# Fusion categories and TQFT: Problem Set 0

## Learning Objectives

- Get familiar with the definition, properties, and various data types of a fusion ring; get comfortable working with small examples.
- Practice the most basic way to construct new fusion rings from old, namely the (Deligne-Kelley) product.
- Begin to appreciate how fusion category theory is connected to linear algebra, abstract algebra, and algebraic number theory.

## Review

**Definition 1.** *A fusion ring* $(F, \otimes, 1, L)$ *is a unital ring* $(F, \otimes, 1)$ *which is free as a* $\mathbb{Z}$*-module with finite basis* $L \subset F$ *together with an involution* $\star : L \to L$ *that lifts to an anti-involution on* $F$ *satisfying*

- $1 \in L$ *and* $1^* = 1$,
- $a \otimes b = \sum_{c \in L} N_{ab}^c c$ *with* $N_{ab}^c \in \mathbb{Z}_{\geq 0}$ *for all* $a, b \in L$
- $N_{ab}^1 = N_{ba}^1 = \delta_{a^* b}$ *for all* $a, b \in L$.

A fusion ring is called *multiplicity-free* if $N_{ab}^c \in \{0, 1\}$ for all $a, b, c \in L$.

### Fusion ring invariants and properties

Recall that the *rank* of $(F, L, \otimes)$ is the number of elements in the basis set $|L|$ and the Frobenius-Perron dimension of $a \in L$ is given by Frobenius-Perron eigenvalue of the matrix $[N_a]_{bc} := N_{ab}^c$.

### Ways to present a fusion ring

There are several different ways to encode the data of a fusion ring: via the *fusion matrices* $\{N_a\}_{a \in L}$, the *fusion rules* $N_{ab}^c \in \mathbb{Z}_{\geq 0}$, or the fusion table whose $(a, b)$ entry is the nonzero entries of the vector $[N_a]_{b\_}$. The function FPdim $: L \to \mathbb{C}$ that sends $a \mapsto \text{FPdim}(a)$ is a ring homomorphism.

### Example

It is standard to write the basis of the Ising fusion ring as $L = \{1, \sigma, \psi\}$. Ising has rank 3 and Frobenius-Perron dimensions 1, $\sqrt{2}$, 1.

**Fusion matrices**

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

**Fusion rules**

$$\begin{cases} \sigma \otimes \sigma = 1 \oplus \psi \\ \sigma \otimes \psi = \psi \otimes \sigma = \sigma \\ \psi \otimes \psi = 1 \end{cases}$$

**Fusion table**

| | $1$ | $\sigma$ | $\psi$ |
|---|---|---|---|
| $1$ | $1$ | $\sigma$ | $\psi$ |
| $\sigma$ | $\sigma$ | $1 \oplus \psi$ | $\sigma$ |
| $\psi$ | $\psi$ | $\sigma$ | $1$ |

## Exercises

1. Consider the rank 2 fusion ring with fusion rule $\tau \otimes \tau = 1 \oplus \tau$.
   (a) Can you express the coefficient of $\tau$ in $\tau^{\otimes n}$ in terms of $n$?
   (b) Compute the Frobenius-Perron dimension of $\tau$.

2. Describe all multiplicity-free fusion rings of rank 3 which contain a non-self dual basis element.

3. Let $(F_1, L_1, \otimes_1)$ and $(F_2, L_2, \otimes_2)$ be fusion rings. Check that $L = L_1 \times L_2$ – which we will write as $L = \{a \boxtimes b\}_{a \in L_1, b \in L_2}$ – gives the free $\mathbb{Z}$-module on $L$ the structure of a fusion ring with respect to the multiplication $\otimes$ on $L$ given by $(a_1 \boxtimes b_1) \otimes (a_2 \boxtimes b_2) = \sum_{c_1 \in L_1, c_2 \in L_2} N_{a_1 b_1}^{c_1} N_{a_2 b_2}^{c_2} c_1 \boxtimes c_2$.

4. Fill in the blanks in the fusion table for the fusion ring Fib $\boxtimes$ Ising.

| | $1 \boxtimes 1$ | $1 \boxtimes \sigma$ | $1 \boxtimes \psi$ | $\tau \boxtimes 1$ | $\tau \boxtimes \sigma$ | $\tau \boxtimes \psi$ |
|---|---|---|---|---|---|---|
| $1 \boxtimes 1$ | $1 \boxtimes 1$ | $1 \boxtimes \sigma$ | $1 \boxtimes \psi$ | $\tau \boxtimes 1$ | $\tau \boxtimes \sigma$ | $\tau \boxtimes \psi$ |
| $1 \boxtimes \sigma$ | $1 \boxtimes \sigma$ | | | | | |
| $1 \boxtimes \psi$ | $1 \boxtimes \psi$ | | | | | |
| $\tau \boxtimes 1$ | $\tau \boxtimes 1$ | | | | | |
| $\tau \boxtimes \sigma$ | $\tau \boxtimes \sigma$ | | | | | |
| $\tau \boxtimes \psi$ | $\tau \boxtimes \psi$ | | | | | |

5. Prove that $\mathrm{FPdim}(X \boxtimes Y) = \mathrm{FPdim}(X)\,\mathrm{FPdim}(Y)$.

6. How do you think you should define a fusion module, i.e. a module over a fusion ring?

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