FACTORIZATION ALGEBRAS: DAY 5

ARAMINTA AMABEL

1. Review

Yesterday we talked about how for TQFTs the factorization algebra of observables (which is $\mathbb{E}_n$) can be situated in the Atiyah-Segal definition of a TQFT. We ended by classifying TQFTs in dimensions 1 and 2.

2. Higher Dimensions

The problem when we try to classify TFTs in higher dimensions is that the objects become too complicated. Up to reversing orientation and taking disjoint unions, the categories $\mathbf{Cob}(1)$ and $\mathbf{Cob}(2)$ have a unique object, P and S^1, respectively. For $n = 3$, there are infinitely many oriented 2-manifolds, one for each genus g. We don't think of genus g surfaces as being that complicated. In fact, we usually think of Σ_g, the genus g surface, as coming from g connect sums of the torus. A closely related way to say this, is that Σ_g has a relatively easy handle-body decomposition. But what happens when we view Σ_g under its handle-body decomposition? We're really viewing it as a composition of cobordisms; i.e., as a morphisms in $\mathbf{Cob}(2)$. Similarly, when we tried to understand the value of a 2-dimensional field theory on S^1, we broke S^1 into the union of two semi-circles, that is to say, into its handle-body decomposition.

If we want to be understand an n-dimensional field theory by breaking manifolds down, using their handle-body decompositions, into lower-dimensional manifolds, we need the TFT to know about manifolds of dimension $< n - 1$. In particular, we would like some sort of data assigned to every $(n - 2)$-dimensional manifold and we would like this data to have something to do with the values on $(n - 1)$-manifolds. In particular, from our discussion yesterday, we expect S^{n-2} to be assigned a category.

The way to encode all this data is the language of higher categories.

Definition 2.1. A *strict n-category* is a category $\mathcal{C}$ enriched over $(n - 1)$-categories.

For $n = 2$, this means that for objects $A, B \in \mathcal{C}$ the morphisms $\mathsf{Hom}_{\mathcal{C}}(A, B)$ is itself a category.

Example 2.2. The strict 2-category $\mathbf{Vect}_2(k)$ has objects cocomplete k-linear categories and morphisms

$$\mathrm{Hom}_{\mathbf{Vect}_2(k)}(C, D) = \mathbf{Fun}_k^{\mathrm{cocon}}(C, D)$$

the functor category of cocontinuos, k-linear, functors.

Example 2.3. The strict 2-category $\mathbf{Cob}_2(n)$ has

- objects: closed, oriented manifolds of dimension $n - 2$.
- morphisms: $\mathrm{Hom}_{\mathbf{Cob}_2(n)}(X, Y) =: \mathcal{C}$ should be the category with
  - objects: cobordisms $X \to Y$
  - morphisms: $\mathrm{Hom}_{\mathcal{C}}(B, B')$ is equivalence classes of bordisms X from $B \to B'$

The big problem here is making the composition law strictly associative. One would like to define composition by gluing bordisms, but get messed up in defining a smooth structure on the result, and things that used to be equalities are now just homeomorphisms. The solution will be to get rid of the "strictness" and move to $(\infty, 2)$-categories.

Let $\mathsf{Cob}_n(n)$ denote an (∞, n)-category version of the cobordism category, see Calaquee-Scheimbauer [CS19].

Definition 2.4. Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category. An *extended* $\mathcal{C}$-valued n-dimensional TFT is a symmetric monoidal functor

$$Z\colon \mathsf{Cob}_n(n) \to \mathcal{C}$$

Example 2.5. Take $\mathcal{C}$ so that $\Omega^n\mathcal{C} = \mathbb{C}$, $\Omega^{n-1}\mathcal{C} = \mathsf{Vect}_{\mathbb{C}}$, and $\Omega^{n-2}\mathcal{C} = \mathsf{LinCat}_{\mathbb{C}}$. Then $Z(S^{n-2})$ is an $(\infty, 1)$-category. It has an $\mathbb{E}_{n-1}$-monoidal structure from the pair of pants bordism. This is *line operators*. Similarly, $Z(S^{n-k})$ is an $\mathbb{E}_{n-k-1}$-monoidal $(\infty, n-k-1)$-category. It describes $(k+1)$-dimensional defects (i.e. operators).

The purpose of this definition is to allow us to reduced n-dimensional TFTs down to information about 1-dimensional TFTs. As we saw before, a 1-dimensional TFT is determined by its value on a point. Thus we might make the following guess.

Guess. An extended field theory is determined by its value on a single point. Moreover, evaluation on a point determines an equivalence of categories between TFTs valued in $\mathcal{C}$ and $\mathcal{C}$.

There's two problems with this guess.

(1) Even in 1-dimension, not every vector space determined a TFT. We needed to restrict to finite-dimensional ones. The analogue in higher dimensions will be something called "fully dualizable objects."
(2) Orientation in dimension 1 is the same as a framing. This isn't true in higher dimensions. We actually wanted a framing, not just an orientation so that we could say that locally M^k was canonically diffeomorphic to $\mathbb{R}^k$ (via the exponential map). Thus we need a version of $\mathsf{Cob}_n(n)$ that works with framed manifolds instead of oriented ones.

The following conjecture is due to Baez and Dolan [BD95].

Conjecture 2.6 (Cobordism Hypothesis: Framed Version)**.** *Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category with duals. Then the evaluation functor $Z \mapsto Z(*)$ induces an equivalence*

$$\mathsf{Fun}^{\otimes}(\mathbf{Bord}_n^{\mathrm{fr}}, \mathcal{C}) \to \mathcal{C}^{\sim}$$

between framed extended n-dimensional TFTs valued in $\mathcal{C}$ and the fully dualizable subcategory of $\mathcal{C}$.

Partial and complete proofs are due to Hopkins-Lurie, Lurie [Lur09], Grady-Pavlov [GP21]. Lots of others have worked on this as well.

Remark 2.7. This leads to the natural question, given $Z(\mathrm{pt})$, how does one obtain $Z(N)$? It's expected that there is a version of factorization homology for (∞, n)-categories so that

$$\int_N Z(\mathrm{pt}) = Z(N)$$

for all manifolds N of dimension $0, \ldots, n$. This is discussed in work of Ayala-Francis; see [AF17]. The upgraded factorization homology is referred to as "β-factorization homology."

Take $\mathcal{C}$ to be a suitable choice of an (∞, n)-category of algebras up to Morita equivalences

$$\mathcal{C} = \mathsf{Morita}_n.$$

See Scheimbaurer's thesis [Sch14] for a factorization homology reconstruction of a fully extended TQFT with Morita target. The fully dualizable objects of Morita_{n-1} are certain types of $\mathbb{E}_{n-1}$-algebras. Thus, you can describe an n-dimensional TQFT by just giving an $\mathbb{E}_{n-1}$-algebra (satisfying certain conditions) that the field theory assigns to a point.

Usually, we are thinking of an n-dimensional TQFT as corresponding to it's $\mathbb{E}_n$-algebra of observables, so what's up with the $\mathbb{E}_{n-1}$-algebra? How do we get from $Z(\mathrm{pt})$ to $Z(S^{n-1})$?

2.1. **Drinfeld Centers.** To answer this question, we are going to use a version of a notion Delaney talked about yesterday.

You saw yesterday that Drinfeld centers created braided monoidal structures from just monoidal structures. You can think of that as saying that the center of an $\mathbb{E}_1$-category is $\mathbb{E}_2$. More generally, we have the higher version of the Deligne conjecture due to Kontsevich, [Kon99].

Theorem 2.8 (Deligne Conjecture; Lurie)**.** *The $\mathbb{E}_n$-Drinfeld center of an $\mathbb{E}_n$-category is and $\mathbb{E}_{n+1}$-category.*

This was proven in full generality in [Lur17, §5.3]. Specifically, see [Lur17, Cor. 5.3.1.15].

One can show that $\mathbb{E}_n$-Drinfeld center of $Z(\mathrm{pt})$ is $Z(S^{n-1})$, answering our previous question.

As a good geometric example, we have the following theorem.

Theorem 2.9 (Ben-Zvi, Francis, Nadler)**.** *Let X be a perfect stack. The center of quasi-coherent sheaves on X is sheaves on the free loop space,*

$$\mathsf{Cent}_{\mathbb{E}_n}(QC(X)) \simeq QC(\mathcal{L}^n X).$$

See [BZFN10, Thm. 1.7 and Cor. 5.12].

3. Holomorphic Field Theories

Now I want to switch gears and talk about non-topological field theories. These will provide our first example of factorization algebras that are not $\mathbb{E}_n$-algebras.

Recall that a function

$$f \colon \mathbb{C} \to \mathbb{C}$$

is *holomorphic* if it is complex differentiable at every point. Equivalently, if we write

$$f(x+iy) = u(x,y) + iv(x,y),$$

and assume f is continuous, then f is holomorphic if and only if f satisfies the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

This is the Looman-Menchoff theorem.

Basically these are incredibly nice complex functions. In order to make sense of holomorphic conditions, our holomorphic field theories will have spacetime $\mathbb{C}^n$.

Topological field theories were particularly nice because of their invariance property on observables,

$$\mathsf{Obs}^q(D_1) \xrightarrow{\sim} \mathsf{Obs}^q(D_2).$$

To be able to get somewhere with holomorphic theories, we will additionally assume an invariance property: translation invariance.

Let V be a holomorphically translation-invariant vector bundle on $\mathbb{C}^n$. This means that we are given a holomorphic isomorphism between V and a trivial bundle. Our space of fields will be

$$\Omega^{0,*}(\mathbb{C}^n, V).$$

Note that by Dolbeault's theorem, this complex has cohomology

$$H^{0,*}(\mathbb{C}^n, V) = H^*(\mathbb{C}^n, \Omega^0 \otimes V)$$

where Ω^0 is the sheaf of holomorphic functions on $\mathbb{C}^n$. Thus, our space of fields is a derived model for the mapping space of holomorphic functions

$$\mathsf{Map}_{\mathrm{hol}}(\mathbb{C}^n, V).$$

Let

$$\eta_i = \frac{\partial}{\partial \bar{z}_i} \vee (-) \colon \Omega^{0,k}(\mathbb{C}^n, V) \to \Omega^{0,k-1}(\mathbb{C}^n, V)$$

be the contraction operator.

Definition 3.1. A field theory on $\Omega^{0,*}(\mathbb{C}^n, V)$ is *holomorphically translation invariant* if the action functional

$$S \colon \Omega^{0,*}(\mathbb{C}^n, V) \to \mathbb{C}$$

is translation-invariant and satisfies

$$\eta_i S = 0$$

for all $i = 1, \ldots, n$.

As with topological theories, we are interested in how the holomorphically translation-invariant condition on a field theory impacts the factorization algebra of observables.

To make the following definition precise requires more background and time than we have. Here is the idea; details can be found in [CG17, Def. 8.1.1] and [CG17, Ch 5. Def. 1.1.1].

Definition 3.2. A factorization algebra $\mathcal{F}$ on $\mathbb{C}^n$ is *holomorphically translation invariant* if we have isomorphisms

$$\tau_x \colon \mathcal{F}(U) \simeq \mathcal{F}(\tau_x U)$$

for all $x \in \mathbb{R}^n$ and open $U \subset \mathbb{R}^n$. These isomorphisms are required to vary holomorphically in x and satisfy

$$\tau_x \circ \tau_y = \tau_{x+y}$$

and commute with the factorization algebra maps.

Note that Obs^q will be a factorization algebra on $\mathbb{C}^n$. The following is [CG21, Prop. 9.1.1.2].

Theorem 3.3 (Costello-Gwilliam)**.** *The observables Obs^q of a holomorphically translation-invariant field theory is a holomorphically translation-invariant factorization algebra.*

Say we are working over $\mathbb{C}$. Assume that our field theory additionally is S^1-invariant. That is, that there is an S^1 action on V which, together with the S^1 action on $\Omega^{0,*}(\mathbb{C})$, gives an action on the space of fields, and all the structures of the field theory are invariant under this.

In this situation, we will get a nice algebraic description of the observables, like we did for locally constant factorization algebras as $\mathbb{E}_n$-algebras.

The following is [CG17, Ch. 5, Thm. 0.1.3].

Theorem 3.4 (Costello-Gwilliam)**.** *A holomorphically translation-invariant and S^1-invariant factorization algebra on $\mathbb{C}$ determines what is called a vertex algebra.*

Definition 3.5 (Borcherds)**.** A *vertex algebra* is the following data:

- a vector space V over $\mathbb{C}$ (the state space);

- a nonzero vector $|0>\in V$ (the vacuum vector);
- a linear map $T\colon V\to V$ (the shift operator);
- a linear map
$$Y(-,z)\colon V\to \mathsf{End}V[[z,z^{-1}]]$$

such that

- (vacuum axiom) $Y(|0>,z)=\mathrm{id}_V$ and
$$Y(v,z)|0>\in v+zV[[z]]$$
for all $v\in V$;
- (translation axiom) $[T,Y(v,z)]=\partial_z Y(v,z)$ for every $v\in V$ and $T|0>=v$;
- (locality axiom) for any pair of vectors $v,v'\in V$, there exists a nonnegative integer N such that
$$(z-w)^N[Y(v,z),Y(v',w)]=0$$
as an element of $\mathsf{End}V[[z^{\pm 1},w^{\pm 1}]]$.

See [BZF04] for a good reference on vertex algebras.

Remark 3.6. Vertex algebras were around before factorization algebras. They have the benefit of being super computational. If you are really good at power series manipulations, or more familiar with representation theory methods, vertex algebras might be better suited for you. Factorization algebras are more geometric and closer to the topologists $\mathbb{E}_n$-algebras.

As motivation for the proof, recall how we got an $\mathbb{E}_n$-algebra A from a locally constant factorization algebra $\mathcal{F}_A$ on $\mathbb{R}^n$. The underlying space of A is given by
$$\mathcal{F}_A(U)$$
where U is any disk in $\mathbb{R}^n$. For convenience, we can take $U=B_1(0)$, the unit ball around the origin. The structure maps are from the inclusions of disjoint disks in to $B_1(0)$,
$$\mathcal{F}_A(B_r(0))\otimes\mathcal{F}_A(B_{r'}(0))\to\mathcal{F}_A(B_1(0)).$$

We will get a vertex algebra from a holomorphically translation-invariant factorization algebra by a similar process, with one important difference.

Vertex algebras act more like Lie algebras.

They have structures like Lie brackets, rather than multiplications like groups do. If evaluating on $B_1(0)$ gave us a group like structure, than to get a Lie algebra structure, we should somehow take the tangent space at 0. This is in analogy with how given a Lie group G, one obtains the Lie algebra as T_eG.

Proof Idea. Let $\mathcal{F}$ be the holomorphically translation-invariant and S^1-invariant factorization algebra on $\mathbb{C}$. Let
$$\mathcal{F}_k(B_r(0))$$
be the weight k eigenspace of the S^1 action. To zoom in on 0, we take the limit
$$V_k=\lim_{r\to 0}H^\bullet(\mathcal{F}_k(B_r(0)).$$
(CG assume the maps in this limit are quasi-isomorphisms.) (They also assume that $V_k=0$ for $k>>$.)

The underlying vector space of the vertex algebra will be
$$V=\bigoplus_{k\in\mathbb{Z}}V_k.$$

The vacuum element is given by the image of the unit in $\mathcal{F}(\emptyset)$.

The translation map is given by the derivation $\frac{\partial}{\partial z}$ from infinitesimal translation in the z direction. The state-field map

$$Y : V \to \mathsf{End}(V)[[z, z^{-1}]]$$

comes from the inclusion of two disjoint disks into a bigger disk, and expanding the holomorphic map into a Laurent series. □

Factorization homology in the language of vertex algebras is called "conformal blocks."

3.1. **Examples.**

Example 3.7. A commutative ring V with derivation T determines a vertex algebra with state space V, translation operator T, and state-field correspondence

$$Y(u, z)v = uv.$$

Example 3.8. Given a manifold X, differential operators Diff_X is an associative algebra, so a type of factorization algebra on $\mathbb{R}$. A 2-dimensional analogue would be something like a factorization algebra on $\mathbb{R}^2$); for example, a vertex algebra. Such a vertex algebra was constructed by Malikov-Schechtman-Vaintrob [MSV99] and is called *chiral differential operators*. The factorization algebra analogue was build by Gorbounov-Gwilliam-Williams [GGW20]. This is an example of a factorization algebra build from a Lie algebra in an enveloping algebra-esque construction.

4. Duality

We understand field theories by studying their observables. This translation is great for a few things:

- a precise definition of topological field theory
- take advantage of the structure of factorization homology
- quantization algebraically becomes deformationo

I want to end by talking about another benefit of this viewpoint. Basically in all fields of math I think about, notions of duality are super exciting. For example, Poincaré duality in manifold theory. In your problem sessions, you saw a version of this called non-abelian Poincaré duality.

Question 4.1. Are there notions of dualities in field theory and factorization algebras?

We are going to start on the algebra side. The cool type of duality for algebras is called *Koszul duality*.

Definition 4.2. Let A be an associative algebra. The *Koszul dual* of A is the linear dual

$$\mathbb{D}(A) = (\mathbb{1} \otimes_A \mathbb{1})^\vee.$$

Here $\mathbb{1}$ denotes the trivial (left or right) A-module.

We can recover this construction using factorization homology. Indeed, there is an equivalence

$$\int_{\mathbb{D}^{\mathbb{1}}} A = \mathbb{1} \otimes_A \mathbb{1}$$

for any $\mathbb{E}_1$-algebra A.

This motivates a definition of Koszul duality for $\mathbb{E}_n$-algebras.

Definition 4.3 (Ayala-Francis)**.** Let A be an $\mathbb{E}_n$-algebra. The *Koszul dual* of A is

$$\mathbb{D}(A) = \left(\int_{\mathbb{D}^n} A \right)^\vee.$$

This is explained in [AF14, Thm. 3.3.2].

Remark 4.4. There is a different original definition of the Koszul dual of an $\mathbb{E}_n$-algebra, due to Ginzburg-Kapranov and Lurie if different contexts. Really Ayala-Francis' result is that the definition I gave above agrees with these previous definitions.

Recently, Ching and Salvatore [CS20] proved a long standing conjecture regarding the Koszul dual of the *operad* $\mathbb{E}_n$.

Theorem 4.5 (Ching-Salvatore, 2020). *The $\mathbb{E}_n$-operad is Koszul dual to itself.*

This was previously known at the level of chain complexes; see [GJ94]. Ching and Salvatore's result aslo recovers a previously known result on the level of algebras.

Corollary 4.6. *The Koszul dual of an $\mathbb{E}_n$-algebra is an $\mathbb{E}_n$-algebra.*

This can be found several places, including [Lur17, §5.2.5].

Example 4.7. Let $\mathfrak{g}$ be a Lie algebra. The Koszul dual of the enveloping algebra $U(\mathfrak{g})$ is the Lie algebra cochains,

$$\mathbb{D}(U_n\mathfrak{g}) = C^\bullet_{\mathrm{Lie}}(\mathfrak{g}).$$

This is also true for the $\mathbb{E}_n$-enveloping algebra $U_n(\mathfrak{g})$. The Koszul dual of $U_n(\mathfrak{g})$ is $C^\bullet_{\mathrm{Lie}}(\mathfrak{g})$, viewed as an $\mathbb{E}_n$-algebra, see [AF19, Cor. 4.2.1].

Thus, you can ask if the factorization homology of dual algebras are related. The following is the main theorem of [AF19].

Theorem 4.8 (Poincaré/Koszul duality, Ayala-Francis). *Under conditions, there is an equivalence*

$$\int_M A \simeq \left(\int_{M^+} \mathbb{D}(A)\right)^\vee.$$

The changing of M to M^+ reflects, for example, the difference in Poincaré duality for manifolds with boundary.

Turning back to the field theory side, note that the Koszul dual of observables on $\mathbb{R}^n$ has the right algebraic structure to be observables of a different field theory on $\mathbb{R}^n$. That is, let $\mathsf{Obs}_{\mathbb{X}}$ denote the observables of a field theory $\mathbb{X}$. We would like to know if there is another field theory $\mathbb{Y}$ so that the Koszul dual of observables on $\mathbb{X}$ is observables on $\mathbb{Y}$; in symbols

$$\mathbb{D}(\mathsf{Obs}_{\mathbb{X}}) \simeq \mathsf{Obs}_{\mathbb{Y}}.$$

Conjecture 4.9 (Costello-Paquette, Costello-Li). *Koszul duality for observables corresponds to AdS/CFT duality.*

See [CP21] and the references therein.

To make this conjecture precise, one would need a notion of Koszul duality for factorization algebras. That is an open problem. Checking this conjecture in examples is an active area of research by lots of people.

5. Exercises

Let Z be a fully extended n-dimensional TQFT valued in $\mathcal{C}$ with $\Omega^n\mathcal{C} = \mathbb{C}$, $\Omega^{n-1}\mathcal{C} = \mathsf{Vect}_{\mathbb{C}}$, and $\Omega^{n-2}\mathcal{C} = \mathsf{LinCat}_{\mathbb{C}}$.

Recall that $Z(S^{n-1})$ is an $\mathbb{E}_n$-algebrain Vect $Z(S^{n-2})$ is an $\mathbb{E}_{n-1}$-algebra in LinCat.

Question 5.1. Show the analogous statement for $Z(S^{n-k})$.

Question 5.2. Let K be a $(n-1)$-manifold with boundary. Show that K determines an object in $Z(\partial K)$.

Question 5.3. Can you describe the unit of $Z(S^{n-2})$?

Question 5.4. Let A be an $\mathbb{E}_n$-algebra. Recall from yesterday's exercises that you constructed an (∞, n)-category from A. Use this construction to write down sensible values for an n-dimensional field theory Z with $Z(S^{n-1}) = A$. In particular, write down $Z(S^{n-k})$ for all k, and $Z(\mathrm{pt})$.

Question 5.5. Show that a vertex algebra (V, Y, T) with

$$Y(u, z) \in \mathsf{End}V[[z]]$$

is equivalent to one formed by a commutative ring with derivation.

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