SPECTRAL SEQUENCES PROBLEM SET 4

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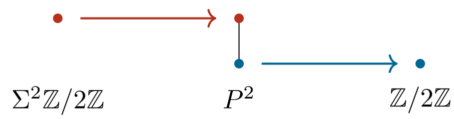
(1) Determine the $\mathcal{A}(0)$-module structures on $H^*(\mathbb{RP}^2;\mathbb{Z}/2)$ and $H^*(\mathbb{RP}^\infty;\mathbb{Z}/2)$. (Recall $H^*(\mathbb{RP}^n;\mathbb{Z}/2)\cong\mathbb{Z}/2[x]/(x^{n+1})$ and $H^*(\mathbb{RP}^\infty;\mathbb{Z}/2)\cong\mathbb{Z}/2[x]$, with $|x|=1$ in both cases.)

(2) Use the $\mathcal{A}(0)$-module structures from the previous problem, together with the Ext charts for $\mathcal{A}(0)$-modules in the reference info below, to run the Adams spectral sequence over $\mathcal{A}(0)$ to compute $H_*(\mathbb{RP}^2;\mathbb{Z})^\wedge_2$ and $H_*(\mathbb{RP}^\infty;\mathbb{Z})^\wedge_2$.

(3) Let P^2 denote the $\mathcal{E}(1)$-module consisting of two $\mathbb{Z}/2$ summands in degrees 0 and 1 joined by a nonzero Sq^1-action from degree 0 to degree 1. Then $\Sigma P^2\cong\widetilde{H}^*(\mathbb{RP}^2;\mathbb{Z}/2)$. The short exact sequence of $\mathcal{E}(1)$-modules $\Sigma\mathbb{Z}/2\to P^2\to\mathbb{Z}/2$, drawn line this:

(0.1)



induces a long exact sequence in $\mathrm{Ext}_{\mathcal{E}(1)}(-,\mathbb{Z}/2)$. Run this long exact sequence (draw the Ext chart!) to show that

$$\mathrm{Ext}_{\mathcal{E}(1)}(P^2;\mathbb{Z}/2)\cong\mathbb{Z}/2[v_1],$$

where $v_1\in\mathrm{Ext}^{1,3}$. See the reference information below for information on the long exact sequence in Ext.

(4) Use the Ext computation from the previous problem to run the Adams spectral sequence over $\mathcal{E}(1)$ computing $ku_*(\mathbb{RP}^2)$. Can you infer some spin^c bordism groups of $\mathbb{RP}^2$?

(5) Try repeating the previous two questions over $\mathcal{A}(1)$ instead of $\mathcal{E}(1)$, and computing $ko_*(\mathbb{RP}^2)$ and $\Omega_*^{\mathrm{Spin}}(\mathbb{RP}^2)$ (the latter in degrees 7 and below).

1. Reference information

Some modules and Ext groups:

- $\mathcal{A}(0)=\langle\mathrm{Sq}^1\rangle=\mathbb{Z}/2[\mathrm{Sq}^1]/((\mathrm{Sq}^1)^2)$. $H^*(H\mathbb{Z};\mathbb{Z}/2)\cong\mathcal{A}\otimes_{\mathcal{A}(0)}\mathbb{Z}/2$, so by the change-of-rings theorem there is an Adams spectral sequence

$$E_2^{s,t}=\mathrm{Ext}_{\mathcal{A}(0)}(H^*(X;\mathbb{Z}/2),\mathbb{Z}/2)\Longrightarrow H_{t-s}(X;\mathbb{Z})^\wedge_2.$$

$\mathrm{Ext}_{\mathcal{A}(0)}(\mathcal{A}(0),\mathbb{Z}/2)$ consists of a single $\mathbb{Z}/2$ at $(s,t)=(0,0)$; $\mathrm{Ext}_{\mathcal{A}(0)}(\mathbb{Z}/2,\mathbb{Z}/2)\cong\mathbb{Z}/2[h_0]$ with $h_0\in\mathrm{Ext}^{1,1}$. The map $\Omega_k^{\mathrm{SO}}(X)\to H_k(X;\mathbb{Z})$ is an isomorphism in degrees 3 and below.

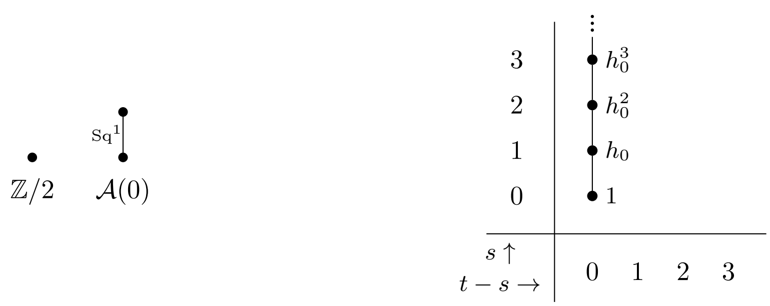


FIGURE 1. Left: common $\mathcal{A}(0)$-modules. Right: $\mathrm{Ext}_{\mathcal{A}(0)}(\mathbb{Z}/2,\mathbb{Z}/2)\cong\mathbb{Z}/2[h_0]$ with $h_0\in\mathrm{Ext}^{1,1}$.

- $\mathcal{E}(1) = \langle Q_0, Q_1\rangle = \mathbb{Z}/2[Q_0, Q_1]/(Q_0^2, Q_1^2)$, where $Q_0 = \mathrm{Sq}^1$ and $Q_1 = \mathrm{Sq}^1\mathrm{Sq}^2 + \mathrm{Sq}^2\mathrm{Sq}^1$. $H^*(ku;\mathbb{Z}/2) \cong \mathcal{A}\otimes_{\mathcal{E}(1)}\mathbb{Z}/2$, so by the change-of-rings theorem there is an Adams spectral sequence
$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{E}(1)}(H^*(X;\mathbb{Z}/2),\mathbb{Z}/2) \Longrightarrow ku_{t-s}(X)^\wedge_2.$$
$\mathrm{Ext}_{\mathcal{E}(1)}(\mathcal{E}(1),\mathbb{Z}/2)$ consists of a single $\mathbb{Z}/2$ at $(s,t) = (0,0)$; $\mathrm{Ext}_{\mathcal{E}(1)}(\mathbb{Z}/2,\mathbb{Z}/2) \cong \mathbb{Z}/2[h_0, v_1]$ with $h_0 \in \mathrm{Ext}^{1,1}$ and $v_1 \in \mathrm{Ext}^{1,3}$. The map $\Omega_k^{\mathrm{Spin}^c}(X) \to ku_k(X)$ is an isomorphism in degrees 3 and below.

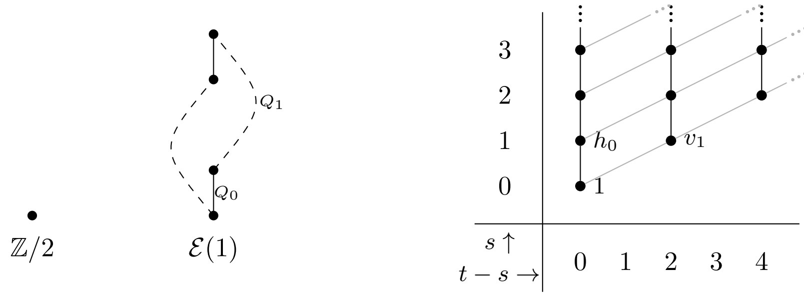


FIGURE 2. Left: common $\mathcal{E}(1)$-modules. Right: $\mathrm{Ext}_{\mathcal{E}(1)}(\mathbb{Z}/2,\mathbb{Z}/2) \cong \mathbb{Z}/2[h_0, v_1]$ with $h_0 \in \mathrm{Ext}^{1,1}$ and $v_1 \in \mathrm{Ext}^{1,3}$.

- $\mathcal{A}(1) = \langle \mathrm{Sq}^1, \mathrm{Sq}^2\rangle$, or explicitly
$$\mathcal{A}(1) \cong \mathbb{Z}/2\langle \mathrm{Sq}^1, \mathrm{Sq}^2\rangle/((\mathrm{Sq}^1)^2 = 0, \mathrm{Sq}^2\mathrm{Sq}^2 = \mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^1, \mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^2 = \mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^1, \mathrm{Sq}^2\mathrm{Sq}^2\mathrm{Sq}^1 = 0)$$
(it's far easier to think of this in terms of the picture Figure 3). $H^*(ko;\mathbb{Z}/2) \cong \mathcal{A}\otimes_{\mathcal{A}(1)}\mathbb{Z}/2$, so by the change-of-rings theorem there is an Adams spectral sequence
$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}(1)}(H^*(X;\mathbb{Z}/2),\mathbb{Z}/2) \Longrightarrow ko_{t-s}(X)^\wedge_2.$$
$\mathrm{Ext}_{\mathcal{A}(1)}(\mathcal{A}(1),\mathbb{Z}/2)$ consists of a single $\mathbb{Z}/2$ at $(s,t) = (0,0)$;
$$\mathrm{Ext}_{\mathcal{A}(1)}(\mathbb{Z}/2,\mathbb{Z}/2) \cong \mathbb{Z}/2[h_0, h_1, v, w]/(h_0h_1, h_1^3, vh_1, h_0^2w - h_1^2),$$
with $h_0 \in \mathrm{Ext}^{1,1}$ and $h_1 \in \mathrm{Ext}^{1,2}$, $v \in \mathrm{Ext}^{3,7}$, and $w \in \mathrm{Ext}^{4,12}$ (it's far easier to think of this in terms of the picture Figure 3). The map $\Omega_k^{\mathrm{Spin}}(X) \to ko_k(X)$ is an isomorphism in degrees 7 and below.

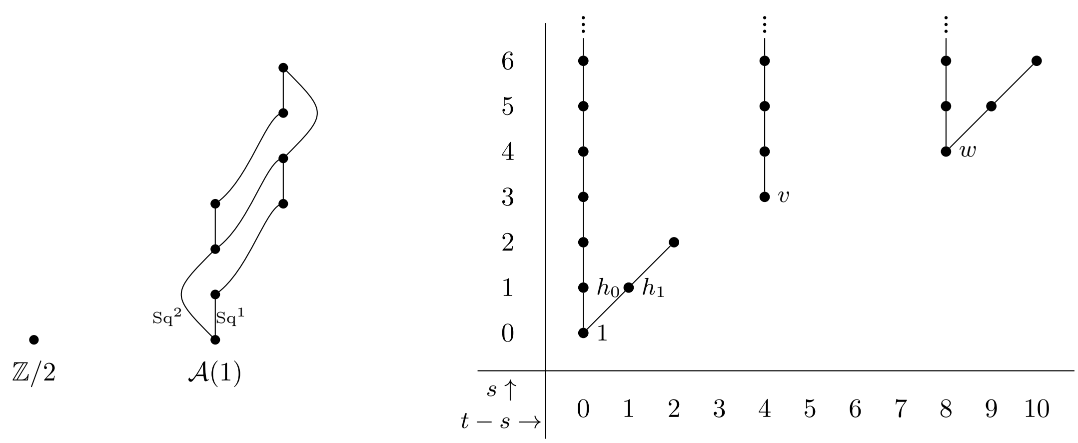


FIGURE 3. Left: common $\mathcal{A}(1)$-modules. Right: $\mathrm{Ext}_{\mathcal{A}(1)}(\mathbb{Z}/2,\mathbb{Z}/2)$.

If B is a graded $\mathbb{Z}/2$-algebra and $0 \to L \to M \to N$ is a short exact sequence of B-modules, there is a long exact sequence
(1.1)
$$\cdots \longrightarrow \mathrm{Ext}_B^{s,t}(N,\mathbb{Z}/2) \longrightarrow \mathrm{Ext}_B^{s,t}(M,\mathbb{Z}/2) \longrightarrow \mathrm{Ext}_B^{s,t}(L,\mathbb{Z}/2) \xrightarrow{\delta} \mathrm{Ext}_B^{s+1,t}(N,\mathbb{Z}/2) \longrightarrow \cdots$$

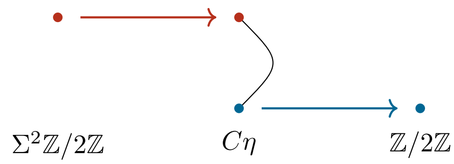
This is a common way to compute $\mathrm{Ext}_B(M, \mathbb{Z}/2)$ in terms of $\mathrm{Ext}_B(L, \mathbb{Z}/2)$ and $\mathrm{Ext}_B(N, \mathbb{Z}/2)$, which usually are simpler.

For example, let $C\eta$ denote the $\mathcal{A}(1)$-module which consists of two $\mathbb{Z}/2$ summands in degrees 0 and 2, joined by Sq^2. Then there is a short exact sequence of $\mathcal{A}(1)$-modules

(1.2a) $$0 \longrightarrow \Sigma^2\mathbb{Z}/2\mathbb{Z} \longrightarrow C\eta \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0,$$

which looks like this:

(1.2b)

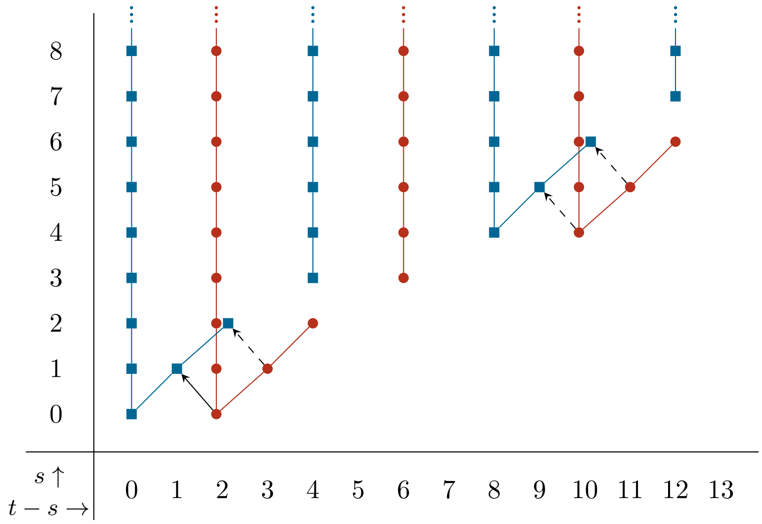


To run the long exact sequence in Ext for $0 \to L \to M \to N \to 0$, put $\mathrm{Ext}_B(L, \mathbb{Z}/2)$ and $\mathrm{Ext}_B(N, \mathbb{Z}/2)$ into the same Adams chart. The boundary maps in the LES have the same degree as if they were d_1s (i.e. one unit to the left, one unit upwards). The boundary maps commute with the action of $\mathrm{Ext}_{\mathcal{A}(1)}(\mathbb{Z}/2, \mathbb{Z}/2)$, e.g. they commute with the actions of h_0 and h_1 (the vertical and diagonal lines).

For $\mathrm{Ext}_{\mathcal{A}(1)}(C\eta, \mathbb{Z}/2)$, we begin by putting $\mathrm{Ext}_{\mathcal{A}(1)}(\Sigma^2\mathbb{Z}/2, \mathbb{Z}/2)$ and $\mathrm{Ext}_{\mathcal{A}(1)}(\mathbb{Z}/2, \mathbb{Z}/2)$ in one chart, which we draw in Figure 4, top. Most boundary maps in the LES are trivial for degree reasons; we draw the four that aren't.

To evaluate the boundary maps: because the long exact sequence commutes with the action of $\mathrm{Ext}(\mathbb{Z}/2\mathbb{Z})$, the three dashed boundary maps are determined by the solid boundary map, via the actions of h_1, w, and $h_1 w$. We will show the solid boundary map is an isomorphism, so that the dashed boundary maps are too.

To show the solid boundary map is an isomorphism, it suffices by exactness to show that $\mathrm{Ext}^{0,2}(C\eta) = 0$, since we already know $\mathrm{Ext}^{1,2}(\Sigma^2\mathbb{Z}/2\mathbb{Z}) = 0$. To show $\mathrm{Ext}^{0,2}(C\eta)$ vanishes, use that it is identified with $\mathrm{Hom}_{\mathcal{A}(1)}(C\eta, \Sigma^2\mathbb{Z}/2\mathbb{Z})$, which vanishes: since $C\eta$ is a cyclic $\mathcal{A}(1)$-module, maps out of $C\eta$ are determined by their values on the generator, which is in degree 0; since $\Sigma^2\mathbb{Z}/2\mathbb{Z}$ has no nonzero elements in degree 0, a map $C\eta \to \Sigma^2\mathbb{Z}/2\mathbb{Z}$ must vanish. This finishes the calculation; we draw the final answer in Figure 4, bottom.



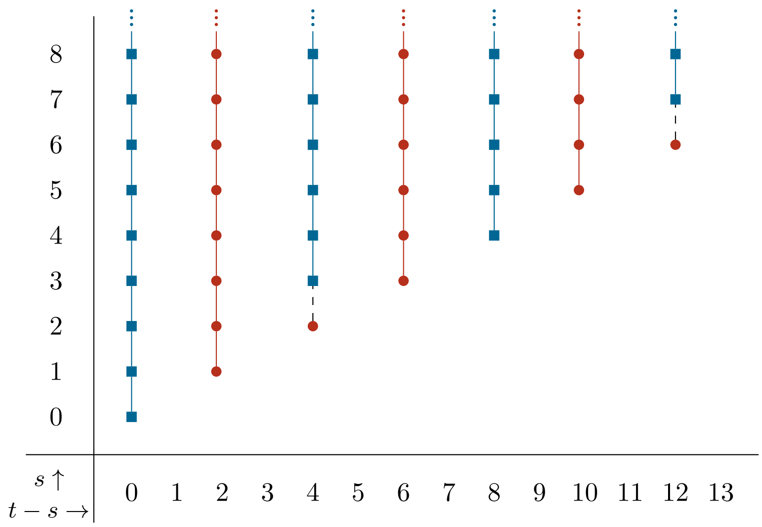


FIGURE 4. Top: the long exact sequence in Ext associated to the short exact sequence of $\mathcal{A}(1)$-modules (1.2). The solid and dashed arrows are boundary maps. Bottom: $\mathrm{Ext}(C\eta)$ as calculated by this long exact sequence. The dashed lines indicate h_0-actions not visible to the long exact sequence, which must be calculated another way.