FACTORIZATION ALGEBRAS: DAY 3

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1. Factorization Homology

1.1. Disk Categories.

Definition 1.1. Let $\mathcal{M}\mathsf{fld}_n$ be the ∞-category of n-manifolds. This has objects n-manifolds and morphisms smooth embeddings.

Let $\mathcal{D}\mathsf{isk}_n \subset \mathcal{M}\mathsf{fld}_n$ be the full ∞-subcategory consisting of manifolds isomorphic to finite disjoint unions of Euclidean spaces.

Remark 1.2. If you are pretending these are just topological categories, the mapping spaces $\mathsf{Emb}(M, N)$ are given the compact-open topology.

Remark 1.3. We consider the empty set to be an object of $\mathcal{M}\mathsf{fld}_n$ for every n.

Remark 1.4. There are several other versions of disk categories (and categories of manifolds) that we will consider during this seminar.

- Framed disks: $\mathcal{D}\mathsf{isk}_n^{\mathrm{fr}}$ will have the same objects of $\mathcal{D}\mathsf{isk}_n$ but with framed embeddings as morphisms. A framed embedding is an embedding $M \to N$ so that the given framing on M and the pulled back framing commute up to a chosen homotopy.

Note that $\mathcal{D}\mathsf{isk}_n$ is a symmetric monoidal ∞-category with tensor product given by disjoint union. Given an n-manifold M, let $\mathcal{D}\mathsf{isk}_{n/M}$ denote the over category. Objects of $\mathcal{D}\mathsf{isk}_{n/M}$ are embeddings $U \hookrightarrow M$ so that U is isomorphic to a finite disjoint union $\sqcup \mathbb{R}^n$ of Euclidean spaces. Morphisms in this category are triangles

$$\begin{array}{ccc} U & \longrightarrow & M \\ \downarrow & \nearrow & \\ V & & \end{array}$$

that commute up to a chosen isotopy. The over category $\mathcal{D}\mathsf{isk}_{n/M}$ comes with a forgetful functor

$$\mathcal{D}\mathsf{isk}_{n/M} \to \mathcal{D}\mathsf{isk}_n$$

Definition 1.5. An n-disk algebra A with values in a symmetric monoidal ∞-category $\mathcal{V}$ is a symmetric monoidal functor

$$A : \mathcal{D}\mathsf{isk}_n \to \mathcal{V}$$

Let $\mathsf{Alg}_n(\mathcal{V})$ denote the category of n-disk algebras.

1.2. $\mathcal{E}_n$-algebras.

Relationship between $\mathcal{E}_n$-algebras and n-disk algebras. If we redo everything above with framed manifolds, framed embeddings, and such, then a *framed* n-disk algebra is the same as an $\mathcal{E}_n$-algebra. The equivalence goes as follows. Given a framed n-disk algebra A with values in $\mathcal{V}$, define an $\mathcal{E}_n$-algebra in $\mathcal{V}$ by $A(\mathbb{R}^n)$ and action

$$\mathsf{Emb}(\coprod_I \mathbb{R}^n, \mathbb{R}^n) \otimes A(\mathbb{R}^n)^{\otimes I} \to A(\mathbb{R}^n)$$

by identifying $A(\mathbb{R}^n)^{\otimes I} \simeq A(\coprod_I \mathbb{R}^n)$ and applying the given embedding.

More precisely, there is an equivalence of categories

$$\mathsf{Alg}_{\mathcal{D}\mathsf{isk}_n^{\mathrm{fr}}}(\mathcal{V}) \cong \mathsf{Alg}_{\mathcal{E}_n}(\mathcal{V})$$

Definition 1.6. Let M be an n-manifold and A an n-disk algebra valued in $\mathcal{V}$. The *factorization homology of M with coefficients in A* is the homotopy colimit

$$\mathsf{colim}\left(\mathcal{D}\mathsf{isk}_{n/M} \to \mathcal{D}\mathsf{isk}_n \xrightarrow{A} \mathcal{V}\right)$$

1.3. **Homology Theories for Manifolds.** Factorization homology satisfy a version, more suited to manifolds, of the Eilenberg-Steenrod axioms for homology theories. The main axiom of such theories is called "$\otimes$-excision."

Definition 1.7. A symmetric monoidal functor

$$F : \mathsf{Mfld}_n \to \mathcal{C}\mathsf{h}_{\mathbb{Q}}$$

satisfies $\otimes$-exision if, for every collar-gluing $U \bigcup_{V\times\mathbb{R}} U' \simeq W$, the canonical morphism

$$F(U) \bigotimes_{F(V\times\mathbb{R})} F(U') \to F(W)$$

is an equivalence.

Here $F(V \times \mathbb{R})$ inherits an $\mathcal{E}_1$-algebra structure from the copy of $\mathbb{R}^1$,

$$\mathsf{Emb}^{\mathrm{fr}}(\coprod_I \mathbb{R}, \mathbb{R}) \otimes F(V \times \mathbb{R})^{\otimes I} \simeq \mathsf{Emb}^{\mathrm{fr}}(\coprod_I \mathbb{R}, \mathbb{R}) \otimes F(V \times (\coprod_I \mathbb{R})) \to F(V \times \mathbb{R})$$

The tensor product

$$F(U) \bigotimes_{F(V\times\mathbb{R})} F(U') \to F(W)$$

is then the tensor product in modules over the $\mathcal{E}_1$-algebra $F(V \times \mathbb{R})$. One reason we have restricted to collar-gluings is so that this tensor product makes sense.

Definition 1.8. The ∞-category of *homology theories for n-manifolds* valued in $\mathcal{C}\mathsf{h}_{\mathbb{Q}}$ is the full ∞-subcategory

$$\mathbb{H}(\mathcal{M}\mathsf{fld}_n, \mathcal{C}\mathsf{h}_{\mathbb{Q}}) \subset \mathsf{Fun}^{\otimes}(\mathcal{M}\mathsf{fld}_n, \mathcal{C}\mathsf{h}_{\mathbb{Q}})$$

of symmetric monoidal functors that satisfy $\otimes$-excision.

Not only is factorization homology a homology theory for n-manifolds, it also is the only such thing.

Theorem 1.9 (Ayala-Francis)**.** *There is an equivalence*

$$\int : \mathsf{Alg}_n(\mathcal{C}\mathsf{h}_{\mathbb{Q}}) \leftrightarrows \mathbb{H}(\mathcal{M}\mathsf{fld}_n, \mathcal{C}\mathsf{h}_{\mathbb{Q}}) : \mathrm{ev}_{\mathbb{R}^n}$$

Remark 1.10. One can replace $\mathcal{C}\mathsf{h}_{\mathbb{Q}}$ with a general symmetric monoidal ∞-category $\mathcal{V}$ as long as $\mathcal{V}$ is "$\otimes$-presentable. For details, see [AF15].

1.4. **Examples.** We compute factorization homology $\int_M A$ for simple choices of M and A.

Example 1.11. Take $M = \mathbb{R}^n$. Then $\mathcal{D}\mathsf{isk}_{n/\mathbb{R}^n}$ has a final object given by the identity map $\mathbb{R}^n = \mathbb{R}^n$. Thus the colimit is given by evaluation on $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n} A = \mathsf{colim}\left(\mathcal{D}\mathsf{isk}_{n/\mathbb{R}^n} \to \mathcal{D}\mathsf{isk}_n \xrightarrow{A} \mathcal{V}\right) = A(\mathbb{R}^n)$$

Example 1.12. Take $M = \coprod_I \mathbb{R}^n$. Then $\mathcal{D}\mathsf{isk}_{n/M}$ again has a final object and as above we obtain

$$\int_{\coprod_I \mathbb{R}^n} A \simeq A(\coprod_I \mathbb{R}^n) \cong A(\mathbb{R}^n)^{\otimes I}$$

Here we are seeing the fact that

$$\int_{(-)} A : \mathcal{M}\mathsf{fld}_n \to \mathcal{C}\mathsf{h}_{\mathbb{Q}}$$

is a symmetric monoidal functor.

Example 1.13. Take $M = S^1$, as a framed manifold. Note that an $\mathcal{E}_1$-algebra A is the same as an associative algebra $\bar{A} := A(\mathbb{R}^1)$. We will use excision to compute $\int_{S^1} A$. Express S^1 as a collar-gluing

$$S^1 \cong \mathbb{R} \cup_{S^0 \times \mathbb{R}} \mathbb{R}$$

By $\otimes$-excision, we have

$$\int_{S^1} A \simeq \left(\int_{\mathbb{R}} A\right) \bigotimes_{\left(\int_{S^0 \times \mathbb{R}} A\right)} \left(\int_{\mathbb{R}} A\right) \simeq \bar{A} \bigotimes_{\bar{A} \otimes \bar{A}^{\mathrm{op}}} \bar{A}$$

where we obtained $\bar{A} \otimes \bar{A}^{\mathrm{op}}$ because the two copies of $\mathbb{R}^1$ in $S^0 \times \mathbb{R}^1 \subset S^1$ are oriented differently. This is the *Hochschild homology* of A.

We have a functor $\mathsf{Alg}_n(\mathcal{V}) \to \mathcal{V}$ given by evaluating on $\mathbb{R}^n$. This is the "forgetful functor."

Definition 1.14. The left adjoint to the forgetful functor is the free functor

$$\mathbb{F}_n \colon \mathcal{V} \to \mathsf{Alg}_n(\mathcal{V})$$

Example 1.15. Consider the free n-disk algebra on $V \in \mathcal{C}\mathsf{h}_{\mathbb{Q}}$. This sends a disjoint union $\coprod_k \mathbb{R}^n$ to

$$\bigoplus_{0 \leq i} C_*(\mathsf{Emb}(\coprod_i \mathbb{R}^n, \coprod_k \mathbb{R}^n)) \bigotimes_{\Sigma_i} V^{\otimes i}$$

Example 1.16. We can similarly define a free framed n-disk algebra. In the framed case,

$$\mathbb{F}_n V(\mathbb{R}^n) = \bigoplus_{i \geq 0} C_* \mathsf{Emb}^{\mathrm{fr}}(\coprod_i \mathbb{R}^n, \mathbb{R}^n) \bigotimes_{\Sigma_i} V^{\otimes i}$$

Since $\mathsf{Emb}^{\mathrm{fr}}(\coprod_i \mathbb{R}^n, \mathbb{R}^n) \simeq \mathsf{Conf}_i(\mathbb{R}^n)$, this agrees with the free $\mathcal{E}_n$-algebra on V.

Proposition 1.17. *For M a framed manifold, and $V \in \mathcal{C}\mathsf{h}_{\mathbb{Q}}$, we have*

$$\int_M \mathbb{F}_n V \simeq \bigoplus_{0 \leq i} C_*(\mathsf{Conf}_i M) \bigotimes_{\Sigma_i} V^{\otimes i}$$

A similar statement is true in the non-framed case, we're just being lazy.

For $U \cong \coprod_I \mathbb{R}^n$ we have

$$\mathbb{F}_n V(U) = \left(\bigoplus_{i \geq 0} C_*(\mathsf{Conf}_i \mathbb{R}^n) \bigotimes_{\Sigma_i} V^{\otimes i} \right)^{\otimes I} \cong \bigoplus_{i \geq 0} C_*(\mathsf{Conf}_i U) \bigotimes_{\Sigma_i} V^{\otimes i}$$

Thus

$$\begin{aligned} \int_M \mathbb{F}_n V &= \underset{U \in \mathcal{D}\mathsf{isk}_{n/M}}{\mathsf{colimit}} \bigoplus_{i \geq 0} \left(C_*(\mathsf{Conf}_i U) \bigotimes_{\Sigma_i} V^{\otimes i} \right) \\ &= \bigoplus_{i \geq 0} \underset{U \in \mathcal{D}\mathsf{isk}_{n/M}}{\mathsf{colimit}} \left(C_*(\mathsf{Conf}_i U) \bigotimes_{\Sigma_i} V^{\otimes i} \right) \end{aligned}$$

Let $\mathsf{Disk}^{\mathrm{fr}}_{n/M}$ denote the *ordinary* category of framed n-disks in M. We'll just show things for the ordinary category $\mathsf{Disk}^{\mathrm{fr}}_{n/M}$, instead of for the ∞-category $\mathcal{D}\mathsf{isk}^{\mathrm{fr}}_{n/M}$. It turns out that this is sufficient:

Theorem 1.18 ([AF15])**.** *The functor $\mathsf{Disk}^{\mathrm{fr}}_{n/M} \to \mathcal{D}\mathsf{isk}^{\mathrm{fr}}_{n/M}$ is a localization. Hence factorization homology can be computed as a colimit over $\mathsf{Disk}^{\mathrm{fr}}_{n/M}$.*

For a more direct proof in the ∞-category case, see Proposition 5.5.2.13 of [Lur17].

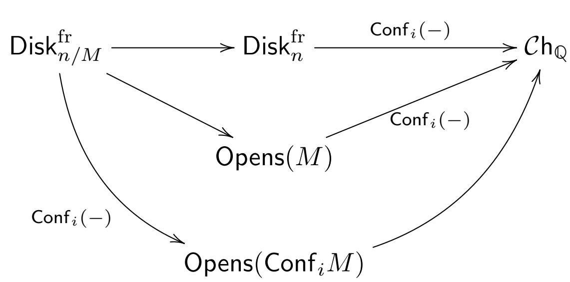
To compute this colimit, we use a hypercover argument. This is theorem A.3.1 in [Lur17]. Also see [**?**] and [**?**].

Theorem 1.19 (Seifert-van Kampen Theorem)**.** *Let X be a topological space. Let $\mathsf{Opens}(X)$ denote the poset of open subsets of X. Let $\mathcal{C}$ be a small category and let $F : \mathcal{C} \to \mathsf{Opens}(X)$ be a functor. For every $x \in X$, let $\mathcal{C}_x$ denote the full subcategory of $\mathcal{C}$ spanned by those objects $C \in \mathcal{C}$ such that $x \in F(C)$. If for every $x \in X$, the simplicial set $N(\mathcal{C}_x)$ is weakly contractible, then the canonical map*

$$\underset{C \in \mathcal{C}}{\mathsf{colim}} \mathsf{Sing}(F(C)) \to \mathsf{Sing}(X)$$

exhibits the simplicial set $\mathsf{Sing}(X)$ as a homotopy colimit of the diagram $\{\mathsf{Sing}(F(C))\}_{C \in \mathcal{C}}$.

To use the Seifert-van Kampen theorem, consider the following commutative diagram,



Let $\bar{x} = (x_1, \ldots, x_i) \in \mathsf{Conf}_i M$. The category $(\mathsf{Disk}^{\mathrm{fr}}_{n/M})_{\bar{x}}$ contains embed disks $U \hookrightarrow M$ so that $\{x_1, \ldots, x_i\}$ is in U. By the Seifert-van Kampen theorem, if $B(\mathsf{Disk}^{\mathrm{fr}}_{n/M})_{\bar{x}} \simeq *$, then

$$\underset{U \in \mathsf{Disk}_{n/M}}{\mathsf{colimit}} \ \mathsf{Conf}_i U \simeq \mathsf{Conf}_i M$$

To show that this category is contractible, we will show it is cofiltered.

Definition 1.20. A nonempty *ordinary* category $\mathcal{C}$ is cofiltered if
1) for every pair $U, V \in \mathcal{C}$ there exists $W \in \mathcal{C}$ and maps $W \to U$ and $W \to V$, and
2) given two maps $u, v : X \to Y$ in $\mathcal{C}$, there exists $Z \in \mathcal{C}$ and a map $w : Z \to X$ so that $uw = vw$.

Computation in the free case. Let $U, V \in (\mathsf{Disk}^{\mathrm{fr}}_{n/M})_{\bar{x}}$. We need to find a finite disjoint union of euclidean spaces $W \to M$ containing $(x_1, \ldots, x_i)$ and maps $W \to U$ and $W \to V$. Note that $U \cap V$ contains $\bar{x}$, but may not be a disjoint union of euclidean spaces. However, we can find a small disk around each x_i and still in $U \cap V$. The second condition is satisfied since $\mathsf{Disk}^{\mathrm{fr}}_{n/M}$ is a poset.

Thus $(\mathsf{Disk}^{\mathrm{fr}}_{n/M})_{\bar{x}}$ cofiltered, and hence contractible. Applying the Seifert-van Kampen theorem, (and adding in a few details about V) we get

$$\int_M \mathbb{F}_n V \simeq \bigoplus_{i \geq 0} C_*(\mathsf{Conf}_i M) \otimes_{\Sigma_i} V^{\otimes i}$$

□

2. Exercises

2.1. **Warm Ups.**

Question 2.1. What is the factorization homology

$$\int_{[0,1]} A$$

of an associative algebra A?

Question 2.2. What can you say about $\int_{S^2} A$ for A a 2-disk algebra?

Hint: Use excision.

Question 2.3. Make sense of the tensor product in excision. How are things modules, algebras, and such?

2.2. **Poincaré Duality for Factorization Homology.** Yesterday, you saw that $\Omega^n X$ was an $\mathbb{E}_n$-algebra.

Question 2.4. Show that $C_\bullet(\Omega^n X)$ is an $\mathbb{E}_n$-algebra.

Question 2.5. Write $C_\bullet(\Omega^n X)$ and $\Omega^n X$ as n-disk algebras (valued in chain complexes and spaces, respectively). Write them as factorization algebras as well.

Note that

$$\Omega^n X = \mathsf{Map}_c(\mathbb{R}^n, X).$$

Question 2.6. Let X be an n-connective space. Convince yourself that compactly supported maps

$$\mathsf{Map}_c(-, X)$$

satisfies $\otimes$-excision. That is, given a collar glueing

$$M \simeq U \bigcup_{V \times \mathbb{R}} U',$$

there is an equivalence

$$\mathsf{Map}_c(U, X) \times_{\mathsf{Map}_c(V \times \mathbb{R}, X)} \mathsf{Map}_c(U', X) \simeq \mathsf{Map}_c(M, X).$$

You can do this by just skimming the proof given in [AF15, Lem. 4.5], if you want; or by trying it out in a few easier cases.

The following is a theorem of Salvatore, Segal, and Lurie, in various contexts. We are following the proof of Ayala-Francis.

Question 2.7 (Nonabelian Poincaré Duality)**.** Let X be an n-connective space. Show that

$$\int_M \Omega^n X \simeq \mathsf{Map}_c(M, X).$$

If curious, ask a friend why for X an Eilenberg-MacLane space, the right-hand side looks like cohomology. This motivates the relationship between nonabelian Poincaré duality and usual Poincaré duality.

2.3. **Enveloping Algebras.** Let $\mathfrak{g}$ be a Lie algebra. Recall the Chevalley-Eilenberg complex $C_\bullet^{\mathrm{Lie}}(\mathfrak{g})$ from yesterday.

Question 2.8. Define a Lie algebra structure on

$$\mathsf{Map}_c(\mathbb{R}^n, \mathfrak{g}).$$

Show that

$$C_\bullet^{\mathrm{Lie}}(\mathsf{Map}_c(\mathbb{R}^n, \mathfrak{g})$$

forms an n-disk algebra. Call it $U_n\mathfrak{g}$.

Question 2.9. Show that $U_1\mathfrak{g}$ is the enveloping algebra $U\mathfrak{g}$.' Check this with your understanding of $U\mathfrak{g}$ as an $\mathbb{E}_1$-algebra from yesterday.

Thus $U_n\mathfrak{g}$ gives us a version of the enveloping algebra in higher dimensions -a "higher enveloping algebra" This is a key example in field theory. Many field theories have observables that look similarl to a higher enveloping algebra construction. For example, any free theory has this property.

Question 2.10. Compute

$$\int_M U_n\mathfrak{g}.$$

Hint: use both the fact that $C_\bullet^{\mathrm{Lie}}$ commutes with factorization homology and nonabelian Poincaré duality.

2.4. **Spare Questions.**

Question 2.11. Show that $\int_M A$ has a canonical action of $\mathsf{Diff}(M)$.

Question 2.12. Let $H \subset \mathrm{GL}(n)$ be a sub-Lie-group. You can think $SO(n)$ if you want. Define a notion of an H-oriented TFT in the functorial setting.

(1) Can you define a notion of an H-oriented $\mathbb{E}_n$-algebra?
(2) How about an H-oriented factorization algebra?

References

[AF15] David Ayala and John Francis. Factorization homology of topological manifolds. *J. Topol.*, 8(4):1045–1084, 2015.

[Lur17] Jacob Lurie. Higher algebra. Preprint available at `math.harvard.edu/~lurie/papers/HA.pdf`, September 2017.