FACTORIZATION ALGEBRAS: DAY 2

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0.1. **Exercises Review.** The following was on yesterday's problem set:

Question 0.1. Given a field theory with space time M, classical observables are a commutative algebra. Show that classical observables form a factorization algebra on M. In fact, any commutative algebra forms a factorization algebra on M.

Let's go over the solution. Let A be a commutative algebra (say in chain complexes). The constant precosheaf defines a functor

$$F_1 \colon \mathsf{Open}(M) \to \mathsf{Ch}$$

sending every open set to A. We can view this as a precosheaf valued in algebras in chain complexes,

$$F_2 \colon \mathsf{Open}(M) \to \mathsf{Comm}(\mathsf{Ch}).$$

We can cosheafify F_2 using the Weiss topology to get a Weiss cosheaf

$$F_3 \colon \mathsf{Open}(M) \to \mathsf{Comm}(\mathsf{Ch}).$$

Since coproducts and tensor products are the same in algebras in chain complexes, we have

$$F_3(U \sqcup V) \simeq F_3(U) \otimes F_3(V).$$

Forgetting back down, we have a factorization algebra

$$F_4 \colon \mathsf{Open}(M) \to \mathsf{Ch}.$$

The functor F_4 is still a Weiss cosheaf since the forgetful functor

$$\mathsf{Comm}(\mathsf{Ch}) \to \mathsf{Ch}$$

preserves reflexive coequalizers.

1. TQFTs

Yesterday we talked about how classical observables form a commutative algebra, and quantum observables form a factorization algebra. Recall the definition of a factorization algebra.

Definition 1.1. A factorization algebra on M is a Weiss cosheaf $\mathcal{F}$ on M with equivalence

$$\mathcal{F}(U \sqcup V) \simeq \mathcal{F}(U) \otimes \mathcal{F}(V).$$

Today I want to talk about how this structure behaves when we have a *topological* field theory.

Informally, a field theory is topological if it does not depend on a metric. Let's investigate what this means for quantum observables. Let M be our spacetime. Then if our theory is topological the cosheaf

$$\mathsf{Obs}^q \colon \mathsf{Open}(M) \to \mathcal{C}$$

cannot know about measurements of size or distance. For example, consider the value $\mathsf{Obs}^q(B_r(0))$ on a ball of radius r. centered at the origin. Since Obs^q does not know about size, it cannot distinguish between balls of different radii. Thus

$$\mathsf{Obs}^q(B_r(0)) \xrightarrow{\sim} \mathsf{Obs}^q(B_{r'}(0))$$

for $r < r'$. Moreover, it does not know distance from the origin. So if we move $B_r(0)$ around inside $B_{r'}(0)$, that will not effect the answer either.

Definition 1.2. A factorization algebra $\mathcal{F}$ on M is *locally constant* if for any inclusion of disks

$$D_1 \subset D_2$$

in M, the induced map

$$\mathcal{F}(D_1) \to \mathcal{F}(D_2)$$

is an equivalence.

We are going to use this as a definition.

Definition 1.3. A field theory is *topological* if Obs^q is locally constant.

Example 1.4. A locally constant factorization algebra on $\mathbb{R}$ is an associative algebra.

Example 1.5. A locally constant factorization algebra on $\mathbb{R}^\infty$ is a commutative algebra.

Example 1.6. The data of a locally constant factorization algebra on $\mathbb{R}^2$ is a vector space V with an S^1-family of multiplications.

Example 1.7. The data of a locally constant factorization algebra on $\mathbb{R}^2$ is a vector space V with many types of multiplications and coherencies.

Locally constant factorization algebras on euclidean spaces give a family of algebra structures starting from associative, and becoming more commutative as the dimension increases.

To precisely describe this structure, we will use the language of operads.

2. Operads

Let $\mathsf{Fin}^{\mathrm{bij}}$ be the category of finite sets and bijections. The category of *symmetric sequences in* Sp is the functor category

$$\mathsf{Sseq}(\mathsf{Spaces}) := \mathsf{Fun}(\mathsf{Fin}^{\mathrm{bij}}, \mathsf{Spaces}).$$

Symmetric sequences can be given the structure of a monoidal category as follows. The composition product $R \circ S$ of two symmetric sequences is

$$(R \circ S)(n) = \bigoplus_i R(i) \otimes_{\Sigma_i} \left(\bigotimes_{j_1 + \cdots + j_i = n} (S(j_1) \otimes \cdots \otimes S(j_i)) \times_{\Sigma_{j_1} \times \cdots \times \Sigma_{j_i}} \Sigma_n \right).$$

The unit of the composition product, denoted $\mathcal{O}_{\mathsf{triv}}$, sends a finite set B to the unit $\mathbb{1}_{\mathsf{Spaces}}$ of Spaces if $|B| = 1$ and to the zero object $*$ of Spaces otherwise.

Definition 2.1. An *operad* in Spaces is a monoid object in $\mathsf{Sseq}(\mathsf{Spaces})$.

An operad $\mathcal{O}$ in spaces has an underlying functor $\mathsf{Fin}^{\mathrm{bij}} \to \mathsf{Spaces}$. For each $i \in \mathbb{N}$, we denote by $\mathcal{O}(i)$ the image of the finite set with i elements $[i]$ under this functor.

Example 2.2 (Little n-Disks Operad)**.** Define an operad $\mathbb{E}_n$ in spaces by

$$\mathbb{E}_n(k) = \mathrm{Conf}_k(\mathbb{R}^n),$$

the configuration space of k distinct points in $\mathbb{R}^n$, topologized as a subset of $(\mathbb{R}^n)^k$. The easiest way to see the product

$$\mathbb{E}_n \circ \mathbb{E}_n \to \mathbb{E}_n$$

is to consider each point in $\mathbb{R}^n$ as a little open disk centered at that point. Then the product is just the inclusion of disks.

We think of the space $\mathcal{O}(k)$ as parameterizing k-ary operations for some type of algebraic structure.

Definition 2.3. An $\mathcal{O}$-algebra in $\mathcal{C}$ is an object $V \in \mathcal{C}$ together with maps

$$\mathcal{O}(k) \otimes V^{\otimes k} \to V$$

for every k, compatible with the multiplication maps for $\mathcal{O}$.

Theorem 2.4 (Lurie)**.** *Locally constant factorization algebras on $\mathbb{R}^n$ are the same as $\mathbb{E}_n$-algebras.*

3. Factorization Homology

Recall that for a factorization algebra $\mathcal{F}$ on M, we had a notion of *factorization homology* given by global sections

$$\int_M \mathcal{F} = \mathcal{F}(M).$$

We want to investigate this local to global structure in the locally constant case. For this, note that there is an equivalence

$$\mathsf{Emb}^{\mathrm{fr}}\left(\bigsqcup_k \mathbb{R}^n, \mathbb{R}^n\right) = \mathsf{Conf}_k(\mathbb{R}^n)$$

which records the center of the disks.

We are going to construct a notion of factorization homology of an $\mathbb{E}_n$-algebra A over a framed n-manifold M;

$$\int_M A.$$