FACTORIZATION ALGEBRAS: DAY 1 EXERCISES

ARAMINTA AMABEL

1. Exercises

There's a lot of exercises here of varying difficulty. You don't need to do all of them, or any of them completely. Just get a feel for things and have fun learning!

1.1. **Cosheaf Condition.**

A good reference for these questions is [**?**, Ch. 6 §1].

Recall that a functor

$$F\colon \mathsf{Open}(M) \to \mathcal{C}$$

is a *cosheaf* if for every open cover $\mathcal{U}$ of M, the diagram

$$\bigsqcup_{i,j} F(U_i \cap U_j) \rightrightarrows \bigsqcup_{k} F(U_k) \to F(M)$$

is a colimit diagram.

Definition 1.1. An open cover $\{U_j\}$ of M is a *Weiss* cover if for every finite set $x_1, \dots, x_k$ of points in M, there exists some U_j so that

$$\{x_1, \dots, x_k\} \subset U_j.$$

Question 1.2. Given an example of an open cover (in the usual topology) of $\mathbb{R}^2$ that is not a Weiss cover.

Question 1.3. Give an example of a Weiss cover that exists on any manifold M.

Definition 1.4. Let $\mathsf{Ran}(M)$ be the set of nonempty finite subsets of M.

Take on faith for a moment that the set $\mathsf{Ran}(M)$ has a reasonable topology.

Question 1.5. Show that a Weiss cover for M determines an ordinary cover for $\mathsf{Ran}(M)$. Conversely, show that an ordinary cover for $\mathsf{Ran}(M)$ determines an Weiss cover on M.

Ignore topology issues for this, just show it on the level of sets.

It turns out that $\mathsf{Ran}(M)$ has a topology so that cosheaves on $\mathsf{Ran}(M)$ are the same as cosheaves on M for the Weiss topology.

1.2. **Examples.**

Question 1.6. Given a field theory with space time M, classical observables are a commutative algebra. Show that classical observables form a factorization algebra on M. In fact, any commutative algebra forms a factorization algebra on M.

Question 1.7. Show that an associative algebra forms a factorization algebra.

Note that I did not tell you what space on which it should be a factorization algebra; this is part of the exercise. Call this space X.

Question 1.8. Let X be the space on which associative algebras determine a factorization algebra. When is a factorization algebra $\mathcal{F}$ on X determined by an associative algebra?

Question 1.9. Let $\mathcal{F}$ be a factorization algebra on $[0,1]$ so that

$$\mathcal{F}|_{(0,1)}$$

comes from an associative algebra. Describe the structure of $\mathcal{F}$ over $[0,1]$ in more familiar terms.

1.3. **Critical Locus.** Recall that the classical observables are functions on the critical locus,

$$\mathsf{Obs}^{\mathrm{cl}} = \mathcal{O}_{\mathrm{EL}}$$

where

$$\mathrm{EL} \subset \mathsf{Map}(M, X)$$

is that the critical locus of the action functional

$$S \colon \mathsf{Map}(M, X) \to \mathbb{R}.$$

For convenience, let's pretend the mapping space $\mathrm{Map}(M, X)$ (the fields) is a smooth manifold Y.

Question 1.10. Let $\Gamma(dS) \subset T^*Y$ denote the graph of dS. Show that the critical locus of S is the intersection

$$\Gamma(dS) \cap \mathrm{Zero}_Y$$

where Zero_M is the zero-section of Y in T^*Y.

Question 1.11. What is EL is $S = 0$?

In reality, we want EL to be a fancier version of the critical locus: the *derived* critical locus. The derived critical locus of $S \colon Y \to \mathbb{R}$ is a dg space on Y, so it is determined by its functions (which are a chain complex).

Definition 1.12. The derived critical locus of $S \colon Y \to \mathbb{R}$ has functions the derived tensor product

$$C^\infty(\Gamma(dS)) \bigotimes^{\mathbb{L}}_{C^\infty(T^*Y)} C^\infty(Y).$$

Question 1.13. What is the derived critical locus of $S = 0$?

Question 1.14. What is the derived critical locus of general S in terms of S?

References

[CG17] Kevin Costello and Owen Gwilliam. *Factorization algebras in quantum field theory. Vol. 1*, volume 31 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2017.