Spectral Sequences Exercises, Day 1

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1. The quaternionic projective space $\mathbb{HP}^n$ is the space of (quaternion) one-dimensional subspaces of $\mathbb{H}^{n+1}$, where $\mathbb{H}$ is the unit quaternions. The fiber bundles

$$\begin{tikzcd} \mathbb{Z}/2 \arrow[r] & S^n \text{ (unit sphere in } \mathbb{R}^{n+1}) \arrow[d] \\ & \mathbb{RP}^n \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} U(1)=S^1 \arrow[r] & S^{2n+1} \text{ (unit sphere in } \mathbb{C}^{n+1}) \arrow[d] \\ & \mathbb{CP}^n \end{tikzcd}$$

have a third cousin, the fiber bundle

$$\begin{tikzcd} SU(2)=Sp(1)=S^3 \arrow[r] & S^{4n+3} \text{ (unit sphere in } \mathbb{H}^{n+1}) \arrow[d] \\ & \mathbb{HP}^n. \end{tikzcd}$$

Use the Serre spectral sequence for this fiber bundle to show $\quad H^*(\mathbb{HP}^n;\mathbb{Z}) \cong \mathbb{Z}[x]/(x^{n+1})$, $|x|=4$.

2. Let ζ_k be a primitive k^{th} root of unity and let $\mathbb{Z}/k$ act on $\mathbb{C}^n$ by
$$(z_1,\dots,z_n) \longmapsto (\zeta_k z_1, \zeta_k z_2, \dots, \zeta_k z_n).$$
Restricted to $S^{2n-1} \subset \mathbb{C}^n$, this is a free action; the quotient, denoted L_k^{2n-1}, is called a *lens space*.

Each $\mathbb{Z}/k$-orbits in S^{2n-1} is contained in a one-dimensional subspace, defining a map $L_k^{2n-1} \to \mathbb{CP}^{n-1}$; the preimage of each point is $\underset{\substack{\uparrow \\ \text{unit complex numbers}}}{S^1}/(\mathbb{Z}/k) \cong S^1$. This defines a fiber bundle

$$\begin{tikzcd} S^1 \cong S^1/(\mathbb{Z}/k) \arrow[r] & L_k^{2n-1} \arrow[d] \\ & \mathbb{CP}^{n-1}. \end{tikzcd}$$

Use the Serre spectral sequence for this fiber bundle to show
$$H^*(L_k^{2n-1};\mathbb{Z}) \cong \mathbb{Z}[c,z]/(kc, c^n, cz, z^2), \quad |c|=2,\ |z|=2n-1.$$
$$H^*(L_k^{2n-1};\mathbb{Z}/k) \cong \mathbb{Z}/k[x,c]/(x^2, c^n), \quad |x|=1,\ |c|=2.$$
(Note: $L_2^{2n-1} = \mathbb{RP}^{2n-1}$.)

3. Use the loop-space–path-space fibration

$$\begin{tikzcd} \Omega X \arrow[r] & \mathrm{Path}(X) \simeq * \arrow[d] \\ & X \end{tikzcd}$$

to compute $H^*(\Omega S^3;\mathbb{Q})$ and $H^*(\Omega S^4;\mathbb{Q})$.
(Bonus: what is the ring structure of $H^*(\Omega S^3;\mathbb{Z})$ and $H^*(\Omega S^4;\mathbb{Z})$?)

4. The unitary group U_n acts on $\mathbb{C}^n$, hence also on S^{2n-1}. The stabilizer of a point $x \in S^{2n-1}$ is isomorphic to U_{n-1} (can you show this?). Therefore there is a fiber bundle

$$\begin{tikzcd} U_{n-1} \arrow[r] & U_n \arrow[d] \\ & S^{2n-1}. \end{tikzcd}$$

Use the Serre spectral sequence for this fiber bundle to inductively show

$$H^*(U_n;\mathbb{Z}) \cong \mathbb{Z}[c_1, c_3, c_5, \dots, c_{2n-1}]/(c_1^2, c_3^2, c_5^2, \dots, c_{2n-1}^2), \quad |c_{2i-1}| = 2i-1.$$

(i.e. $H^*(U_n;\mathbb{Z})$ is an exterior algebra on odd-degree generators.)
Base case: $U_1 \cong S^1$.

5. Can you run the same argument as in problem 4, but with $Sp_n \curvearrowright \mathbb{H}^n$ and S^{4n-1}, to show $H^*(Sp_n;\mathbb{Z}) \cong \mathbb{Z}[c_3, c_7, c_{11}, \dots, c_{4n-1}]/(c_3^2, c_7^2, \dots, c_{4n-1}^2)$, $|c_{4i-1}| = 4i-1$? (Base case: $Sp_1 \cong S^3$.) What about SU_n?

6. Using what you learned in problems 4 and 5 and the fibrations

$$\begin{tikzcd} U_n \arrow[r] & EU_n \simeq * \arrow[d] \\ & BU_n \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Sp_n \arrow[r] & ESp_n \simeq * \arrow[d] \\ & BSp_n \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} SU_n \arrow[r] & ESU_n \simeq * \arrow[d] \\ & BSU_n, \end{tikzcd}$$

what can you deduce about $H^*(BU_n;\mathbb{Z})$, $H^*(BSp_n;\mathbb{Z})$, and $H^*(BSU_n;\mathbb{Z})$?

7. Let $\begin{tikzcd} V \arrow[d] \\ X \end{tikzcd}$ be a rank-n vector bundle, so that the unit sphere bundle $S(V)$ belongs to a fiber bundle

$$\begin{tikzcd} S^{n-1} \arrow[r] & S(V) \arrow[d, "\pi"] \\ & X. \end{tikzcd}$$

Assume X is simply connected.
Use the Serre spectral sequence for this fiber bundle, and its edge homomorphisms, to deduce the Gysin long exact sequence

$$\cdots \to H^k(X;\mathbb{Z}) \to H^{k+n}(X;\mathbb{Z}) \xrightarrow{\pi^*} H^{k+n}(S(V);\mathbb{Z}) \to H^{k+1}(X;\mathbb{Z}) \to \cdots$$

Can you identify the other two maps in this long exact sequence?