Fusion Categories

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1 Lecture 1: 2023 May 1

Connections between fusion categories and TQFT. We understand TQFTs formally as functors

$$Z : \mathbf{Bord} \to \mathcal{S}$$

which are symmetric monoidal. For example, when $d = 2$, the objects in $\mathbf{Bord}_2$ are surfaces and the morphisms are manifolds M with boundaries Σ_1 and Σ_2. The symmetric monoidal structure on Z asks that

$$Z\left(\Sigma_1 \coprod \Sigma_2\right) = Z(\Sigma_1) \otimes Z(\Sigma_2).$$

The Bordism Hypothesis says that TQFTs are detremined by where they send a point, i.e., based on

$$\bullet \mapsto \text{Fully-dualizable object} X \in \mathcal{S}\,.$$

Definition 1.0.1. Fusion categories are the fully-dualizable objects in some category $\mathcal{S}$.

In what follows, $3D = (d+1)D$ where $d = 2$.

Type of Fusion Category	Type of TQFT
(Spherical) Fusion	$3D$ (Fully Extedend) Turaev-Viro TQFT
Modular Fusion	$3D$ Reshetikhin TQFT
GX Modular Fusion Category	$3D$ Homotopy Quantul Field Theory
	$4D$ Invertible Crave-Yetter Theory

Rouph Schedule:

1. Day Zero: Overview, motivation, and fusion rings. The idea of a fusion category and the decategorification of a fusion category.
2. Day One: Fusion categories; string diagrams; pivotal, sphereical, and unitary structures; module categories over fusion categories.
3. Day Two: Skeleatel fusion categories (in particulare the Turaev-Viro State Sum TQFT and Levin-Wen Hamiltonian FOR lattice TQFT)
4. Day Three: Drinfeld Centres of Fusion categories (in particular tube algebras) and braided fusion categories (Reshetikhin-Turaev TQFT and Topological order — bosnic, ermionic)
5. Day Four: Symmetries of Modular Fusion Categories, Syemmtry TQFTs, Enriched Fusion Categoies, Higher Fusion Categories.

1.1 Day Zero

The holistic idea of a fusion category is as follows: A fusion category is like a "quantum" finite group.

To compare:

Finite group $(G, \cdot)$	"Quantum" finite group $(\mathcal{F}, \otimes)$
Finite number of elements	finitely many indecomposable objects in $\mathscr{F}$ and $\mathcal{F}(X, Y)$ is a finite Hilbert space.
Binary operation $\mu : G \times G \to G$	Bifunctor $\otimes : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$
Unit $e \in G$ such that $eg = g = ge$ for all g	$\mathbb{1} \in \mathcal{F}$ such that $\mathbb{1} \otimes X \cong X \cong X \otimes \mathbb{1}$ for all $x \in \mathcal{F}$
Inverses for all $g \in G$ there is g^{-1} such that $gg^{-1} = e = g^{-1}g$	Inverses For $X \in \mathcal{F}$ there are duals X^* and ${}^*X$ in $\mathcal{F}$ such that $\mathbb{1} \in X^* \otimes X$, etc.
Associativity $(gh)k = g(hk)$	$(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$

Example 1.1.1. Given the group $S_3 = \{(), (12), (13), (23), (123), (132)\}$ we then produce the category $S_3\mathbf{Rep}$ of K-representations. The idea is to go from a finite group to the category of representations (and more — there's more than *just* rep theory of finite groups in the story of fusion categories).

Remark 1.1.2. A large part of fusion category theory is only concered with things "up to isomorphism."

Definition 1.1.3 (Fusion Rings)**.** A *fusion ring* is a ring F which is free as a $\mathbb{Z}$-module with respect to a basis $L = \{a, b, c, \cdots\}$ (whcih we call our label set) satisfying:

- L is finite.
- F is unital.
- $1 \in L$.

- For any $a, b \in L$,

$$ab = \sum_{c \in L} N_{ab}^c c$$

and $N_{ab}^c \in \mathbb{N}$.

- There is an involution $* : L \to L$ which extends to an anti-involution $* : F \to F$, i.e., $*^2 = *$ and $*(ab) = *(b) * (a)$.

Example 1.1.4. The trivial fusion ring is the fusion ring F with the label set $L = \{1\}$ and fusion rule $1 \cdot 1 = 1$.

Example 1.1.5 (Ising Fusion Ring)**.** The fusion ring is F is the ring where $L = \{1, \sigma, \psi\}$ and the fusion rules are $\sigma^2 = 1 + \psi$, $\sigma\psi = \psi\sigma = \sigma, \psi^2 = 1$.

Example 1.1.6 (Toric Code)**.** This is the fusion ring with label set and fusion rules equal to the Klein 4-group.

Remark 1.1.7 (Fusion Rings (Combintaorially))**.** L $= \{1, a, a^*, b, b^*, \cdots\}, N_{ab}^c \in \mathbb{N}$ such that:

- Asosciativity (which is encoded by)

$$\sum_x N_{ab}^x N_{xc}^d = \sum_x N_{bc}^x N_{ax}^x$$

- Unitality is

$$N_{1a}^b = \delta_{ab} = N_{a1}^b$$

- Duality

$$N_{a^*b}^1 = \delta_{ab} = N_{ba^*}^1$$

- Frobenius reciprocity

$$N_{ab}^c = N_{c^*b}^b = N_{cb^*}^a$$

Definition 1.1.8. Let F be a fusion ring with label set L. Then:

- The *rank* of F is $|L|$.
- The *multiplicity* of F is answer of "yes" or "no" as to whether or not

$$N_{ab}^c > 1$$

for $a, b, c \in L$.

- The Frobenius-Perron Dimensions are defined to be

$$FP\dim(a) = \text{Frobenius-Perron Eigenvalue of the matrix} N_a = \left(N_a\right)_{bc} = N_{ab}^c.$$

Note that $FP\dim(a) = \max\{\lambda \in \mathbb{R} \mid (N_a)_{bc}\mathbf{v} = \lambda\mathbf{v}\}$ and $FP\dim$ determines a ring homomorphism $F \to \mathbb{C}$. Also for any m we get that

$$FP\dim(m)^2 = \sum_{a \in A} FP\dim(a) + FP\dim(a)$$

so $FP\dim$'s are algebraic numbers.

Example 1.1.9. Here are some invariants of fusion rings as illustrated by the example of

$$\mathrm{TY}(A) = \begin{cases} a \otimes m = m = m \otimes a \\ m \otimes m = \oplus_{a \in A} a \end{cases}$$

where A is a finite Abelian group. Note that $L = \{a \mid a \in A\} \cup \{m\}$. Also TY stands for Tamadara-Yamagami. Then

$$\mathrm{rank}(\mathrm{TY}(A)) = |A| + 1.$$

As $\mathrm{TY}(A)$ has no multiplicity, it is multiplicity-free.

Definition 1.1.10 (Fusion Categories (Non-definition))**.** A fusion category is a categorification of a fusion ring

$$(F, +, \times) \xrightarrow{\text{categorified}} (\mathscr{F}, \oplus, \otimes) :$$

Given N_{ab}^c, we want $\mathscr{F}$ with:

- $|L|$-many isomorphism classes of indecompables
- $\dim(\mathscr{F}(A \otimes B, C)) = N_{ab}^c$
- And is natural

and for which there is a decategorification

$$(\mathscr{F}, \oplus, \otimes) \xrightarrow{\text{decategorification}} (F, +, \times).$$

Theorem 1.1.11 (Ocneanu Rigidty)**.** *There are only finitely many categorifications of a fixed fusion ring to a fusion category.*

A natural and good question: "How to tell different categorifications of the same fusion ring apart?"

Remark 1.1.12. Most of the invariants we use to study fusion categories are just invariants of their fusion rings.

Theorem 1.1.13 (Generalized Tannaka-Krein Reconstruction)**.** *Every Fusion category is* $\mathbf{Rep}(H)$ *for* H *is a weak Hopf algebra.*

2 Day 2: 2023 May 2

2.1 Lecture 1

Today's Schedule:

- Fusion Categories: Definition and examples
- STring diagrams
- Pivotal, Spherical Structures (probably not, but potentially, unitary structures)
- Module Categories over Fusion Categories (deferred to exercises)

Definition 2.1.1. A fusion category $\mathscr{C}$ is a k-linear Abelian category which is

- Finite
- Semisimple

- Monoidal
- Rigid

for which $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ is bilinear on morphisms and the monoidal unit $\mathbb{1}$ is simple.

Remark 2.1.2. After seeing the definition of a fusion category, we saw an awesome diagram of relations between the various categorical adjectives that appear in the defintion. This was an impossibility to TeX so here's a dinosaur pun instead:

What do pterodactyls do when they fly? They dino-soar!

Remark 2.1.3. In today's story, the field $k = \mathbb{C}$. While other fields are possible, we will not focus on these gadgets.

We will focus on the following chain of structures:

$$\begin{array}{c} \text{Category } \mathscr{C} \\ \big\downarrow \text{Has a tensor-product} \\ \text{Monoidal category } (\mathscr{C}, \otimes, \mathbb{1}, \lambda, \rho, \alpha) \\ \big\downarrow \text{Has duals} \\ \text{Rigid monoidal category} \\ \big\downarrow \mathscr{C} \text{ is finite and } \otimes \text{ is Bilinear} \\ \text{Fusion category} \end{array}$$

We also saw string diagrams. I cannot live TeX these and present great shame.

Definition 2.1.4. A *monoidal category* is a 6-tuple $(\mathscr{C}, \otimes, \mathbb{1}, \rho, \lambda, \alpha)$ where:

- $\otimes$ is a bifunctor;
- $\mathbb{1}$ is an object of $\mathscr{C}$ called the monoidal unit;
- There are natural isomorphisms $\rho : \mathrm{id}_{\mathscr{C}} \otimes \mathbb{1} \to \mathrm{id}_{\mathscr{C}}$ and $\lambda : \mathbb{1} \otimes \mathrm{id}_{\mathscr{C}} \to \mathrm{id}_{\mathscr{C}}$ called the (left and right) unitors;
- There is a natural isomorphism $\alpha : (X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z)$ called the associator.

and the diagrams

$$\begin{array}{ccc} (X \otimes \mathbb{1}) \otimes Y & \longrightarrow & X \otimes (\mathbb{1} \otimes Y) \\ & \searrow \quad \swarrow & \\ & X \otimes Y & \end{array}$$

and

$$\begin{array}{ccccc} & & (W \otimes X) \otimes (Y \otimes Z) & & \\ & \nearrow & & \searrow & \\ ((W \otimes X) \otimes Y) \otimes Z & & & & W \otimes (X \otimes (Y \otimes Z)) \\ \downarrow & & & & \uparrow \\ (W \otimes (X \otimes Y)) \otimes Z & & \longrightarrow & & W \otimes ((X \otimes Y) \otimes Z) \end{array}$$

commute.

Remark 2.1.5. We always assume that the unit is strict, i.e., that $\mathbb{1} \otimes X = X = X \otimes \mathbb{1}$.

Remark 2.1.6. We saw that in the string calculus that tensor producting things is horizontal composition of things.

Example 2.1.7. Consider the category $G\mathbf{Vect}$, the category of G-graded k-vector spaces. The (isomorphism classes of) simple objects here are the objects δ_g for $g \in G$ and satisfy the equations

$$\delta_g \otimes \delta_h = \delta_{gh}$$

for all $g, h \in G$. To see the pentagon rule in action we consider the case when $X = \delta_g$, $Y = \delta_h$, $Z = \delta_k$, and $W = \delta_\ell$ for $g, h, k, \ell \in G$. Then the diagram

$$\begin{array}{ccccc} & & (W \otimes X) \otimes (Y \otimes Z) & & \\ & \overset{\alpha_{W \otimes X, Y, Z}}{\nearrow} & & \overset{\alpha_{X,Y,Z \otimes W}}{\searrow} & \\ ((W \otimes X) \otimes Y) \otimes Z & & & & W \otimes (X \otimes (Y \otimes Z)) \\ {\scriptstyle \alpha}\downarrow & & & & \uparrow \\ (W \otimes (X \otimes Y)) \otimes Z & & \longrightarrow & & W \otimes ((X \otimes Y) \otimes Z) \end{array}$$

shows that the numbers α satisfy a 3-cocycle condition on G. Thus fusion categories relate to group cohomology $H^\bullet(G; -) = R^n(-)^G$.

Example 2.1.8 (Tambara-Yamagami Fusion Category)**.** This example starts with $\mathrm{TY}(A)$ and categorifiex it to $\mathrm{TY}(A, \chi, \tau)$ whereχ is a non-degenerate symmetric bicharacter and

$$\tau \in \left\{ \pm \frac{1}{\sqrt{|A|}} \right\}.$$

I could not capture this in time.

We now want to talk about about the structure of fusion categories. However, we will be using MacLane's Strictness Theorem to simplify the story and write every structure isomorphism as an identity.

Theorem 2.1.9 (MacLane's Strictness Theorem)**.** *Every monoidal category* $(\mathscr{C}, \otimes, \mathbb{1}, \alpha)$ *is monoidally equivalent to a strict monoidal category, i.e., one in which the associators*

$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$$

are all identities.

Remark 2.1.10. This is super helpful for string diagrams! It means that we don't have to worry about the literal physical position and bend in wires that appear in string diagrams.

Definition 2.1.11. An object X^* is a *left dual* of X if there exist morphisms $\mathrm{eval}_X : X^* \otimes X \to \mathbb{1}$ and $\mathrm{coeval}_X : \mathbb{1} \to X \otimes X^*$ satisfying the zig-zag axioms

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{coeval}_X} & (X \otimes X^*) \otimes X \\ \| & & \downarrow {\scriptstyle \alpha_{X, X^*, X}} \\ X & \xleftarrow[\mathrm{eval}_X]{} & X \otimes (X^* \otimes X) \end{array}$$

and dually.

Remark 2.1.12. The zig-zag axioms are monoidal versions of the triangle identities which define adjunctions. The left dual X^* is a left adjoint to X in this language, coeval is the unit, and eval is the counit.

Definition 2.1.13. A pivotal structure on a fusion category is a family of natural isomorphisms $\psi_X : X \to X^{**}$ which is a monoidal natural isomorphism between X and $(-)^{**}$. In particular, $\psi_{X\otimes Y} = \psi_X \otimes \Psi_Y$.

We can use this to define the (left and right) trace of a morphism $f \in \mathscr{C}(X, X)$. The definition is via string diagrams so here's another dinosaur pun:

What do you call a dinosaur that recently broke up with their partner? A tyrannosaurus ex!

Definition 2.1.14. Sphericality is a property of ψ that says

$$\text{trace}^R(f) = \text{trace}^L(f).$$

If ψ satisfies this property we say that the fusion category $\mathscr{C}$ is spherical.

Definition 2.1.15. The quantum dimension of an object X in a spherical fusion category

$$d_X = \text{trace}(\text{id}_X).$$

Definition 2.1.16. If $\mathscr{C}$ is a spherical fusion category, we define the dimension of $\mathscr{C}$ to be

$$D := \sum_{X\in\mathscr{C}^{\text{simple}}_{/\text{iso}}} d_X.$$

Daniel asked a question which amounted to the answer that if X isn't simple in the graphical calculus we can still define trace and stuff visually by just taking the direct sum decomposition of the object into simples (this uses semisimplicity) and then taking a formal sum of the various strings.

3 Day 3: 2023 May 3

3.1 Recap of Yesterday's Lecture and Loose Ends

Recall that a fusion category is a rigid $\mathbb{C}$-linear monoidal category with some nice finiteness/semisimplicity properties. In particular, a fusion category has finitely many isomorphism classes of simple objects. We define

$$\text{Irr}(\mathscr{C}) := \text{set of representatives of isomorphism casses containing the unit } \mathbb{1}.$$

Each simple (isomorphism class) has an associated "quantum dimension." In particular, for spherical fusion category $\mathscr{C}$ and a simple $X \in \mathscr{C}$,

$$d_X = \text{trace}(\text{id}_X).$$

As a string diagram, this is a circle from $\mathbb{1} \to \mathbb{1}$ and so gives a morphism in $\mathscr{C}(\mathbb{1}, \mathbb{1}) \cong \mathbb{C}$. As such, when we say that d_x is a number we mean that under the isomorphism above, $d_x = \text{trace}(\text{id}_X)$ can be regarded as a scalar.

3.2 Lecture 2

Two Powerful Methods for Fusion Categories

1. "Strictification:" Wolog we can assume that all associators $\alpha_{X,Y,Z} = \text{id}_{X\otimes Y\otimes Z}$.
2. "Skeletaization" (orthogonal to strictification) also called "combinatorialization."

It would be nice to do all these structural tricks at once, but *you can't use both at the same time.*

Definition 3.2.1 (Skeletization)**.** Here is how to do skeletization as a process:

1. Pick one object per isomorphism class of simple object. Pass to the label set

$$\mathcal{L} = \{\mathbb{1}, a, a^*, b, b^*, \cdots\}$$

2. Pick a basis for every "trivalent" hom space

$$\mathscr{C}(a \otimes b, c) \qquad \text{“Trivalent fusion space”}$$

and

$$\mathscr{C}(c, a \otimes b) \qquad \text{“Trivalent splitting space.”}$$

Both have the same dimension

$$\dim \mathscr{C}(a \otimes b, c) = \dim \mathscr{C}(c, a \otimes b) =: N_{ab}^{c}$$

so we need to pick N_{ab}^{c} basis elements; say

$$\{|a, b, c; \mu\rangle\}_{\mu \in 1, \cdots, N_{ab}^{c}} \to \{\alpha\}$$

where the symobol used is a new graphical calculus.

3. This induces a choice of basis on all hom spaces, like so:

$$\mathscr{C}\left((a \otimes b) \otimes c, d\right) \cong \bigoplus_{m} \mathscr{C}(a \otimes b, m) \otimes \mathscr{C}(m \otimes c, d)$$

which we think of as the image:

However, since we only have one object per isomorphism class, $(a \otimes b) \otimes c = a \otimes (b \otimes c)$ as objects and similar with our trivalent hom spaces. So we can get

$$\mathscr{C}\left((a \otimes b) \otimes c, d\right) = \mathscr{C}\left(a \otimes (b \otimes c), d\right) = \bigoplus_{n} \mathscr{C}(b \otimes c, n) \otimes \mathscr{C}(a \otimes n, d)$$

Consequently there are two natural bases for our trivalent hom spaces. These are

and we define the change of basis matrix to be the matrix

$$\left(F_{d}^{abc}\right)_{(n,\gamma,\delta);(m,\mu,\nu)}.$$

In pictures we get

We will now derive the pentagon equations which was done visually. I cannot TEX this. We do, however, get the identity I was not able to capture in time

$$\sum_{h,\sigma\psi,\rho} \left(F_{g}^{abc}\right)\left(F_{e}^{abd}\right)$$
$$= \sum_{\delta}$$

There were more rules for duals and rigidity in this new calculus. I again could not TEX this.

What did we do? Well: we took a spherical fusion category and produced combinatorial data

$$\{N_{c}^{ab}, \left(F_{d}^{abc}\right)_{(n,\gamma,\delta);(m,\alpha,\beta)}, t_a\}$$

where the N_{c}^{ab} are the fusion rules, the (F_{d}^{abc}) are the F-symbols, and the t_a are our spherical constants which satisfy equations. We can use these numbers and equations to look for solutions to the various identities required of fusion rings (so in particular find all possible categorifications of a given fusion ring) and pentagon equations in order to find and search for computer-assisted classification of fusion categories. Physicists are very good at this!

In Colleen's notes there is a summary of the trivalent graphical calculus. It is worth taking a look! Of note is the Bubble Popping rule and the quantum dimension rules.

How do we use these? Well' let's study here the Turaev-Viro state sum TQFT. Assume that we have a multiplicity-free skeletal fusion category $\mathscr{C}$. Think of this state sum as a black box which takes as input the skeletal data $\{N_c^{ab}, (F_d^{abc})_n, t_a\}$ and a triangulation of a 3-manifold (X^3, T) in the sense that we write

$$X^3 \cong \left(\coprod_{i=1}^{n} \Delta_i\right)_{/\sim}$$

which allows us to see X^3 as a collection of tetrahedra glued together. As output we get a number $Z_{\mathscr{C}}(X^3) \in \mathbb{C}$ which is an invariant of the manifold.

Definition 3.2.2. Given a triangulated manifold (X, T) define $T^{(0)}$ to be the vertices of the triangulation, $T^{(1)}$ to be the edges, $T^{(2)}$ to be the edges, and $T^{(3)}$ the tetrahedra.

Definition 3.2.3. A state is a map $s :: T^{(1)} \to \mathscr{L}$ where $T^{(1)}$ is a set of edges edge and $\mathscr{L}$ is our label set.

Then we get that

$$Z_{\mathscr{C}}(X) := \sum_s \frac{\prod_{T^{(3)}} \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} \prod_{T^{(1)}} d_a}{\prod_{T^{(2)}} \theta(a,b,c) \prod_{T^{(0)}} D}.$$

The general idea of the proof that $Z_{\mathscr{C}}(X^3)$ is an invariant of the manifold is to show that $Z_{\mathscr{C}}(X^3, T) = Z_{\mathscr{C}}(X^3, T')$ for any two triangulations of X^3. The crucial observation is that any two triangulations of X^3 are related by $2-3$ and $1-4$ Pachner moves. As such, it suffices to show that there is a trivalent graph which encodes the Pachner moves. You can prove that the equivalence of $2-3$ moves corresponds to the pentagon equations and allow us to prove that it is an invariant.

We've seen now the diference between our two techniuqes for studying fusion categories via strictification and skeletization. In both languages we can exxpress inavriants like N_{ab}^c, FP dim, global dimension, quantum dimension, and the circle diagrams.

4 Day 4: 2023 May 4

Today:

- Braided Fusion Categories
- Drinfeld Cenres of Fusion Categories
- Drinfeld Centres of BFCs.

Example 4.0.1 (Fib Fusion Category)**.** $\mathscr{L} = \{1, \tau\}, \tau \otimes \tau = 1 \oplus \tau$, and

$$(F_\tau^{\tau\tau\tau}) = \begin{pmatrix} \phi^{-1} & \phi^{-1/2} \\ \phi^{-1/2} & \phi^{-1} \end{pmatrix}$$

and all other F-symbols are 1. Note $\phi = (1+\sqrt{5})/2$.

Example 4.0.2 (Ising Fusion Category)**.** $\mathscr{L} = 1, \sigma, \psi\}$ and I missed the rest :(

4.1 Motivation: From Fusion Categories to Braided Fusion Categories

A $3D$ TQFT can be thought of in a 1-categorical version as $Z : \mathbf{Bord}_3 \to \mathbf{K\text{-}Vect}$ which sends $X^3 \to Z_{\mathscr{C}}(X^3, T) = Z_{Z(\mathscr{C})}(X^3, L)$ where $Z(\mathscr{C})$ is the Drinfeld centre of $\mathscr{C}$.. In the $(\infty, 3)$-categorical version this is a functor $\mathbf{Bord}_3 \to \mathbf{Alg}_1(\mathsf{Cat})$. This ∞-functor sends 0-cells to $\mathscr{C}$, 1-cells to ${}_{\mathscr{C}}\mathscr{C}_{\mathscr{C}}$, and 2-cells to $\mathrm{End}_{\mathscr{C}\,\mathbf{bimod}}({}_{\mathscr{C}}\mathscr{C}_{\mathscr{C}})$ where $\mathscr{C}$ is a spherical fusion category. Note that

$$\mathrm{End}_{\mathscr{C}\,\mathbf{bimod}}({}_{\mathscr{C}}\mathscr{C}_{\mathscr{C}}) \simeq Z(\mathscr{C})$$

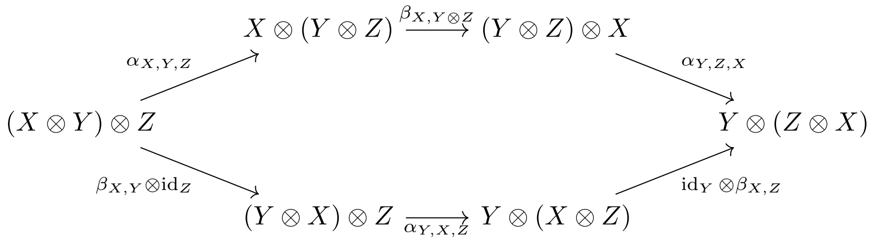
where $Z(\mathscr{C})$ is again the Drinfeld centre.

Algebra	Braided fusion categories are “like” finite quantum Abelian groups
Topology	I missed the explanation :(

Definition 4.1.1. A monoidal category $\mathscr{C}$ is *braided* if there are natural isomorphisms

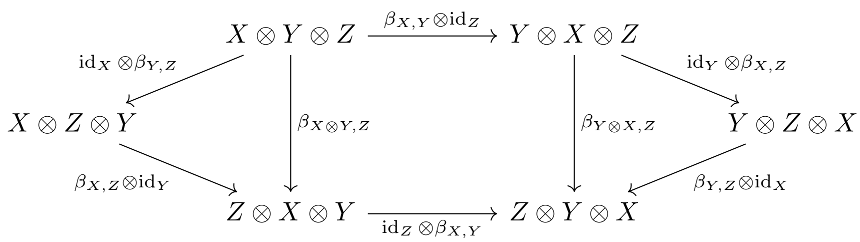
$$\beta_{X,Y} : X \otimes Y \to Y \otimes X$$

called the braiding isomorphisms which satisfy hexagon axioms



and one which expresses how we can commute Z outside the tensor.

Theorem 4.1.2. *In a strict braided monoidal category, the diagram*



This means we can now draw braid diagrams as string diagrams and cup/cap/link/and more!

Recall the Reidemister Theorem:

Theorem 4.1.3. *Two link diagrams represent the same link if they can be related by a finite number of Reidemeister moves (RI, RII, RIII which are strings and I cannot TEX them live) and isotopy.*

We have already shown that RII and RIII are legit. However, there is a *framed* version of everything and that’s the version that we’ll need which uses Reidemeister moves RII, RIII, and a new unlinking version which we’ll sort out in a future moment.1

4.2 Degeneracy of Braided Structure and Examples

These categories lie on a spectrum: on one end we have things being “more boring” (this is where things are symmetric and $\beta_{X,Y} = \beta^{-1}_{Y,X}$) and “less boring” to the other end. This is non-symmetric direction is the direction where the braiding is significantly less degenerate and the braid isomorphisms become .

For the symmetric case Deligne’s Theorem tells us these are morally all vector space representations of groups or supergroups.

Theorem 4.2.1 (Deligne)**.** *Symmetric BFCs are monoidally equivalent to either* $\mathbf{Rep}(G)$ *or* $\mathbf{Rep}(G, Z)$*.*

The nondegeneracy in braided monoidal categories is measured by the symmetric centre $Z_2(\mathscr{C})$, i.e., the subcategory of $\mathscr{C}$ generated by objects X for which $\beta_{X,Y} = \beta^{-1}_{Y,X}$ for all $Y \in \mathscr{C}$ (so essentially X centralizes the category).

In a spherical braided fusion cateegory we can measure this instead with the “S-matrix”

$$S_{X,Y} = \mathrm{trace}(\beta_{Y,X^{"*}} \circ \beta_{X^*,Y})$$

Definition 4.2.2. If $S_{X,Y}$ has determinant 0 we say that $\mathscr{C}$ is nondegenerate.

4.3 The Drinfeld Centre of a Monoidal Category

Let $Z(\mathscr{C})$ be the category whose objects are pairs $(Z, \{b_{X,Z}\}_{X\in\mathscr{C}})$ of objects $Z \in \mathscr{C}$ and sets of half-braidings[1] of Z and X. Note that $b_{X,Z} : X \otimes Z \to Z \otimes X$ are the ways of moving Z past X and we ask that the hexagon

$$\begin{array}{ccccccc} & & X\otimes(Y\otimes Z) & \xrightarrow{\mathrm{id}_X \otimes b_{Y,Z}} & X\otimes(Z\otimes Y) & & \\ & \overset{\alpha_{X,Y,Z}}{\nearrow} & & & & \overset{\alpha^{-1}_{X,Z,Y}}{\searrow} & \\ (X\otimes Y)\otimes Z & & & & & & (X\otimes Z)\otimes Y \\ & \underset{b_{Z,X\otimes Y}}{\searrow} & & & & \underset{b_{X,Z}\otimes \mathrm{id}_Y}{\nearrow} & \\ & & Z\otimes(X\otimes Y) & \xrightarrow[\alpha^{-1}_{Z,X,Y}]{} & (Z\otimes X)\otimes Y & & \end{array}$$

commutes for any $X, Y \in Z(\mathscr{C})$ (so note that $b_{Z,X}$ is part of the data of X) and morphisms are $f : Z \to Z'$ in $\mathscr{C}$ such that for all X the diagrams

$$\begin{array}{ccc} X\otimes Z & \longrightarrow & X\otimes Z' \\ \downarrow & & \downarrow \\ Z\otimes X & \longrightarrow & Z'\otimes X \end{array}$$

commute.

Generically speaking, $Z(\mathscr{C})$ inerits structure from $\mathscr{C}$:

- Because $\mathscr{C}$ is monoidal, so is $Z(\mathscr{C})$;
- If $\mathscr{C}$ is (pivotal/spherical) then $Z(\mathscr{C})$ is pivotal
- If $\mathscr{C}$ is fusion so is $Z(\mathscr{C})$.

But $Z(\mathscr{C})$ is braided even if $\mathscr{C}$ is not braided!

4.4 Notions of Equivalence

A monoidal equivalecne $F\mathscr{C} \to \mathscr{D}$ is a functor together with natural isomorphisms $J_{X,Y} : F(X \otimes Y) \cong F(X) \otimes F(Y)$. There are other equivalences which are important, however, like asking

$$Z_{\mathscr{C}}(X) = Z_{Z(\mathscr{C})}(X).$$

If we have another category with $\mathscr{C}'$ with $Z(\mathscr{C}' \simeq Z(\mathscr{C})$ then

$$Z_{\mathscr{C}}(X) = Z_{Z(\mathscr{C})}(X) = Z_{Z(\mathscr{C}')}(X) = Z_{\mathscr{C}'}(X)$$

and gives a notion of Morita equivalence for fusion categories. In particular:

Theorem 4.4.1. *For fusion categorie s$\mathscr{C}$ and $\mathscr{D}$,*

$$\mathscr{C} \underset{Morita}{\simeq} \mathscr{D}$$

if and only if

$$Z(\mathscr{C}) \simeq Z(\mathscr{D})$$

as braided fusion categories.

Example 4.4.2. There is such an equivalence

$$G\mathbf{Vec} \underset{\text{Mortia}}{\simeq} \mathbf{Rep}(G)$$

even though these categories are absolutely not strictly equivalent as fusion categories!

[1]This is notation that I have chosen because b looks like half a β so I can visually think of it as half a β.

5 Day 5: 2023 May 5, Revenge of the Fifth

Today:

1. Skeletal BFCs
2. I missed the last three points :(

5.1 Lecture 4

Let $(\mathscr{C}, \otimes, \mathbb{1}, \alpha, \beta, \psi, \text{coeval}, \text{eval})$ be a braided fusion category. To skeletize such a category we pick out first the F-symbols from α, the pivotal coefficients from ψ, and also get to build what are called R-sybols from β. These are symbols $[R_c^{ab}]_{\mu\nu}$ which allow us to introduce braidings into our trivalent graph moves. In particular, uncrossing a braid of a and b which intersect at a vertex μ introduces a multiplication by $[R_c^{ab}]_{\mu\nu}$, replaces μ with ν, and un-crosses a and b. Such a move is called an R-move. There is also an R-move which reverses the order in a stacked diagram.

There is an equation of R and F moves which records the fact that $\mathscr{C}$ satisfies the heexagon axioms (and in particulary visually allows us to see crossed and braided fishbone diagrams).

Example 5.1.1. By writing down the Fibonacci fusion cateegory, we can also skeletize it as a braided fusion category by settingg $R_1^{\tau\tau} = e^{-4\pi 1/5}, R_\tau^{\tau\tau} = e^{3\pi i/5}$.

Example 5.1.2. There was the skeletization of the Ising braided fusion category, but I missed teh details :(

There is another way of working with braided fusion categories, which is through what are called ribbon fusion categories, a.k.a. "premodular" fusion categories.

Definition 5.1.3. Let $\mathscr{C}$ be a braided, rigid, monoidal category. Twists on $\mathscr{C}$ are natural isomorphisms $\theta : \text{id}_\mathscr{C} \Rightarrow \text{id}_\mathscr{C}$ for which

$$\theta_{X\otimes Y} = (\theta_X \otimes \theta_Y) \circ \beta_{Y,X} \circ \beta_{X,Y}.$$

for all $X, Y \in \mathscr{C}$.

Definition 5.1.4. We say that a braided, rigid, monoidal category with a twist θ. We say that θ is *ribbon* if $\theta_X^* = \theta_{X^*}$ for all $X \in \mathscr{C}$ (where θ_X^* is a shorthand for the complex conjugate of the operator). We say that $\mathscr{C}$ is *ribboned* if there is a choice of a ribbon on θ.

Remark 5.1.5. It is possible for a ribboned category to have multiple non-equiavlent ribbonings.

The relation between sphereical braided fusion categories and ribboned fusion categories is in what is called the *Drinkfeld isomorphism*. Define the drinfeld isomorphism $u_X : X \to X^{**}$ is defined as a string diagram that looks like a kidney bean (that I can't draw :(in TEX live).

Proposition 5.1.6. *If $\psi_X : X \to X^{**}$ ar our pivotal isomorphisms then $\theta_X := u_X^{-1} \circ \psi_X$ is a twist on $\mathscr{C}$. Furthermore, if ψ is sphereical then θ is ribbon.*

Remark 5.1.7. When working in a ribboned category, you have ribbon string diagrams.

5.2 Premodular Fusion Categories

Let $(\mathscr{C}, \otimes, \mathbb{1}, \alpha, \text{coeval}, \text{eval}, \beta, \psi)$ (or) $(\mathscr{C}, \otimes, \mathbb{1}, \alpha, \text{coeval}, \text{eval}, \beta, \theta)$ be a preomodular fusion category. We can define the modular data:

$$S_{X,Y} = \text{trace}(\beta_{Y,X} \circ \beta_{X,Y})$$

for X and Y representatives of isomorphism classes of simples in $\mathscr{C}$

Remark 5.2.1 (ACHTUNG)**.** Sometimes you'll see this information defined as

$$C_{X,Y} = \text{trace}(\beta_{Y,X^*} \circ \beta_{X^*,Y}).$$

Definition 5.2.2. If $\mathscr{C}$ is a premodular category then the modular information is the matrix

$$\left(T_{X,Y}\right) = \mathrm{diag}(\theta_X) = \delta_{XY}\theta_X$$

Definition 5.2.3. We say that a premodular category is modular if S is nonsingular. If $\mathscr{C}$ is also a fusion category then $\mathscr{C}$ is a modular fusion category.

When $\mathscr{C}$ is modular, there is a nice formula

$$S_{X,Y} = \frac{1}{\theta_X\theta_Y}\sum_c N_{XY}^C d_C\theta_C.$$

Remark 5.2.4. The modular data is an invariant of a premodular category.

Modular data is an avatar of a modular fusion category.

Remark 5.2.5 (ACHTU?NG)**.** The data $\{S, T\}$ do **not** uniquely determine modular fusion categories (although in low rank they sometimes do).

However, from the modular data you can extract fusion rules!

Example 5.2.6. Consider $G\mathbf{Vec}^\omega$ where ω is a 3-cocycle on G which witnesses how we view $G\mathbf{Vec}$ as a monoidal category. If G is Abelian then $G\mathbf{Vec}^\omega$ admits a braiding with (as derived from Hexagon Axiom 1)

$$\frac{b(g,hk)}{b(g,h)b(g,k)} = \frac{\omega(g,h,k)\omega(h,k,g)}{\omega(k.g.h)}.$$

Such a pair (ω, b) is called an *Abelian* 3*-cocycle.* The modular information is

$$S_{g,h} = b(g,h)b(g,h)$$

and

$$\theta_g = \chi(g)b(g,g) \in \{\pm 1\}.$$

Note that χ is a character of G and relates to a pivotal structure.

Example 5.2.7. Let's compute the Drinfeld centre $Z(G\mathbf{Vec}^\omega) \simeq \mathbf{Rep}(D^\omega G)$ which is a modular fusion category. Note that $D^\omega G$ isa twisted quantum double Hopf algebra. Also when ω is simple we should have

$$Z(G\mathbf{Vec}_G^\omega) \underset{\text{Morita}}{\simeq} Z(\mathbf{Rep}G).$$

The simple objects of $Z(G\mathbf{Vec}_G^\omega)$ are pairs $([t], \rho_t)$ where $[t]$ is a conjugacy class in G and ρ_t is a ψ-projective irrep of the centralizer $C_G(t)$ and $\psi_t(x,y)$ is a function satisfying

$$\psi_t(x,y) = \frac{\omega(t,x,y)\omega(x,y,(xy)^{-1}txy)}{\omega(x,x^{-}tx.y)}.$$

The modular data is:

$$S\big(([a],\rho_a),([b],\rho_b)\big) \sim \sum_{\substack{g\in[a]\\ h\in[b]\cap C_G(g)}} \chi^*_{\rho_g}(h)\chi^*_{\rho_h}(g).$$

Similarly,

$$\theta([a],.\rho_a) = \frac{\chi_{\rho_a}(a)}{\chi_{\rho_a}(1_G)}.$$

5.3 3D RT TQFT

Any knot or link diagram internal to our modular fusion category gets assigned a number.

Example 5.3.1. The Hopf link for simples X and Y. If we want to compute the RT invariant. If we have a skeletization of MFC then we cna use the ribbon trivalent graphical calculus to compute that, modulo normalizations, a sum of R-symbols times ribboned diagrams we can bubble pop. So up to quantum dimensions the Hopf link is equal to a sum of other puppings and spherical tricks. This has some really cool pictures that I have missed.

If we have a more complicated braid, we can still use the RT evaluation algorithm and proceed similarly. It will be hard, but we can still rewrite crossed strands in terms of trivalent diagrams with R-moves that we can then apply a lot of F moves and bubble popping to get a number out. It can be done, but it's involved!

5.4 RT 3-manifold invariant

Let $\mathscr{C}$ be modular. Assume that M is presented as (integral) surgery diagram on a framed link L. Then if ω be the Kirby colour (which is not a simple object but is $\sum_{X\in\mathrm{Irr}(X)} d_X X$),

$$Z_{\mathscr{C}}(X^3, L_\omega) \sim \sum_{X\in\mathrm{Irr}(\mathscr{C})} d_X Z_{\mathscr{C}}(L_X).$$

So:

$$Z_{\mathscr{C}}^{TV}(X^3, T) = Z_{Z(\mathscr{C})}^{RT}(X^3, L)$$

Note that $\mathscr{C}$ is a spherical fusion category, but $Z(\mathscr{C})$ is a modular fusion category.

5.5 MFCs as Algebraic Tehories of (Gapped) Bosonic Topological Phases of Matter

The idea is to think of string diagrams (morphisms in an MFC) as spacetime trajectories of pointlike anyons. Under this dictionary:

MFC	Topological Phases
Simple objects	Anyon
Iso Classes	Anyon flavours
Morphisms	Anyon process
coeval and eval	Creation, Annihilation
Braid	Exchange

So topological phases of matter are quantum systems govered by a 3D TQFT.