FACTORIZATION ALGEBRAS: DAY 3 EXERCISES

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1. Exercises

1.1. Warm Up.

Question 1.1. What is the factorization homology

$$\int_{[0,1]} A$$

of an associative algebra A?

Question 1.2. What can you say about $\int_{S^2} A$ for A a 2-disk algebra?

Hint: Use excision.

Question 1.3. Make sense of the tensor product in excision. How are things modules, algebras, and such?

1.2. Poincaré Duality for Factorization Homology.

Yesterday, you saw that $\Omega^n X$ was an $\mathbb{E}_n$-algebra.

Question 1.4. Show that $C_\bullet(\Omega^n X)$ is an $\mathbb{E}_n$-algebra. Write $C_\bullet(\Omega^n X)$ and $\Omega^n X$ as n-disk algebras (valued in chain complexes and spaces, respectively). Write them as factorization algebras as well.

Note that

$$\Omega^n X = \mathsf{Map}_c(\mathbb{R}^n, X).$$

Question 1.5. Let X be an n-connective space. Convince yourself that compactly supported maps

$$\mathsf{Map}_c(-, X)$$

satisfies $\otimes$-excision. That is, given a collar glueing

$$M \simeq U \bigcup_{V\times\mathbb{R}} U',$$

there is an equivalence

$$\mathsf{Map}_c(U, X) \times_{\mathsf{Map}_c(V\times\mathbb{R}, X)} \mathsf{Map}_c(U', X) \simeq \mathsf{Map}_c(M, X).$$

You can do this by just skimming the proof given in [AF15, Lem. 4.5], if you want; or by trying it out in a few easier cases.

The following is a theorem of Salvatore, Segal, and Lurie, in various contexts. We are following the proof of Ayala-Francis.

Question 1.6 (Nonabelian Poincaré Duality)**.** Let X be an n-connective space. Show that

$$\int_M \Omega^n X \simeq \mathsf{Map}_c(M, X).$$

Say M is compact. From Debray's talks and exercises, if X an Eilenberg-MacLane space, the right-hand side looks like cohomology. This motivates the relationship between nonabelian Poincaré duality and usual Poincaré duality.

1.3. **Enveloping Algebras.** Let $\mathfrak{g}$ be a Lie algebra. Recall the Chevalley-Eilenberg complex $C_\bullet^{\mathrm{Lie}}(\mathfrak{g})$ from yesterday.

Question 1.7. Define a Lie algebra structure on

$$\mathsf{Map}_c(\mathbb{R}^n, \mathfrak{g}).$$

Show that

$$C_\bullet^{\mathrm{Lie}}(\mathsf{Map}_c(\mathbb{R}^n, \mathfrak{g})$$

forms an n-disk algebra. Call it $U_n\mathfrak{g}$.

Question 1.8. Show that $U_1\mathfrak{g}$ is the enveloping algebra $U\mathfrak{g}$.' Check this with your understanding of $U\mathfrak{g}$ as an $\mathbb{E}_1$-algebra from yesterday.

Thus $U_n\mathfrak{g}$ gives us a version of the enveloping algebra in higher dimensions -a "higher enveloping algebra" This is a key example in field theory. Many field theories have observables that look similarl to a higher enveloping algebra construction. For example, any free theory has this property.

Question 1.9. Compute

$$\int_M U_n\mathfrak{g}.$$

Hint: use both the fact that $C_\bullet^{\mathrm{Lie}}$ commutes with factorization homology and nonabelian Poincaré duality.

1.4. **Spare Questions.**

Question 1.10. Show that $\int_M A$ has a canonical action of $\mathsf{Diff}(M)$.

Question 1.11. Let $H \subset \mathrm{GL}(n)$ be a sub-Lie-group. You can think $SO(n)$ if you want. Define a notion of an H-oriented TFT in the functorial setting.

(1) Can you define a notion of an H-oriented $\mathbb{E}_n$-algebra?
(2) How about an H-oriented factorization algebra?

References

[AF15] David Ayala and John Francis. Factorization homology of topological manifolds. *J. Topol.*, 8(4):1045–1084, 2015.