SPECTRAL SEQUENCES PROBLEM SET 2

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(1) The loop-space-path-space fibration $\Omega X \to \mathcal{P}X \to X$, together with the identification $\Omega K(A,n) \simeq K(A,n-1)$ and the contractability of $\mathcal{P}(X)$, leads to a fibration $K(A,n-1) \to * \to K(A,n)$. Use the Serre spectral sequence associated to this fibration to show that

$$H^*(K(\mathbb{Z},3);\mathbb{Q}) \cong \mathbb{Q}[c]/(c^2), \qquad |c| = 3.$$

(2) Use the Serre spectral sequence for the same fibration to show that

$$H^*(K(\mathbb{Z},4);\mathbb{Q}) \cong \mathbb{Q}[d], \quad |d| = 4.$$

(3) A *2-group* $\mathbb{G}$ is a group object in the 2-category of categories, characterized by the data of a group $G \coloneqq \mathrm{Obj}(\mathbb{G})$, an abelian group $A \coloneqq \mathrm{Hom}_{\mathbb{G}}(e,e)$, an action of G on A, and a class $k \in H^3(BG;A)$ called the *k-invariant*. This data makes the classifying space of $\mathbb{G}$ into a fiber bundle over BG with fiber B^2A:

$$\begin{array}{ccc} B^2A & \longrightarrow & B\mathbb{G} \\ & & \downarrow \\ & & BG. \end{array} \tag{0.1}$$

For any Eilenberg-Mac Lane space $K(A,n)$, there is a natural identification $[X, K(A,n)] \cong H^n(X;A)$ given by pulling back the tautological class $t \in H^n(K(A,n);A)$ that we defined in lecture using the Hurewicz homomorphism.

Suppose that in the data for a 2-group $\mathbb{G}$, G acts trivially on A, so that the Serre spectral sequence does not need local coefficients and is multiplicative. Suppose also that A is discrete, so $B^2A = K(A,2)$. Show that in the Serre spectral sequence, the tautological class $t \in E_2^{0,2} = H^2(K(A,2);2)$ transgresses to $k \in H^2(BG;A)$.

(4) If $A = \mathrm{U}_1$, so $BA = B^2\mathbb{Z}$, the same story takes place in one degree higher: the k-invariant is valued in $H^4(BG;\mathbb{Z})$, and the tautological class in $H^3(K(\mathbb{Z},3);\mathbb{Z})$ transgresses to the k-invariant in the fiber bundle (0.1).

Assume $n \geq 5$. The *string group* String_n is the 2-group for which $G = \mathrm{Spin}_n$, $A = \mathrm{U}_1$, the G-action on A is trivial, and k is the generator of $H^4(B\mathrm{Spin}_n;\mathbb{Z}) \cong \mathbb{Z}$. There is an isomorphism $H^*(B\mathrm{Spin}_n;\mathbb{Q}) \cong \mathbb{Q}[p_1,\dots,p_n]$ with $|p_i| = 4i$. What can you determine about $H^*(B\mathrm{String}_n;\mathbb{Q})$?

(5) In class, we used the Serre spectral sequence of a sphere bundle to produce the Gysin long exact sequence. Suppose instead we are given a fibration

$$\begin{array}{ccc} F & \longrightarrow & E \\ & & \downarrow \\ & & S^n. \end{array} \tag{0.2}$$

Assume for simplicity $n > 1$. Use the Serre spectral sequence associated to this fibration to deduce the following exact sequences:

$$0 \to E_\infty^{0,k} \longrightarrow H^k(F;A) \xrightarrow{d_n} H^{k-n+1}(F;A) \longrightarrow E_\infty^{n,k-n+1} \longrightarrow 0.$$
$$0 \longrightarrow E_\infty^{n,k-n} \longrightarrow H^k(E;A) \longrightarrow E_\infty^{0,k} \longrightarrow 0.$$

Paste these two together to obtain the *Wang long exact sequence*:

$$\cdots \to H^k(F;A) \to H^{k-n+1}(F;A) \to H^{k+1}(E;A) \to H^{k+1}(F) \to \cdots$$

(6) The *Wu manifold* is the quotient $\mathrm{SU}_3/\mathrm{SO}_3$, which is a compact orientable smooth manifold. There is a diffeomorphism $\mathrm{SO}_3 \cong \mathbb{RP}^3$, meaning $H^*(\mathrm{SO}_3; \mathbb{Z}/2) \cong \mathbb{Z}/2[x]/(x^4)$. Describe $H^*(\mathrm{SU}_3/\mathrm{SO}_3; \mathbb{Z}/2)$. Can you compare the map of Serre spectral sequences for the fibrations $\mathrm{SO}_3 \to \mathrm{SU}_3 \to \mathrm{SU}_3/\mathrm{SO}_3$ with the universal SO_3-bundle $\mathrm{SO}_3 \to E\mathrm{SO}_3 \to B\mathrm{SO}_3$ to determine some characteristic classes of the principal SO_3-bundle $\mathrm{SU}_3 \to \mathrm{SU}_3/\mathrm{SO}_3$?