FACTORIZATION ALGEBRAS: DAY 1

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1. Plan

I am supposed to teach you about factorization algebras. There's a lot to say about these algebraic gadgets, so I won't come close to covering everything. I hope you come away with three skills:

(1) knowing what a factorization algebra *is* (definition and examples),
(2) being able to relate them to other notions of field theories, and
(3) a curiosity for learning more about the subject.

The five lectures will roughly cover the following themes:

Day 1. physical motivation
Day 2. topological field theories
Day 3. functorial perspective
Day 4. holomorphic field theories
Day 5. applications

For today, we have the following goal.

Goal 1.1. Discover "factorization algebras" in nature.

The quotes are because I haven't told you the definition of factorization algebras yet, and part of the goal is to stumble upon the definition ourselves. Nature means physics here.

2. Classical Story

If you have not heard of a field theory before, the picture I want you to have in mind is the following. Imagine we have some container X (a manifold or more general space) and a particle moving around in X.

For $I \subset \mathbb{R}$ a time interval, the space of maps

$$\mathrm{Map}(I, X)$$

is all the paths a particle could take. So $t \in I$ maps to to the position of the particle at time t.

This system of a particle in X might be subject to constraints in the real world.

Example 2.1. Only paths of least energy.

Example 2.2. $X = \mathbb{R}^2$ and only straight line paths.

Example 2.3. $X = S^2$ the sphere, and only great circle paths.

Remark 2.4. These are examples of the general case where paths are geodesics.

The physical constraints on what paths a particle can take are called the *equations of motion* or the *Euler-Lagrange equations.* These can be written as a PDE, and they determine a map

$$S\colon \mathrm{Map}(I, X) \to \mathbb{R}$$

called the *action functional*. The allowable paths are then

$$\mathrm{EL} \subset \mathrm{Map}(I, X)$$

the subset of paths so that

$$\mathrm{EL} = \{f\colon I \to X : (dS)(f) = 0\}.$$

This is the *critical locus* of S.

More generally, we might be interested in how the particle moves around over some time interval I and as we move around some parameter space N. Then $N \times I = M$ is "spacetime" and we are interested in parameterized paths

$$\mathsf{Map}(M, X).$$

Definition 2.5. A *classical field theory* is a space of fields $\mathsf{Map}(M, X)$ and a set of equations of motion that can be encoded by the critical locus of a map

$$S\colon \mathsf{Map}(M, X) \to \mathbb{R}.$$

The *dimension* of the field theory is the dimension of M.

Remark 2.6. We already have several equivalent descriptions of a given classical field theory. One could hand you the space of fields and the action functional, or the equations of motion.

Question 2.7. What if someone just told you the critical locus EL? Would that be equivalent data to the whole field theory?

Example 2.8 (Classical mechanics, massless free theory)**.** Say $I = [a, b]$. Take fields

$$\mathsf{Map}(I, \mathbb{R}^n).$$

The action functional sends a field f to

$$S(f) = \int_a^b \langle f(t), \frac{d^2}{dt^2} f(t)\rangle dt.$$

Here, we are taking the inner product on $\mathbb{R}^n$. In this case, the critical locus is straight lines,

$$\mathrm{EL} = \{\text{straight lines}\}.$$

Question 2.9. Can you extend this notion to a massless free theory on a general n-manifold M? What extra structure will you need M to have?

Example 2.10 (Gauge Theory)**.** Given a Lie group G and a spacetime M, G gauge theory on M which I'll denote Gauge_G^M has fields

$$\mathsf{Map}(M, BG) = \mathsf{Bun}_G^\nabla M.$$

There are many different choices for action functionals, giving different gauge theories.

Example 2.11 (Yang-Mills)**.** One type of gauge theory Gauge_G^M is Yang-Mills. A field is then a principal G-bundle P on M with connection A. The action functional is

$$S(A) = -\frac{1}{2}\int_M \mathrm{tr}(dA \wedge *dA).$$

Here $*$ is the Hodge star operator, and tr is an invariant scalar product on $\mathrm{Lie}(G)$. If we assume G is semi-simple, we can use the Killing form for this.

This is particularly studied when $G = SU(n)$ and M has dimension 4.

We can alter this action functional to produce new theories by adding another term in the integrand.

$$S(A) = -\frac{1}{2}\int_M \mathrm{tr}(dA \wedge *dA) + \theta \int_M \mathrm{tr}(dA \wedge dA)$$

This additional term is called the "theta term."

Remark 2.12. The action functional S must be local: it must arise as the integral over M of some polynomial in a field ϕ and its derivatives.

Question 2.13. Can you show that the examples of action functionals given above are local?

Remark 2.14. The space of fields does not have to be a mapping space. When it is, it is referred to as a sigma model.

3. Quantization

Knowing a particle is moving in a box is good, but we want to know more.

Example 3.1. Say you hand a little kid a box containing a spider (i.e. a scary particle). They look up at you really scared. You tell them,

> Don't worry, I can tell you what type of paths the spider can take. It only moves along straight lines!

They're still scared. And they have questions.

- Where in the box is the spider right now?
- How fast is the spider moving?
- Is the box open?
- Are there holes in the box?
- Can the spider get out of the box?

Remark 3.2. Notice that these questions are of three different types. The first two are about the particle and the equations of motion. They can maybe be answered locally.

The third and fourth questions are about the topology of the spacetime. That's global information.

Answering the last question requires knowledge about the whole field theory. And this is the most important question.

These questions are types of *measurements* one can make on the system. In classical field theory, one can (in theory) answer all of these questions.

This changes in quantum field theory!

Theorem 3.3 (Heisenberg Uncertainty Principle)**.** *In quantum field theory, one cannot precisely know both the position and the momentum of a particle at the same time.*

Goal 3.4. Formulate this uncertainty mathematically.

3.1. Observables.

Definition 3.5. Given a classical field theory

$$\mathrm{EL} \subset \mathrm{Maps}(I, X),$$

the classical observables are the real functions on the critical locus,

$$\mathrm{Obs}^{\mathrm{cl}} = \mathcal{O}_{\mathrm{EL}}.$$

So an observable takes in an allowed path and spits out a real number, the measurement on that path.

Remark 3.6. I did not say what kind of functions we are taking. This will depend on the context. We defined EL to be the critical locus of the action functional S. Since S is local, it looks like the integral of a polynomials in derivatives. Thus

$$\{dS = 0\}$$

has the same flavor of a space built from zeroes of a polynomial. It lives in the world of algebraic geometry. Thus, functions on EL are going to be something like algebraic functions. When things are nice, holomorphic functions also makes sense.

The vector space of functions

$$\mathrm{Obs}^{\mathrm{cl}} = \mathcal{O}_{\mathrm{EL}}$$

form a commutative algebra;

$$f, g \colon \mathrm{EL} \to \mathbb{R}$$

combine to give

$$(fg)(u) = f(u)g(u).$$

Physically, given two measurements f, g, we can preform them at the same time to get a new measurement fg.

Note: This is exactly what fails in the quantum world!

There, we *cannot* take two measurements (like position and momentum) simultaneously.

Ok, what can we do? Say $I = [a, b]$ is our spacetime. Let f, g be observables. We can form a new measurement on I that on (t_1, t_2) does f, on (t_3, t_4) does g and does nothing in between. (Nothing means the measurement sending every path to zero.)

So we have a way of combining observables on disjoint open intervals. (Open here because we want a continuous map.)

More generally, say we were looking at a spacetime

$$M = I \times N$$

where N has dimension n. We can preform different measurments on disjoint disks. Given two embeddings

$$\mathbb{D}^{n+1} \simeq I \times \mathbb{D}^n \to I \times N$$

we do f on one and g on the other.

Upshot. Whatever quantum field theory is, the set of measurements (or observables) one can make on it have a weird multiplication structure controlled by disjoint open subsets of spacetime.

Question 3.7. What should we call this new algebraic gadget? What type of algebra?

4. Factorization Algebras

Definition 4.1. Let M be a manifold. A *factorization algebra on M* is a functor

$$\mathcal{F} \colon \mathsf{Open}(M) \to \mathsf{Ch}$$

together with isomorphisms

$$\mathcal{F}(U \sqcup V) \simeq \mathcal{F}(U) \otimes \mathcal{F}(V)$$

for disjoint unions of opens $U, V \subset M$, such that $\mathcal{F}$ satisfies a (Weiss) cosheaf condition.

Let's break that all down.

A factorization algebra is the data of:

- for an inclusion of disjoint disks
$$\coprod_{i=1}^{k} D_i \hookrightarrow M,$$
a vector space
$$\mathcal{F}\left(\coprod_{i=1}^{k} D_i\right)$$
and an isomorphism
$$\mathcal{F}\left(\coprod_{i=1}^{k} D_i\right) \simeq \bigotimes_{i=1}^{k} \mathcal{F}(D_i),$$
and
- for an inclusion
$$\coprod_{i=1}^{k} D_i \subset D \hookrightarrow M,$$
a map
$$\mathcal{F}\left(\coprod_{i=1}^{k} D_i\right) \to \mathcal{F}(D).$$

The weird multiplication comes from putting this data together:
$$\bigotimes_{i=1}^{k} \mathcal{F}(D_i) \simeq \mathcal{F}\left(\coprod_{i=1}^{k} D_i\right) \to \mathcal{F}(D)$$
tells us how to multiply observables on the disjoin disks D_i.

The last piece was the cosheaf condition. This is going to be on the exercises.

Theorem 4.2 (Costello-Gwilliam)**.** *The quantum observables Obs^q of a field theory on spacetime M has the structure of a factorization algebra on M.*

The factorization algebra structure on Obs^q means we get a vector space
$$\mathsf{Obs}^q(U)$$
for every open subset U of spacetime. These are the *local* observables at U. When we were discussing classical observables, we had a single vector space
$$\mathsf{Obs}^{\mathrm{cl}} = \mathrm{Hom}(\mathrm{EL}, \mathbb{R}).$$
One could ask how to get a single object $\mathsf{Obs}^q(M)$ from the factorization algebra Obs^q. This should be some sort of "global" observables. Since Obs^q is a type of cosheaf, we can take its global sections.

Definition 4.3. Let $\mathcal{F}$ be a factorization algebra on M. The *factorization homology* of $\mathcal{F}$ is the global sections
$$\int_M \mathcal{F} = \mathcal{F}(M).$$

References

[CG17] Kevin Costello and Owen Gwilliam. *Factorization algebras in quantum field theory. Vol. 1*, volume 31 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2017.