# SPECTRAL SEQUENCES PROBLEM SET 3

ARUN DEBRAY
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(1) Recall that $H^*(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}[c]/(c^{n+1})$ with $|c| = 2$. This and the universal coefficient theorem imply $H^*(\mathbb{CP}^n;\mathbb{Z}/2) \cong \mathbb{Z}/2[\overline{c}]/(\overline{c}^{n+1})$, where $\overline{c} \coloneqq c \bmod 2$. Determine $\mathrm{Sq}(\overline{c})$, $\mathrm{Sq}(\overline{c}^2)$, and $\mathrm{Sq}(\overline{c}^3)$.

(2) Similarly, if $k$ is even, the mod 2 cohomology of the lens space $L_k^{2n-1}$ is $H^*(L_k^{2n-1};\mathbb{Z}/2) \cong \mathbb{Z}/2[x,y]/(x^2,y^n)$, with $|x| = 1$ and $|y| = 2$, where $y$ is the pullback of $\overline{c} \in H^2(\mathbb{CP}^{n-1};\mathbb{Z}/2)$ via the quotient $L_k^{2n-1} \to \mathbb{CP}^{n-1}$ from the first problem set. (How much of this can you see from the Serre spectral sequence? What changes if $k$ is odd?) Determine $\mathrm{Sq}(x)$ and $\mathrm{Sq}(y)$.

(3) Use the Atiyah-Hirzebruch spectral sequence to show that $K^*(\mathrm{SU}_3) \cong H^*(\mathrm{SU}_3;\mathbb{Z}) \otimes K^*(\mathrm{pt})$. (Recall $H^*(\mathrm{SU}_3;\mathbb{Z}) \cong \mathbb{Z}[c_3, c_5]/(c_3^2, c_5^2)$ with $|c_i| = i$.)

(4) Try running the Atiyah-Hirzebruch spectral sequence for $K^*(\mathbb{RP}^3)$ and $K^*(\mathbb{RP}^4)$. Are there any differentials? Are there extension problems? For reference: $H^*(\mathbb{RP}^3;\mathbb{Z}) \cong \mathbb{Z}[x,y]/(2x, x^2, y^2, xy)$ with $|x| = 2$ and $|y| = 3$, and $H^*(\mathbb{RP}^4;\mathbb{Z}) \cong \mathbb{Z}[x]/(2x, x^3)$ with $|x| = 2$.

(5) Compute $\Omega_k^{\mathrm{SO}}(B\mathbb{Z}/2)$ for $k \le 6$. (Recall $H_*(B\mathbb{Z}/2;\mathbb{Z})$ consists of a $\mathbb{Z}$ in degree 0 together with a $\mathbb{Z}/2$ in every positive odd degree, and $H_*(B\mathbb{Z}/2;\mathbb{Z}/2)$ consists of a $\mathbb{Z}/2$ in every nonnegative degree, and that $\Omega_*^{\mathrm{SO}}$ begins with $\mathbb{Z}$ in degrees 0 and 4, $\mathbb{Z}/2$ in degree 5, and no other summands below degree 8.) Compare $\widetilde{\Omega}_k^{\mathrm{SO}}(B\mathbb{Z}/2)$ and $\Omega_{k-1}^{\mathrm{O}}$ — their similarity is not a coincidence!

(6) Use the Atiyah-Hirzebruch spectral sequence to obtain a long exact sequence for supercohomology:

$$\cdots \to H^k(X;\mathbb{Z}) \to \mathrm{SH}^k(X) \to H^{k-2}(X;\mathbb{Z}/2) \xrightarrow{\mathrm{Bock}\circ\mathrm{Sq}^2} H^{k+1}(X;\mathbb{Z}) \to \cdots$$