FACTORIZATION ALGEBRAS: DAY 4

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1. Functorial Things

As we have seen in the past two days, factorization algebras, $\mathbb{E}_n$-algebras, and n-disk algebras are all related by the fundamental transformation of thinking of embedded disks, or points at their centers. By similar reasoning, observables are sometimes called point observables.

To record what we learned, we have a Costello-Gwilliam approach and a special case when the field theory is topological. The observables are a factorization algebra in general, and in the topological case we get a $\mathbb{E}_n$-algebra in Ch.

Indeed, say our field theory is topological and lives on $\mathbb{R}^n$. The factorization algebra of observables only depends on the data of

$$\mathcal{F}(\mathbb{D}^n) \in \mathsf{Ch}$$

and the multiplications coming from inclusions of disks into bigger disks. If we draw this, we see that $\mathcal{F}(\mathbb{D}^n)$ is the value at a disk centered at some point and then inclusion of disks can be pulled out to be a bordism between spheres. This sphere is the linking sphere of the disk. The result, which I'll denote

$$\mathcal{F}(\mathbb{D}^n) = \mathcal{Z}(S^{n-1})$$

for its dependence on the linking sphere is the $\mathbb{E}_n$-algebra in Ch.

1.1. Definition of Line Operators.

Observables gave us some cool algebras to think about, but there's some structure of the field theory that it misses.

Example 1.1. Consider G-gauge theory on M. Given a loop C in M, we can define a map

$$\mathsf{Bun}_G^{\nabla} M \to \mathbb{R}$$

sending a principal G-bundle $P \to M$ to the trace of the holonomy map

$$\mathrm{Hol}_C(P) \colon \mathfrak{g} \to \mathfrak{g}.$$

This is called a Wilson loop operator.

So given a loop in spacetime, we get a function on fields, which is something observables could know about, but observables doesn't see any of the dependence of loops in M.

In general, this type of construction giving observables depending on loops or lines in spacetime are called *line operators*.

Remark 1.2. I know this is super vague. It's my understanding that line operators are still partially in the physics art stage rather than being fully mathematically understood.

Let's think about what type of structure the set of line operators would have.

One piece of structure comes from stacking lines. This gives us a way of composing line operators. We should get a *category* of line operators with

- objects: line operators (a pair of a line in spacetime and the operator), and
- morphisms: $\mathsf{Hom}(L, L')$ is the set of point observables that can be inserted between L and L' to form a new line operator.

Question 1.3. Is there any algebraic structure on the category of line operators?

For observables, the $\mathbb{E}_n$-algebra structure came from looking at the linking sphere of the point we were at. Let's replicate that with lines. In $\mathbb{R}^n$, the linking sphere of a line is S^{n-2}. Now instead of including disks into bigger disks, we have lines colliding (analogous to points colliding). The result is a bordism between copies of S^{n-2}. This is a $\mathbb{E}_{n-1}$-algebra structure.

Claim 1.4. *For a topological theory on $\mathbb{R}^n$, line operators form an $\mathbb{E}_{n-1}$-monoidal category $\mathcal{Z}(S^{n-2})$.*

Let's add this to our table.

1.2. **Line Operator Constructions.** How do we say anything about line operators on the non-topological side?

Idea 1.5. Let F be a field theory on M. There is an ansatz from physics that a line operator for $C \subset M$ is the same as the data of boundary theory for F restricted to $M \setminus \mathsf{Norm}(C)$.

That is, the field theory works the same away from C and has a "defect" at C.

Thinking in terms of observables, in the topological $\mathbb{R}^n$ case, we would like an $\mathbb{E}_n$-algebra away from C.

Question 1.6. What is the data of observables of a boundary theory for F on $M \setminus \mathsf{Norm}(C)$?

Theorem 1.7 (Ayala-Franics-Tanaka)**.** *Let A be an $\mathbb{E}_n$-algebra. The data needed to produce a new $\mathbb{E}_n$-algebra that agrees with A on $\mathbb{R}^n \setminus \mathbb{R}^k$ is an object in*

$$\mathsf{LMod}\left(\int_{S^{n-k-1}\times\mathbb{R}^{k+1}} A\right).$$

Remark 1.8. To state this theorem, we are using the $\mathbb{E}_1$-algebra structure on

$$\int_{S^{n-k-1}\times R^{k+1}} A.$$

This comes from the $\mathbb{R}^{k+1}$ direction by stacking. This is related to the exercise from yesterday on making sense of excision.

For line operators, we are looking at modules over

$$\int_{S^{n-2}} A.$$

This S^{n-2} is the linking sphere of the line C that we encountered before.

Thus, we have an approximation to the category of line operators

$$\mathsf{LineOp} \simeq \mathsf{Mod}\left(\int_{S^{n-2}} A\right).$$

2. Atiyah-Segal

Returning to the topological side, we start to see a pattern. Continuing down, we could ask for the date of an $\mathbb{E}_n$-algebra $\mathcal{Z}(S^{n-1})$, an $\mathbb{E}_{n-1}$-monoidal category $\mathcal{Z}(S^{n-2})$, and so on.

We will use a functorial TQFT perspective to package together into a single functor.

Notation 2.1. For X an oriented manifold, let $\overline{X}$ denote X with the opposite orientation.

Definition 2.2. Let n be a positive integer. We define a category $\mathbf{Cob}(n)$ as follows:

- objects: $(n-1)$-dimensional oriented manifolds

- morphisms from M to N are given by equivalence classes of n-dimensional oriented manifolds with boundary B together with an an orientation-preserving diffeomorphism
$$\partial B \simeq \overline{M} \sqcup N$$
Two morphisms B and B' are equivalent if there is an orientation-preserving diffeomorphism $B \to B'$ that restricts to the identity on the boundaries,
$$\begin{tikzcd} B \arrow[r] \arrow[d, "\simeq"] & B' \arrow[d, "\simeq"] \\ \overline{M} \sqcup N \arrow[r, equal] & \overline{M} \sqcup N \end{tikzcd}$$

Composition is given by gluing cobordisms along their shared boundary. View $\mathbf{Cob}(n)$ as a symmetric monoidal category under disjoint union.

For k a field, let $\mathbf{Vect}(k)$ denote the symmetric monoidal category of k-vector spaces with tensor product.

Definition 2.3 (Atiyah, Segal)**.** Let k be a field. A *topological field theory* (TFT) of dimension n is a symmetric monoidal functor $Z\colon \mathbf{Cob}(n) \to \mathbf{Vect}(k)$.

In particular, $Z(\emptyset) = k$.

Lemma 2.4. *The value $Z(S^{n-1})$ is an $\mathbb{E}_n$-algebra.*

Proof. The pair of pants bordism gives the
$$\mathbb{E}_n(2) \otimes Z(S^{n-1})^{\otimes 2} \to Z(S^{n-1})$$
map. More legs, means more points. □

Remark 2.5. Note that Atiyah's definition considers all possible spacetimes at once, instead of working on one specific n-manifold at a time. We could instead consider the category of bordism submanifolds within a fixed manifold.

Notation 2.6. For V a k-vector space, let $V^\vee$ denote the linear dual, $V^\vee := \mathrm{Hom}(V, k)$.

Let Z be an n-dimensional TFT. Given an oriented $(n-1)$-manifold M, the product manifold $M \times [0, 1]$ can be viewed as a morphism in $\mathbf{Cob}(n)$ in multiple ways.

- As a morphism from $M \to M$, the product $M \times [0, 1]$ maps to the identity map
$$\mathrm{id}\colon Z(M) \to Z(M).$$
- As a morphism $M \sqcup \overline{M} \to \emptyset$, the product $M \times [0, 1]$ determines an evaluation map
$$\mathrm{ev}\colon Z(M) \otimes Z(\overline{M}) \to k.$$
- As a morphism $\emptyset \to \overline{M} \sqcup M$, the product $M \times [0, 1]$ determines a coevaluation map
$$\mathrm{coev}\colon k \to Z(\overline{M}) \otimes Z(M).$$

Recall that a pairing $V \otimes W \to k$ is perfect if it induces an isomorphism $V \to W^\vee$.

Proposition 2.7. *Let Z be a topological field theory of dimension n. For every $(n-1)$-manifold M, the vector space $Z(M)$ is finite dimensional. The evaluation map $Z(M) \otimes Z(\overline{M}) \to k$, induced from the cobordism $M \times [0, 1]$, is a perfect pairing.*

3. Classifying Topological Field Theories

3.1. **Low Dimensions.**

Example 3.1 (Dimension 1)**.** Let $Z\colon \mathbf{Cob}(1) \to \mathbf{Vect}(k)$ be a 1-dimensional TFT. Let P denote a single point with positive orientation and $Q = \overline{P}$. Let $Z(P) = V$. This finite-dimensional vector space, determines Z on objects. By Proposition 2.7, $Z(Q) = Z(\overline{P}) = V^\vee$. A general object of $\mathbf{Cob}(1)$ looks like

$$M = \coprod_{S_+} P \sqcup \coprod_{S_-} Q$$

for S_+, S_- sets. Since Z is symmetric monoidal, we have

$$Z(M) = \coprod_{S_+} V \sqcup \coprod_{S_-} V^\vee$$

What about morphisms? A morphism in $\mathbf{Cob}(1)$ is a 1-dimensional manifold with boundary B. Using the monoidal structure, it suffices to describe $Z(B)$ where B is connected. There are five possibilities

- B is an interval viewed as a morphism $P \to P$. Then $Z(B) = \mathrm{Id}_V$.
- B is an interval viewed as a morphism $Q \to Q$. Then $Z(B) = \mathrm{Id}_{V^\vee}$.
- B is an interval viewed as a morphism $P \sqcup Q \to \emptyset$. Then $Z(B)\colon V \otimes V^\vee \to k$. By Proposition 2.7, this is the canonical pairing of V and $V^\vee$,

  $$Z(B)(x \otimes f) = f(x)$$

  Under the isomorphism $V \otimes V^\vee \cong \mathrm{End}(V)$, the morphism $Z(B)$ corresponds to taking the trace.
- B is an interval viewed as a morphism $\emptyset \to P \sqcup Q$. Then $Z(B)$ is the map

  $$k \to V \otimes V^\vee \cong \mathrm{End}(V)$$

  sending $\lambda \in k$ to $\lambda \mathrm{Id}_V$.
- $B = S^1$ is a circle viewed as a morphism $\emptyset \to \emptyset$. Then $Z(S^1)$ is a linear map $k \to k$; i.e., multiplication by some $\gamma \in k$. To determine γ, view S^1 as the union of two semi-circles along $P \sqcup Q$. This determines a decomposition of S^1 into the composite of two cobordisms,

  $$\emptyset \to P \sqcup Q \to \emptyset$$

  By the above cases, Z maps this to the composite

  $$k \to \mathrm{End}(V) \to k$$

  of the map $\lambda \mapsto \lambda \mathrm{Id}_V$ and the trace map. Thus $Z(S^1)$ is the scaling by $\mathrm{Tr}(\mathrm{Id}_V) = \dim(V)$ map.

Remark 3.2. In 1-dimension, the observables are given by $Z(S^0) = \mathrm{End}(V)$. Notice that this has the structure of an associative algebra.

Thus in dimension 1, we see that the vector space $Z(P)$ determines the TFT Z. Does every $V \in \mathbf{Vect}(k)$ appear as $Z(P)$ for some 1-dimensional TFT? Nope, only the finite-dimensional ones. We get an equivalence of categories

$$\mathrm{Fun}^{\otimes}(\mathbf{Cob}(1), \mathbf{Vect}(k)) \to \mathbf{Vect}^{\mathrm{fin}}(k)$$

by evaluating on the point.

Let's try to do something similar in dimension 2.

Example 3.3 (Dimension 2)**.** Let Z be a 2-dimensional TFT. The only objects in $\mathbf{Cob}(2)$ are the empty set and disjoint unions of copies of S^1. We don't get a new object $\overline{S^1}$ since the circle has an orientation-reversing diffeomorphism. The observables, $A = Z(S^1)$ determines Z on objects.

What about morphisms? A morphism in $\mathbf{Cob}(2)$ is 2-dimensional oriented manifold with boundary.

- The pair of pants cobordism determines a map $m\colon A \otimes A \to k$. One can check that m defines a commutative, associative multiplication on A.
- The disk $\mathbb{D}^2$ viewed as a cobordism $S^1 \to \emptyset$ determines a linear map $\mathrm{Tr}\colon A \to k$.
- The disk $\mathbb{D}^2$ viewed as a cobordism $\emptyset \to S^1$ determines a linear map $k \to A$. The image of $1 \in k$ under this map acts as a unit for the multiplication. Indeed, we can glue $\mathbb{D}^2$ to one of the legs of the pants. The resulting manifold is diffeomorphic to $S^1 \times [0,1]$. But $S^1 \times [0,1]$ maps to Id_A under Z.

Note that the composite of

$$A \otimes A \xrightarrow{m} A \xrightarrow{\mathrm{Tr}} k$$

comes from the cobordism $S^1 \times [0,1]$ viewed as a map $S^1 \sqcup S^1 \to \emptyset$. By Proposition 2.7, the map $\mathrm{Tr} \circ m$ is a nondegenerate pairing.

Definition 3.4. A *commutative Frobenius algebra* over k is a finite-dimensional commutative k-algebra A, together with a linear map $\mathrm{Tr}\colon A \to k$ such that the bilinear form $(a,b) \mapsto \mathrm{Tr}(a,b)$ is nondegenerate.

Theorem 3.5. *The category of 2-dimensional TFTs is equivalent to the category of Frobenius algebras.*

Remark 3.6. In 2-dimensions, the observables $Z(S^1)$ of a TFT Z has the structure of a Frobenius algebra.

4. Exercises

Let Z be an n-dimensional field theory.

Question 4.1. Show that $Z(S^{n-1})$ acts on $Z(N)$ for every n-manifold N.

Question 4.2. For G-gauge theory on M, we defined a Wilson loop operator. Given a representation of G, define a version of a Wilson loop operator on $\mathsf{Gauge}_G M$.

Remark 4.3. For G-gauge theory on M, the category of line operators should be

$$\mathsf{Rep}(G) \times \mathsf{Rep}(G^L)$$

where G^L is the Langlands dual group. ($SO(2n)^L = SO(2n)$, $SL(n)^L = PGL(n)$).

Question 4.4. Let A be an $\mathbb{E}_n$-algebra. Build two different (∞, n)-categories from A.

Question 4.5. If we think of the factorization algebra of observables as point operators, we see this "pointyness" in the Weiss cover condition. If we wanted a notion of open covers that accounted for both point and line operators, what would that look like?

References