Spectral Sequences

Lectures and Material by Arun Debray
Live $\TeX$ by Geoff Vooys

May 5, 2023

Contents

1 Lecture 1: 2023 May 1

- Today: Generalities and Serre Spectral Sequences. Note: These things have many different names, so your milage may vary on the literal terminology.
- Tomorrow: Serre
- Wednesday: Atiyah-Hirzebruch
- Thursday/Friday: Adams Spectral Sequence

1.1 What is a spectral sequences

In the spirit of Ravi Vakil: Spectral sequences are so called because they are like spirits and are scary. This is undeserved.

Cohomolog — "well-oiled machine" You can use this to make a bunch of computations. Spectral sequences are in the same vein and work in largely the same way (a big machine) but more complicated. We'll learn how to drop problems in the SS machine.

This machine computes Abelian groups, vector spaces, more generally objects in an Abelian category.

Starting point: E_2-page ($E_2^{*,*}$). This is a bigraded array of objects in our Abelian category together with data of differentials.

The differentials are

$$d_2 : E_2^{p,q} \to E_2^{p+2,q-1}$$

and satisfy $d_2^2 = 0$.

Once you know this data,

$$H(E_2^{\bullet,\bullet}, d_2) =: E_3\text{-page}$$

which has further differentials

$$d_3 : E_3^{p,q} \to E_3^{p+3,q-2}.$$

Now rinse and repeat:

$$H(E_3^{\bullet,\bullet}, d_3) =: E_4\text{-page}$$

with

$$d_4 : E_4^{p,q} \to E_4^{p+4,q-3}$$

and so on!

Theo asked if knowing all the E_n and d_n do we determine d_{n+1} on the E_{n+1}-page? Arun said not for free.

We hope that this process stabilizes at some point. When this happens we get an E_∞-page. It does stabilize in many cases of interest. Ty[oca;;y $E_\infty^{\bullet;\bullet}$ is the associated graded of the thing you wanted to compute with respect to *some filtration*.

Remark 1.1.1. Often this filtration appears in the construction of the spectral sequence.

A good reference: Boardman's Conditionally Convergent Spectral Sequences. This is for when looking at spectral sequences which don't necessarily converge but still might be useful.

Example 1.1.2 (The Serre Spectral Sequnce for the Fibre Bundle)**.** Consider

$$\begin{array}{ccc} U(1) & \longrightarrow & EU(1) \\ & & \downarrow \\ & & BU(1) \end{array}$$

which is

$$\begin{array}{ccc} \mathbb{S}^1 & \longrightarrow & \mathbb{S}^\infty \\ & & \downarrow \\ & & \mathbb{C}\mathbb{P}^\infty \end{array}$$

or

$$\begin{array}{ccc} K(\mathbb{Z},1) & \longrightarrow & * \\ & & \downarrow \\ & & K(\mathbb{Z},2) \end{array}$$

We'll have that

$$E_2^{*,*} = H^p\big(BU(1); H^q(U(1);\mathbb{Z})\big)$$

converges to

$$H^\bullet(\mathbb{S}^\infty, \mathbb{Z}) = \mathbb{Z};$$

note the last statement holds because $\mathbb{S}^\infty$ is contractible. Here's how to see this

$$
\begin{array}{c|cccc}
q* & & & & \\
1 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\
0 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\
\hline
 & & & & *p \\
 & 0 & 2 & 4 & 6 \cdots
\end{array}
$$

Now d_2 gives a diagonal downwords map $\searrow$: $d_2 : \mathbb{Z} \to \mathbb{Z}$ which vanishes in degree $q = 0$. Turns out that

$$d_2 : E_2^{2k,1} \to E_2^{2k+3,0}$$

is an isomorphism. Take homology to get 0 and hence the E_3-page:

$$
\begin{array}{c|c}
 & \\
 & \\
0 & \mathbb{Z} \\
\hline
 & 0
\end{array}
$$

Thus we have only one nonzero term and so $E_3 = E_4 = \cdots = E_\infty$. In terms of generators and relations:

$$
\begin{array}{c|cccc}
1 & x & c_1 x & c_1^2 x & c_1^3 x \\
 & & & & \\
0 & 1 & c_1 & c_2 &
\end{array}
$$

Then

$$d_2(x) = c_1, d_2(c_1) = 0, d_2(c_1 x) = d_2(c_1 x) + (-1)c_1 d_2(x) = -c_1^2$$

Definition 1.1.3. A *fibre bundle* is a map $p : E \to B$ of topological space which locally in B looks like $B \times F$ (in the sense that there is a local trivialization in the open topology on B). If F is a vector spaces and the transition functions are linear this is a *vector bundle.*

Definition 1.1.4. If $F \cong G$ for a topological gorup G and the transition functions commute with the action of G on itself by left multiplication then this is a *principal G-bundle* (also known as a (left) G-torsor).

Remark 1.1.5. There is a homotopical analogue of fibre bundles which are called fibrations. For the purpose of this week, we can interchange the two terms. Many homotopical properties of fibre bundles carry over to fibrations.

Example 1.1.6. Consider:

$$
\begin{array}{ccc}
\mathbb{Z}/2\mathbb{Z} & \longrightarrow & \mathbb{S}^n \\
 & & \downarrow \\
 & & \mathbb{R}\mathbb{P}^n
\end{array}
$$

which is a principal $\mathbb{Z}/2\mathbb{Z}$-bundle. Consider also

$$
\begin{array}{ccc}
U(1) & \longrightarrow & \mathbb{S}^{2n-1} \\
 & & \downarrow \\
 & & \mathbb{C}\mathbb{P}^n
\end{array}
$$

as a principal $U(1)$-bundle and

$$
\begin{array}{ccc}
\mathrm{Sp}(1) & \longrightarrow & \mathbb{S}^{4n-1} \\
 & & \downarrow \\
 & & \mathbb{H}\mathbb{P}^n
\end{array}
$$

Let (X, x) be a pointed topological space. There is then a fibration

$$\begin{array}{ccc} \Omega X & \longrightarrow & PX \\ & & \downarrow \\ & & X \end{array}$$

where ΩX is the based loop space of X and PX is the path space of X (where we assume that paths begin at the basepoint x). We also have, for A an Abelian group, the fibration

$$\begin{array}{ccc} K(A, n-1) & \longrightarrow & * \\ & & \downarrow \\ & & K(A, n) \end{array}$$

For any topological group G there is a universal principal G-bundle

$$\begin{array}{ccc} G & \longrightarrow & EG \\ & & \downarrow \\ & & BG \end{array}$$

in the sense that for every G-torsor P, there is a unique $f : X \to BG$ (uniqueness up to homotopy) for which there is a pullback diagram

$$\begin{array}{ccc} P & \longrightarrow & EG \\ \downarrow & & \downarrow \\ X & \xrightarrow[f]{} & BG \end{array}$$

Facts:

- Producing the space BG is functorial in G.
- When G is discrete at least, $H^\bullet(BG)$ is the group cohomology of G.

Other fibrations: If we have an SES

$$1 \longrightarrow G \longrightarrow H \longrightarrow K \longrightarrow 1$$

there is a fibration

$$\begin{array}{ccc} BG & \longrightarrow & BH \\ & & \downarrow \\ & & BK \end{array}$$

Now the "Leray-Serre spectral sequence" or "Serre spectral sequence."

Theorem 1.1.7 (Leray, Serre, Lyndon, Hochschild; note Leray worked on this while a prisoner of war under the tyranny of the Nazis)**.** *If we have a fibration*

$$\begin{array}{ccc} F & \longrightarrow & E \\ & & \downarrow \\ & & B \end{array}$$

then there is a spectral sequence, for any commutative ring A,

$$E_2^{p,q} = H^p\big(B; H^q(F; A)\big) \Rightarrow H^{p+q}(E; A).$$

Note that the $\implies$ arrow states that E_∞-page exists and is the associated graded of a filtration on $H^\bullet(E; A)$.

Note that

$$d_r : E_r^{p,q} \to E_r^{p+r,q+1-r}$$

and the spectral sequences is multiplicative in the sense that

$$d_r(xy) = d_r(x)y + (-1)^{|x|} x d_r(y)$$

Remark 1.1.8. Some of this assumes that B is simply connected. If it is not, then we need to take the twisted cohomology of B which is messy. We need to twist by local systems and do stuff locally (remark by Geoff: the monodromy representations sneak their heads into the story).

1.2 Edge Homomorphisms

Consider

$$\begin{array}{ccc} F & \longrightarrow & E \\ & & \downarrow \\ & & B \end{array}$$

THen we get morphisms $H^p(B) \to E_\infty^{p,0}$ from the horizontal line of the sequence and this is $H^p(B) \to H^b(E)$ and likewise for the vertical line when $p = 0$ and $H^\bullet(E) \to H^\bullet(F)$.

Example 1.2.1. Consider the fibration:

$$\begin{array}{ccc} \mathbb{S}^1 & \longrightarrow & \mathbb{S}^{2n-1} \\ & & \downarrow \\ & & \mathbb{C}\,\mathbb{P}^n \end{array}$$

and note that

$$H^\bullet(\mathbb{S}^n; \mathbb{Z}) = \mathbb{Z}[x]/(r^2),\ |n| = r$$

Work backwards: Use the chomology of $\mathbb{S}^2, \mathbb{S}^{2n-1}$ to infer $H^\bullet(\mathbb{C}\,\mathbb{P}^{n-1})$. If $d_2(x) = 0$ then x maps to 0_∞.

Observation from a question: The edge homomorphisms are a way of computing that pullback.

2 Lecture 2: 2023 May 2

Serre Spectral Sequence Part 2

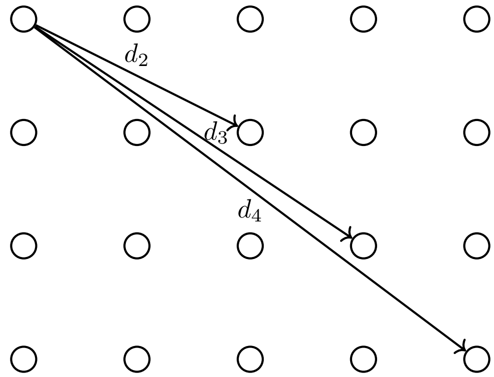
Recall: Associated to a fibration

$$\begin{array}{ccc} F & \longrightarrow & E \\ & & \downarrow \\ & & B \end{array}$$

there is the Serre spectral sequence

$$E_2^{p,q} = H^p\big(B; H^q(F; A)\big) \Rightarrow H^{p+q}(E; A)$$

where A is a commutative ring. The differentials are “generalized knight’s moves:



The Serre spectral sequences is also multaplicative[1] in the sense that

$$d_n(xy) = d_n(x)y + (-1)^{|x|} x d_n(y).$$

This is natural for pullbacks of fibrations/fibre bundles in the sense that given two fibrations and a diagram

$$\begin{array}{ccc} F & \longrightarrow & F' \\ \downarrow & & \downarrow \\ E & \longrightarrow & E' \\ \downarrow & & \downarrow \\ B & \longrightarrow & B' \end{array}$$

in which the bottom square is a pullbakc then there is a morphism

$$E_2(E' \to B') \to E_2(E \to B).$$

In particular the pullback of E_2 will be E_2 of the pullback.

2.1 Gysin Long Exact Sequence

Assume that B is simply connected and consider a sphere bundle

$$\begin{array}{ccc} \mathbb{S}^{n-1} & \longrightarrow & E \\ & & \downarrow \\ & & B \end{array}$$

Note that if B is not simply connected then $E \to B$ is oriented. A counter example that shows that you cannot freely drop simple connectivity comes from taking E to be a unit sphere bundle insade a verctor bundle.

Let's throw the Serre spectral sequence at this fibration. Get:

$\mathbb{Z}$	$n-1$	$H^0(B)$	$H^1(B)$	$H^2(B)$	$\cdots$
$\mathbb{Z}$	0	$H^0(B)$	$H^1(B)$	$H^2(B)$	$\cdots$

Just because of where things live on E_2, the only possible nonzero differential is d_n. By multiplicativity, if $x \in H^k(B)$, regarded as living in $E_n^{k,n-1}$ then

$$d_n(x) = x d_n(1) + 1 d_n(x) = x d_n(1)$$

[1] Some terms and conditions apply. Your mileage may vary.

so $d_n(x) = xd_n(1)$. Note that we used that $x \in E_n^{k,0}$. Consequently, let $e(E) := d_n(1) \in H^n(B)$. This is the *Euler class* of E. This is a *natural* character class of sphere bundle. Thus we get that, from the naturality of Serre spectral sequence

$$e(f^*B) = f^*e(B).$$

Now consider the map $d_n : H^k(B) \to H^{k+n}(B)$ which is given by multiplication by the Euler class. The homology with respect to d gives us $E_{n+1} = E_\infty$ and so we get an exact sequence:

$$- \longrightarrow E_\infty^{k,n-1} \longrightarrow H^k(B) \xrightarrow{d_n} H^{k+n}(B) \longrightarrow E_\infty^{k+n,0} \longrightarrow 0$$

Filtration on $H^*(E)$ conming from the E_∞-page, we get a short exact sequence

$$0E_\infty^{k,0} \longrightarrow H^k(E) \longrightarrow E_\infty^{0,k} \longrightarrow 0$$

Glue to build an exact sequence

$$H^k(B) \xrightarrow{\cup e(E)} H^{k+n}(B) \xrightarrow{\pi^*} H^{k+n}(E)$$

where the π^* is induced by the edge homomorphism. Repeating mutatis mutandis for the right side of the sequence we get the long exact sequence

$$\cdots \longrightarrow H^k(B) \xrightarrow{\cup e(E)} H^{k+n}|(B) \xrightarrow{\pi^*} H^{k+n}(B) \xrightarrow{\pi_*} H^{k+1}(B) \longrightarrow \cdots$$

Note that the pushforward is given by integration on the fibre and this is called the *Gysin long exact sequence.*

Remark 2.1.1. If instead we have a fibration

$$\begin{array}{ccc} F & \longrightarrow & E \\ & & \downarrow \\ & & \mathbb{S}^n \end{array}$$

then we get the *Wang long exact sequence.*

Remark 2.1.2. The most general version of Gysin is *very* general. A long exact sequence contianing the *Smith homomorphism*[2] in place of $e(E)$. This comes up in physics in the study odefects, symmetry breaking, etc.

One feature of sepctral sequences we may have notices is that "the last possible differential" is the most important. In particular, if d_n is the last differential and $d_n(x) = y$, $x \in E_2^{0,n}$, $y \in E_2^{n+1}, 0$ we say that x transgress to y. Some examples: Gysin, $H^*(\mathbb{C}\mathbb{P}^n)$, $H^*(\mathbb{H}\mathbb{P}^n)$, $H^*(BU_n), \cdots$.

Example 2.1.3. Eilenberg-MacLane spaces. Consider that

$$\pi_k(K(A,n)) = \begin{cases} A & \text{if } k = n; \\ 0 & \text{else.} \end{cases}$$

Then also we get that $\Omega K(A,n) = K(A, n-1)$. Now consider, if $x \in X$ is a base point, then the fibration

$$\Omega X \longrightarrow P(x) \quad \simeq \quad *$$

$$X$$

[2]A generalization of the Euler class.

implies that we have a fibration

$$\begin{array}{ccc} K(A, n-1) & \longrightarrow & * \\ & & \downarrow \\ & & K(A, n) \end{array}$$

which powers inductive computations of $H^\bullet(K(A,n))$. So:

$$H^\bullet(K(A,n); A) = \begin{cases} 0 & 1 \le k \le n; \\ A & k = n; \\ ???? & k > n. \end{cases}$$

Now from the Serre spectral sequence applied to the fibration

$$\begin{array}{ccc} K(A, n-1) & \longrightarrow & * \\ & & \downarrow \\ & & K(A, n) \end{array}$$

as

$$\begin{array}{c|ccc} n-1 & A & & \\ & & & A \\ \hline & & & n \end{array}$$

we see that d_n is the only way to kill these two "tautological" copies of n. Thus E_∞ has to be zero there and so the transgression must be an isomoprhism in this case.

Theorem 2.1.4 (Borel's Transgression Theorem)**.** *If G is a connected Lie group and A is a commutative ring such that $H^\bullet(G; A)$ is a nil-square algebra on odd-degree generateors*[3]*. Then $H^\bullet(BG; A) \cong A[y_1, \cdots, y_k]$ where each y_i is in even degree and in the Serre spectral sequence for*

$$\begin{array}{ccc} G & \longrightarrow & EG \\ & & \downarrow \\ & & BG \end{array}$$

x_i transgresses to y_i.

Example 2.1.5. $G = \mathrm{SU}_n, \mathrm{U}_n, \mathrm{Sp}_n$ and $A \in$ **Cring**.

Example 2.1.6. $G = \mathrm{SO}_n, \mathrm{Spin}_n, \mathrm{O}_n$ with A with 2 a unit in A.

Example 2.1.7. $G = \mathrm{PSU}_n, \mathrm{PSp}_n, \mathrm{PSO}_n, A = \mathbb{Q}$.

Example 2.1.8. Consider $G = \mathrm{SU}_3$. Then there is a fibration

$$\begin{array}{ccc} \mathrm{SU}_2 & \longrightarrow & \mathrm{SU}_3 \\ & & \downarrow \\ & & \mathrm{SU}_3 / \mathrm{SU}_2 \end{array}$$

where $\mathrm{SU}_3 / \mathrm{SU}_2 \cong \mathbb{S}^5$ is simply regarded as a coset space. Now we find that the Serre sepctral sequence takes the form

[3]That is $H^\bullet(G; A)$ is an exterior algebra on odd-degree generators.

3	y	yz
0	1	z
	0	n

and all the diferentials vanish. So

$$H^\bullet(\mathrm{SU}_3;\mathbb{Z}) = \frac{\mathbb{Z}[y,z]}{(y^2,z^2)}$$

where $|y| = 3, |z| = 5$. Now this generalizes to give us $H^*(\mathrm{SU}_n)$.

Example 2.1.9. Consider the firbation

$$\begin{array}{ccc} \mathrm{SO}_3 & \longrightarrow & * \\ & & \downarrow \\ & & B\,\mathrm{SU}_3 \end{array}$$

Bore's Transgression Theorem gives $H^*(B\,\mathrm{SU}_3;;\mathbb{Z}) = \mathbb{Z}[d_4, d_6]$. For SU_n the trick is the exact same, while Sp_n and U_n are similar. For SO_n and Spin_n we have to work harder to find the transgressing differential.

Some cohomology rings:

- $H^\bullet(B\,\mathrm{U}_n;\mathbb{Z}) = \mathbb{Z}[c_2,\cdots,c_n], |c_i = 2i$
- $H^\bullet(B\,\mathrm{SU}_n;\mathbb{Z}) = \mathbb{Z}[c_2,c_3,\cdots,c_n]$.

3 Lecture 3: 2023 May 3

3.1 The Atiyah-Hirzebruch Spectral Sequences

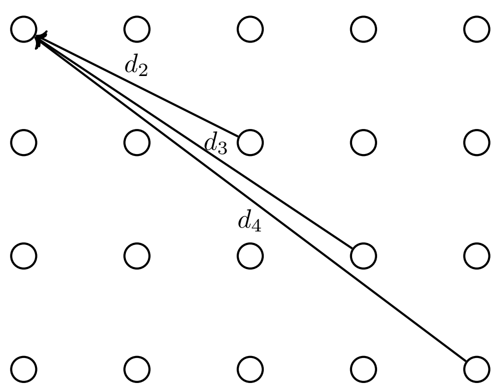
Recall the Serre spectral sequences is

$$E_2^{p,q} = H^p\big(B, H^q(F)\big) \Rightarrow H^{p+q}(E)$$

has a dual homological version

$$E^2_{p,q} = H_p\big(B; H_q(F)\big) \Rightarrow H_{p+q}(E)$$

with differentials given by reverse generalized knight's moves:



Remark 3.1.1. There is no ring structure on H_k but the homological Serre spectral sequences is a spectral sequence of coalgebras.

Definition 3.1.2 (Eilenberg-Steenrod Axioms)**.** Ordinary homology theory $H_\bullet : \mathbf{Top} \to \mathbf{Ab}$ satisfies:

1. Homotopy equivalent functions induces the same map on homology, i.e., if $f \simeq g$ then $H_\bullet(f) = H_\bullet(g)$.
2. Excision: Given inclusions $U \subseteq A \subseteq X$ with $\overline{U} \subseteq \text{int}(A)$ the inclusion
$$(X \setminus U, A \setminus U) \to (X, A)$$
induces an isomorphis on homology.
3. Coproducts get set to direct sums in homology.
4. There is a long exact sequence of $A \to X \to X/A$.
5. $H_\bullet(*) = A$ in degree zero only

Remark 3.1.3. These five axioms uniquely characterize homology. What happens if we remove Axiom 5?

Definition 3.1.4. A functor that satisfies Eilenberg-Steenrod axioms $1-5$ is a generalized homology pair.

Remark 3.1.5. There is likewise a notion of ageneralized cohomology theory.

Lots of these generalized homology theories are merely homotical rather than geometric. But there are plenty of geometric ones!

3.2 K-Theory

Let $\text{Vec}(X)$ denote the monoid of vector bundles on X under direct sum. Define the 0-th K-group of X to be the group completion of the monoid
$$K^0(X) := \widetilde{\text{Vec}(X)}$$
which has formal rule $E - F \simeq E \oplus G - F \oplus G$. This is in fact a ring under tensor product for multiplication. $KO^0(X)$ is the same thing with real vector bundles.

Both of these generalize to generalized cohomology theories K^*, KO^* representable in degres in terms of bundles of Clifford-modules.

The periodicity of Clifford says that $K^*(X)$ is 2-periodic in the sense that $K^n(X) \cong K^{n+2}(X)$ and $KO^*(X)$ is 8-periodic.

Bott-periodicity says that
$$K^*(*) = \mathbb{Z}, 0, \mathbb{Z}, 0, \mathbb{Z}, 0, \cdots$$
and $K^*(*) = \mathbb{Z}[\beta, \beta^{-1}]$ with degree $|\beta| = 2$. Similarly,
$$KO^*(*) = \mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, 9, \mathbb{Z}, 0, 0, 0, \mathbb{Z}, \mathbb{Z}/2, \cdots .$$
You can remember this by singing the Bott song!

3.3 Bordism

Generalized homology theory $\Omega_*^O(X) = \{n-\text{manifolds that bound} <\!\!\to X\}$.

Remark 3.3.1. If $M = \partial N$ for a compact $(n+1)$-manifold N then $M \to X$ extends to $N \to X$.

If $X = \{*$ then

The torus $\mathbb{T}^2$ bounds a "solid donut." In particular $[\mathbb{T}^2] = 0$ in $\Omega_2^0 *)$ but $[\mathbb{RP}^2] \neq 0$ in $\Omega_2^O(*)$.

	Ω_k^O	Ω_k^{SO}	Ω_k^{Spin}	Ω_k^{String}	$\Omega_k^{\text{pin}-}$
	o data	Orientation	Spin	String	Pin-
0	$\mathbb{Z}/2 \cdot *$	$\mathbb{Z} \cdot *$	$\mathbb{Z} *$	$\mathbb{Z}$	$\mathbb{Z}/2$
1	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
2	$\mathbb{Z}/2 \cdot \mathbb{RP}^2$	0	$\mathbb{Z}/2 \cdot \mathbb{T}^2$	$\mathbb{Z}/2 \cdot \mathbb{T}^2$	$\mathbb{Z}/8 \cdot \mathbb{RP}^2$
3	0	0	0	$\mathbb{Z}/24 \cdot \mathbb{S}^3$	0
4	$\mathbb{Z}/2\,\mathbb{CP}^2 \oplus \mathbb{Z}/2\,\mathbb{OP}^4$	$\mathbb{Z}\,\mathbb{CP}^2$	$\mathbb{Z} \cdot K3$	0	0
5	$\mathbb{Z}/2 \cdot \text{SU}_3/\text{SO}_3$	$\mathbb{Z}/2 \cdot \text{SU}_3/\text{SO}_3$	0	0	0

Other generalized homology theories:

- Kitoev advocates this idea that the groups of SPT phases for a symmetry G are the values of a generalized homology theory on BG (bosonic fermionic will be different theories; cf. Xiang, Goiotto – Johnson-Freyd).
- Freed-Hopkins-Teknon: Freed-Hopkins provide an implementation of this idea using the genralized cohomology theory of invertible field theories of manifolds of a map to X.

3.4 The Atiyah-Hirzebruch Spectral Sequence

Let X be a space and R a generalized cohomology theory. This is a generalized cohomology theory

$$E_2^{p,q} = H^p(X; R^q(*)) \Rightarrow R^{p+q}(X).$$

Some facts:

- This is natural in X and in R.
- Homological version: If R is a generalized homological theory

$$E^2_{p,q} = H_p(X; R_q(*)) \Rightarrow R_{p+q}(X).$$

- This is multiplicative if R is a *ring spectrum*.

The Leray-Serre-Atiya-Hirzebruch spectral sequence takes a firbation

$$\begin{array}{ccc} F & \longrightarrow & E \\ & & \downarrow \\ & & B \end{array}$$

(assume that B is simply connected) and for R a generalized cohomology theory, there is a spectral sequence

$$E_2^{p,q} = H^p(B, R^q(F)) \Rightarrow R^{p+q}(E)$$

which is natural and which is multipliciative whenever R is.

Example 3.4.1. LEt's compute $K^*(\mathbb{C}\mathbb{P}^\infty)$ and input $H^*(\mathbb{C}\mathbb{P}^\infty; \mathbb{Z}) = \mathbb{Z}[c]$ where $|c| = 2$ and $K^*(*) = \mathbb{Z}[\beta, \beta^{-1}]$ for $|\beta| = 2$. Now:

	q					
$\mathbb{Z}\beta^2$	4	β^2	$\beta^2 c$	$\beta^2 c^2$	$\beta^2 c^3$	
	3					
$\mathbb{Z}\beta$	2	β	βc	βc^2	βc^3	
	1					
$\mathbb{Z}$	0	1	c	c^2	c^3	
		0	1	2	3	p
	-1					
$\mathbb{Z}\beta^{-1}$	-2	β^{-2}	$\beta^{-1}c$	$\beta^{-1}c^2$	$\beta^{-1}c^3$	
	-3					

We get that $E_2 = E_\infty$ because all elements in even total degree and differentials raise the degree by 1 and so all differentials vanish. Thus

$$K^*(\mathbb{C}\mathbb{P}^\infty) = \mathbb{Z}[c] \otimes_{\mathbb{Z}} \mathbb{Z}[\beta, \beta^{-1}] \cong \mathbb{Z}[\beta, \beta^{-1}, c]$$

where $|c| = |\beta| = 2$.

Some useful facts about the AHSS:

- I totally missed this so here's a dinosaur pun: what do you call a ds/dt-raptor? A velocity-raptor!
- Another useful fact: The AHHS for $R^*(-) \otimes \mathbb{Q}$ *always* collapses. So in particular, there are no differentials and no extension problems.

So what happens when we do have differentials? Maunder identifies some AHSS differentials in terms of stable cohomology operations.

Definition 3.4.2. A *stable cohomology operation* is a natrual transformation

$$H^\bullet(-; A) \to H^{\bullet+k}(-; B)$$

that commutes with the suspension isomorphism.

Example 3.4.3. Here is a table of examples and nonexamples:

Examples	Nonexamples
Reduction modulo p	Cup products!
Bockstein	Pontryagin square
STeenrod squares	

Remark 3.4.4. The Steenrod squares are the generators of the ring A (the Steenrod algebra) of stable cohomology operations for $A = B = \mathbb{Z}/2$. It has an axiomatic characterization

1. Sq^i is natrual in pullback and raises degree by i.
2. $Sq^0 = \mathrm{id}$ and

$$Sq^i(x) = \begin{cases} 0 & |x| < i \\ x^2 & |x| = i; \\ ??? & |x| > i \end{cases}$$

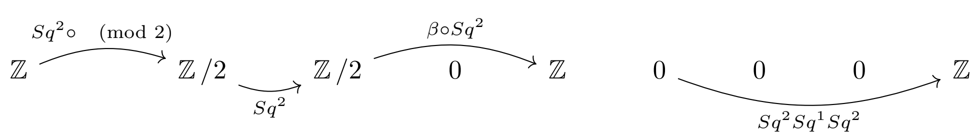
3. Sq satisfies the Cartan formula $Sq(xy) = Sq(x)Sq(y)$, and

$$[Sq^n(xy) = \sum_{i+j=n} Sq^i(x)Sq^j(y).$$

Theorem 3.4.5 (Maunder)**.** *The first nonzero differential in AHSS for R is a stable cohomology operation coming from data of R.*

Example 3.4.6. For $R = KU$ we consider that $d_3 : H^p \to H^{p+3}$ is a stable cohomology operation! Specifically: it is $\beta \circ Sq^2 \pmod 2$ where $\beta : H^k(-; \mathbb{Z}/2) \to H^k(-; \mathbb{Z})$. Higher differentials are more mysterious.

Example 3.4.7. Consider for KO:



I missed the Ω^*_{Spin} :(

4 Lecture 4: 2023 May 4

4.1 The Adams Spectral Sequence

There is a big diagram that is for show and too hard to TEX. It is really cool and depicts the stable homotopy groups of the spheres! That is, it depicts

$$\varinjlim_k \pi_{n+k}(\mathbb{S}^k).$$

For applications to mathy physics, you don't have to use the full machinery. Where to find introductions? Beuadry-Cambell[4], arXiv:1801.07530.

This is a hyperlink!

Recall from Wednesday (Day 3): The Steenrod squares Sq^i act on mod 2 chomology,

$$H^*(X; \mathbb{Z}/2)$$

and the mod 2 Steenrod algebra,

$$A = \langle \mathrm{Sq}^i : i \in \mathbb{N} \rangle$$

acts naturally on $H^*(X; \mathbb{Z}/2)$ but does not allow us to witness $H^*(X; \mathbb{Z}/2)$ as an A-algebra. Let's think about homological algbera in the category of A-modules. Now note that A is $\mathbb{Z}$-graded with $|\mathrm{Sq}^i| = i$. In particular, we can compute $\mathrm{Ext}_A(M, N)$ for A-modules.

Remark 4.1.1. Recall that for any ring R, $\mathbf{R\text{-}Mod}(-,-)$ need not be an exact functor. Consequently there is a corresponding derived functor

$$\mathrm{Ext}^n_R(-,-) : \mathbf{R\text{-}Mod} \to \mathbf{Ab}$$

which can be calculated by resolving the contravariant variable in projectives and resolving the covariant variable in injectives. In particular, $\mathrm{Ext}^n_R(-,-) = R^n \operatorname{Hom}(-,-)$.

TL;DR Ext is a functor built from Hom in a way that Ext takes a short exact sequence of R-modules and produces a long exact sequence of ABelian groups in Ext (and also $\mathrm{Ext}^0(-,-) \cong \mathrm{Hom}(-,-)$).

Since A is $\mathbb{Z}$-graded, $\mathrm{Ext}^{s,t}_A(M, N)$ is *bigraded* where:

- s is the homological grading
- t comes from the grading of A.

In particular we get that

$$\mathrm{Ext}^{s,t}_A(M, N) \cong A\mathbf{Mod}(\Sigma^t N, M) = A\mathbf{Mod}(N[t], M)$$

where we regard N and M as $\mathbb{Z}$-graded Abelian groups which commute with the STeenrod operations. Note that $\Sigma^t N = N[t]$ is the "shift the grading by t" operator.

The (well, 'an") Adams spectral sequence is a spectral sequence with signature

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_A\left(H^*(X; l\,\mathbb{Z}/2), \mathbb{Z}/2\right) \Rightarrow \pi^s_{t-s}(X)^\wedge_2$$

where

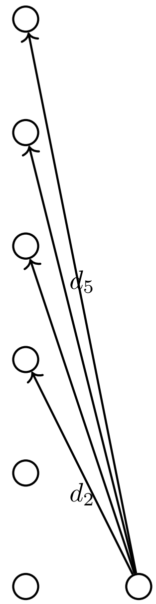
$$\pi^s_{t-s}(X)^\wedge_2 := \left(\varinjlim_k \pi_{t-s+k}(\Sigma^k X)\right)^\wedge_2$$

is the stable homotopy group completed at 2, i.e.,

$$\pi^s_{t-s}(X)^\wedge_2 \cong \varprojlim_{n\in\mathbb{Z}} \frac{\pi^s_{t-s}(X)}{2^n \pi^s_{t-s}(X)} \cong \pi^s_{t-s}(X) \otimes_{\mathbb{Z}} \mathbb{Z}_2\,.$$

This is typically graded with grading $(t-s, s)$ where d_r moves 1 unit to the left and r units up:

[4] I probably misspelled a name.



This is hard. Let's do something simple and more relevant to physics. Idea: Replace π_*^s with other gerneralized homology theories such that E_2-pages satisfies

$$H\mathbb{Z} \qquad \underbrace{k\mathrm{o}, k\mathrm{u}}_{\text{connective } K\text{-theory}} \qquad , tmf, tmf_1(3)$$

These compute stuff we want! $H\mathbb{Z}$ approximates Ω_*^{SO} (isomorphism in degrees ≤ 3); $k\mathrm{u}$ approximates $\Omega_*^{\mathrm{Spin}^c}$ (isomorphism in degrees ≤ 3); $k\mathrm{o}$ approximates Ω_*^{Spin} (isomorphism in degrees ≤ 5); tmf approximates $\Omega_*^{\mathrm{String}}$ (isomorphism in degrees ≤ 15); and unknown for $tmf_1(3)$.

Remark 4.1.2. Why would we want to compute bordism groups? Freed-Hopkins showed that reflection-positive invertible TFTs are classified by bordism groups. This means anaolalies, SPTs are classified in terms of bordism groups.

Example 4.1.3. Fermionic phases SPT with G-symmetry are classified by $\Omega_*^{\mathrm{Spin}}(BG)$ (under mild technical conditions on the symmetry).

What does the Adams spectral sequence mean in physics? So in particular what is an intermpretation fo the Adams SS data in terms of physics/invertible field theory/spins..

Now the the simplification! The Change-of-Rings Theorem says that

$$\mathrm{Ext}_B\left(B \otimes_C M, N\right) \cong \mathrm{Ext}_C(M, N).$$

For example, note that

$$H^*(H\mathbb{Z}; \mathbb{Z}/2) \cong A \otimes_{A[0]} \mathbb{Z}/2$$

so by Künneth we get

$$H^*(H\mathbb{Z} \wedge X; \mathbb{Z}/2) \cong A \otimes_{A[0]} H^*(X; \mathbb{Z}/2)$$

where

$$A[0] = \frac{(\mathbb{Z}/2)[\mathrm{Sq}^1]}{(\mathrm{Sq}^1)^2}.$$

Then by applying the Change-of-Rings, we get an Adams SS with

$$E_2^{s,t} = \mathrm{Ext}_{A[0]}^{s,t}\left(H^*(X; \mathbb{Z}/2), \mathbb{Z}/2\right) \Rightarrow H_{t-s}(X; \mathbb{Z})_2^\wedge$$

Likewise

$$H^*(k\,\mathrm{o};\mathbb{Z}/2) = A \otimes k[1] \otimes \mathbb{Z}/2$$

where

$$A[1] = \langle \mathrm{Sq}^1, \mathrm{Sq}^2 \rangle.$$

There is a cool visual drawing of this algebra that I missed, but $A[1]$ is an 8-dimensional noncommutative algebra. Then

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{A[1]}\left(H^*(A;\mathbb{Z}/2), \mathbb{Z}/2\right) \Rightarrow k\,\mathrm{o}_{t-s}(A)^{\wedge}_2$$

Similarly, for $k\,\mathrm{u}$ we get $H^*(k\,\mathrm{u}) = A \otimes_{E[1]}$ where $E[1] = \langle Q_1, Q_4 \rangle$ which simplifies the story by using $\mathrm{Ext}_{E[1]}$. also $H^*(tmf) = A \otimes_{A[2]} \mathbb{Z}/2$ where $A[2] = \langle \mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^4 \rangle$.

Example 4.1.4. Consider $k\,\mathrm{u}_*(M)$ and

$$\mathrm{Ext}^{s,t}_{E[1]}(\mathbb{Z}/2, \mathbb{Z}/2) \Rightarrow k\,\mathrm{u}_{t-s}(*).$$

Then we get an "Ext chart" which I'm not able to draw in time. $d_n = 0$ always so all differentials vanish and E_2 is thus E_∞ the sequence thus collapses and the extension gets hard. In this case h_0 lifts to multiplication by 2 in the characteristic zero setting while the tower of h_0's must be a $\mathbb{Z}$ if you know your group is finitely generated [5]. Thus

$$k\,\mathrm{u}_* = \mathbb{Z}, 0, \mathbb{Z}, 0, \mathbb{Z}, 0, \cdots$$

In an Ext chart you can have differentials commute with h_0-actions.

Example 4.1.5. Using the Ext chart for $k\,\mathrm{o}_*$ we get that the differentials also collapse, Our extension problem has the pattern of the Bott song and learned the first seven spin bordism to get $\Omega_k^{\mathrm{Spin}} \cong k\,\mathrm{o}_k$ for $k \leq 7$.

5 Lecture 5: 2023 May 5, Revenge of the Fifth

5.1 Day 5: Twists

Yesterday: We used the Adams spectral sequence and change of rings to compute

$$\Omega_*^{\mathrm{SO}}(X), \text{approximated by } H_*(X;\mathbb{Z}), \text{computed by } \mathrm{Ext}_{A(0)}\left(H^*(X;\mathbb{Z}/2), \mathbb{Z}/2\right)$$

Bordism Group	Approximated by	Computed by
$\Omega_*^{\mathrm{Spin}^c}(X)$	$k\,\mathrm{u}_*(X)$	$\mathrm{Ext}_{E(1)}$
$\Omega_*^{\mathrm{Spin}}(X)$	$k\,\mathrm{o}_*(X)$	$\mathrm{Ext}_{A(1)}$
$\Omega_*^{\mathrm{String}}(X)$	$\mathrm{tmf}_*(X)$	$\mathrm{Ext}_{A(2)}$

But lots of things are missing from this story. For example, "$\mathbb{Z}_4^{TF}$ (mod 4) symmetry" T such that T is anti-unitya, i.e., $T^2 = -1)^F, T^4 = 1$. However, cannot write the corresponding tangential as spin but you should get Pin +.

Remark 5.1.1. O is to SO what $\mathrm{Pin}^{\pm}$ is to Spin but is Tring? to String? >:(

But is it the case that

$$\Omega_*^{\mathrm{Pin}+} \stackrel{?}{=} \Omega_*^{\mathrm{Spin}}(X)$$

hols? Not quite... but it is true for *twisted* spin bordism and twisted spin bordism can be computed with AHSS and Adams SS in nearly the same way! It's also true for twisted SO, Spin, String...

[5] Otherwise you would get a $\mathbb{Z}_2$ — I think this appeal can be made due to the fact that $\mathbb{Z}$ is dense in $\mathbb{Z}_2$ and $\mathbb{Z}$ finitely generated? This is Geoff's comment only.

5.2 What is Twisted Spin Bordism?

Definition 5.2.1. Pick a space X and a vector bundle V on X. An (X,V)-twisted spin structure on a vector bundle $E \to M$ is the data of:

1. A map $f : M \to X$;
2. A spin structure on $E \oplus f^*V \to M$.

Remark 5.2.2. You can define twisted orientation, twisted spin^c structure, or any wtwisted G structure for any G in the same way. The point is to twist by G-bundles!

The bordism groups of (X,V)-twisted spin structures are written

$$\Omega_{*+V}^{\mathrm{Spin}}(X).$$

Example 5.2.3 (($BU(1), \mathcal{L}_{\mathrm{taut}}$)-twisted Spin structure)**.** Consider

$$\begin{array}{ccc} & & \mathcal{L}_{\mathrm{taut}} \\ & & \downarrow \\ M & \xrightarrow[f]{} & B\,\mathrm{U}(1) \end{array}$$

Now $TM \oplus f^*\mathcal{L}_{\mathrm{taut}}$ is spin if and only if there is the data of a cpa line bundle in characteristic classes. The whitney sm gives

$$w_2(TM) = w_2(L) = w_2(L) \pmod 2$$

and $c_1(L) \iff L$. So a $(B\,\mathrm{U}(1), L_{\mathrm{taut}})$-twisted spin structure is the data of an integer lift of w_2, i.e., a Spin^c structure.

Other examples:

G	Twist of Spin Bordism
$\mathrm{Pin}^+, w_2 = 0$	$(B\,\mathbb{Z}/2, 3\,\mathrm{sgn})$
$\mathrm{Pin}^-, w_2 = w_1^2$	$(B\,\mathbb{Z}/2, \mathrm{sgn})$
$\mathrm{Pin}^c, \beta(w_2) = 0$ $\mathrm{Spin} \times_{\{\pm 1\}} \mathbb{Z}/2n, \omega_2(n) = w_2(\beta)$	$(B\,\mathbb{Z}/2, \mathrm{sgn})$-twisted Spin^c structure $(B2/n, \mathbb{C})$
$\mathrm{Spin} \times_{\{\pm 1\}} \mathrm{SU}_2, w_2(m) = w_2(E)$ for E an SO_3 bundle	$(B\,\mathrm{SO}_3, V_{\mathrm{taut}})$

Remark 5.2.4. The twisted spin spin bordism in condesed matter says that $w_1 =?$ if and only if $G_b \to \mathbb{Z}/2$ is unitary or antiunitary. So $w_2 =?$ if and only if there is an extension

$$1 \longrightarrow (\mathbb{Z}/2)^F \longrightarrow G_f \longrightarrow C_3 \longrightarrow 1$$

The classifying space for (X,V)-twisted spin structure is

$$\begin{array}{ccc} & & B \\ & \nearrow & \downarrow \\ M & \xrightarrow[E]{} & B\,\mathrm{O} \end{array}$$

We'd like that a lift of E to be exactpyl an (X,V)-twisted spin structure on E. The answer iis we need to write

$$\begin{array}{c} B\,\mathrm{Spin} \times X \\ \big\downarrow {\scriptstyle V_{\mathrm{taut}} \oplus (V - \mathrm{rank}(V))} \\ B\,\mathrm{O} \end{array}$$

By the Pontryagin-Thom Theorem, we know how to access the (X, V)-twisted Spin bordism groups: they are the homotopy groups of the Thom spectrum

$$\mathrm{Thom}\big(\mathrm{B\,Spin} \times \mathrm{X}, \mathrm{V_{taut}} \oplus (\mathrm{V\text{-}\,rank(V)})\big)\,.$$

But because our boundle is an extermal direct sum, i.e., a direct sum of osmething on $B\,\mathrm{Spin}$ and something on X, the Thom spectrum factors and we learn

$$\Omega^{\mathrm{Spin}}_{*+V}(X) \xrightarrow{\cong} \Omega^{\mathrm{Spin}}_{*}(X^{V-\mathrm{rank}\,V})$$

where the object $X^{V-\mathrm{rank}\,V}$ is the spectrum of X and V (in the homotopy sense of spectrum).

Example 5.2.5. Whever $BZ/2)^{3\,\mathrm{sgn}\,-2}$ is, its Spin bordism groups are equal to the Pin^{+} bordism of a point.

Remark 5.2.6. We can use the Thom spectrum for other types of twisted generalized cohomology theories.

We now want to feed $\Omega^{\mathrm{Spin}}_{*}(X^{V-\mathrm{rank}\,V})$ to AHSS or Adams. To do so, we need:

1. $H^*(X^{V-\mathrm{rank}\,V};\mathbb{Z}), H_*(X^{V-\mathrm{rank}\,V};\mathbb{Z})$;
2. $H^*(X^{V-\mathrm{rank}\,V};\mathbb{Z}/2), H_*(X^{V-\mathrm{rank}\,V};\mathbb{Z}/2)$;
3. The operators Sq^i on the mod 2 cohomology.

All of these are not hard! For Number 2: The Thom Isomorphism Theorem states that there is an isomorphism

$$H^*(X;\mathbb{Z}/2) \xrightarrow{\cong} H^*(X^{V-\mathrm{rank}\,V};\mathbb{Z}/2).$$

This is an isomorphism of $H^*(X;\mathbb{Z}/2)$-modules and the image of 1 is what is called the Thom class.

For Number 1, the isomorphism need an oriteation to work over $\mathbb{Z}$. In general, where ewe use twisted cohomology,

$$H^*(X;\mathbb{Z}_{w_1 as}) \xrightarrow{\cong} H^*(X^{V-\mathrm{rank}\,V};\mathbb{Z})$$

and then proceeds mutatis mutandis.

For number 3 we use the W_u formula to get

$$\mathrm{Sq}^i(U) = Uw_i(V), \quad \mathrm{Sq}^i(Ux) = \sum_{j+k=i} Uw_j(V)\,\mathrm{Sq}^k(U).$$

Example 5.2.7 **(The AHSS for $\Omega^{\mathrm{Pin}+}_{*} = \Omega^{\mathrm{Spin}}_{*}(B\,\mathbb{Z}/2)^{3\,\mathrm{sgn}\,-2}$).** So from the generating equations I missed we get the diagram:

	q					
$\mathbb{Z}$	4	$\mathbb{Z}/2$		$\mathbb{Z}/2$		$\mathbb{Z}/2$
	3					
$\mathbb{Z}/2$	2	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$\mathbb{Z}/2$	1	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$\mathbb{Z}$	0	$\mathbb{Z}/2$		$\mathbb{Z}/2$		$\mathbb{Z}/2$
		0	1	2	3	4

Because this is a horizontal SS the differential goes up and so the is a left and up knight's move. Now the inclusion of the line $p = 0$ is the edge hom and so d_2 in row 0 is an isomorphism. Geometric information I missed says

$$\Omega^{\mathrm{Spin}}_1 \to \Omega^{\mathrm{pin}+}_1$$

is an isomorphism and so d_2 in row 1 is an isomorphism as well. We can show, however, that

$$\Omega^{\mathrm{Pin}+}_0 = \mathbb{Z}/2, \Omega^{\mathrm{Pin}+}_1 = 0, \Omega^{\mathrm{Pin}+}_2 = \mathbb{Z}/2, \Omega^{\mathrm{Pin}+}_3 = ??, \Omega^{\mathrm{Pin}+}_4 = ??$$

Another option: Adams. A drawing of the Sq^1 and Sq^2 actions tell us that we can modify by a twist. So we get cool stuff I wrote by hand bout could not TEX. Allows us to deduce that

$$\Omega^{\mathrm{Pin}+}_3 = \mathbb{Z}/2, \quad \Omega^{\mathrm{Pin}+}_4 = \mathbb{Z}/16..$$