\hbar]]$$

with the usual multiplication. That's not telling us anything about the field theory We need to ask for the deformation to encode more informaiton.

Example 1.3. In our running example, we have

$$\mathsf{Obs}^{\mathrm{cl}} = C^\infty(T^*\mathbb{R}) = \mathbb{R}[p, q].$$

We have an interesting structure on $T^*\mathbb{R}$: the symplectic form. On functions, the symplectic form gives a Poisson bracket,

$$\{p, q\} = 1.$$

Remark 1.4. In general, the derived critical locus EL of S has the structure of a (-1)-shifted symplectic stack, so $\mathcal{O}_{\mathrm{EL}}$ is a P_0-algebra.

We can ask for deformations of classical observables that respect this Poisson bracket. That is, a deformation $R = T[[\hbar]]$ so that

$$[f, g] = \hbar\{f, g\}$$

up to higher order terms in $\hbar$, for $f, g \in T$.

Example 1.5. The quantum observables of our example theory is the Weyl algebra

$$\mathsf{Obs}^q = \mathbb{R}[[\hbar]][p, q]$$

with multiplication so that $[p, q] = \hbar$.

More generally, we could have a theory with fields

$$\mathrm{Map}(\mathbb{R}, V)$$

where V was a symplectic manifold. Then $\mathcal{O}_{\mathrm{EL}}$ again has a Poisson bracket, and quantum observables are a deformation respecting this bracket.

In higher dimensions, we ask for the space of fields to have a shifted symplectic structure, and use this to deform the observables.

Thus in the language of factorization algebras, quantization is deformation.