FUSION CATEGORIES & TQFT OVERVIEW

LECTURE 0

- OVERVIEW
- MOTIVATION
- FUSION RINGS

IDEA OF A FUSION CATEGORY

LECTURE 1

- FUSION CATEGORIES
- STRING DIAGRAMS
- PIVOTAL, SPHERICAL, UNITARY STRUCTURES
- MODULE CATEGORIES OVER FUSION CATEGORIES

LECTURE 2

- SKELETAL FUSION CATEGORIES
  - ↳ TURAEV-VIRO STATE SUM TQFT
  - ↳ LEVIN-WEN HAMILTONIAN FOR LATTICE TQFT

LECTURE 3

- DRINFELD CENTERS OF FUSION CATEGORIES
  - ↳ TUBE ALGEBRAS
- BRAIDED FUSION CATEGORIES
  - ↳ RESHETIKHIN-TURAEV TQFT
  - ↳ TOPOLOGICAL ORDER
    - BOSONIC, FERMIONIC

LECTURE 4

- SYMMETRIES OF MODULAR FUSION CATS
  - ↳ SYMMETRY PROTECTED, SYMMETRY ENRICHED TOPOLOGICAL ORDER
  - ↳ SYMMETRY GAUGING

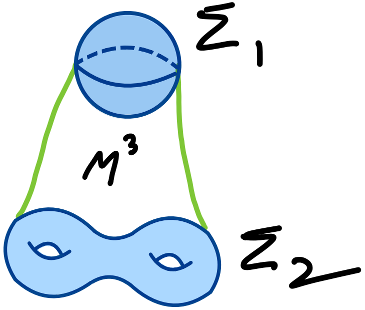
SYMMETRY TQFTs
ENRICHED FUSION CATEGORIES
HIGHER FUSION CATEGORIES } ?

MOTIVATION: FUSION CATEGORIES AND TQFT

$d=2$

$$Z : \mathrm{BORD} \longrightarrow \mathfrak{S}$$

SYMMETRIC MONOIDAL FUNCTOR



$$\bullet \longmapsto \text{FULLY-DUALIZABLE OBJECT } X \in \mathfrak{S}$$

FUSION CATEGORIES ARE FULLY-DUALIZABLE OBJECTS IN SOME $X \in \mathfrak{S}$

$$3D = (\underset{d}{2}+1)$$

TYPE OF FUSION CATEGORY	TYPE OF TQFT
(SPHERICAL) FUSION	3D (FULLY-EXTENDED) TURAEV-VIRO TQFT
MODULAR FUSION	3D RESHETIKHIN-TQFT
G× MODULAR FUSION CATEGORIES	3D HOMOTOPY QUANTUM FIELD THEORY
	4D INVERTIBLE THEORY CRANE-YETTER TQFT

LECTURE 0

~~OVERVIEW~~

~~MOTIVATION~~

FUSION RINGS

HOLISTIC IDEA OF A FUSION CATEGORY:

a fusion category is like
a "quantum" finite group

ALGEBRA $\mathbb{C}[G]$		
FINITE GROUP $(G, \times)$	⇝	**"QUANTUM" FINITE GROUP $(\mathcal{F}, \otimes)$**
		$X \otimes Y$ IN $\mathcal{F}$
FINITE # OF ELEMENTS $g \in G$	⇝	FINITELY MANY INDECOMPOSABLE OBJECTS*, $X \in \mathcal{F}$
$g_1 = g_2$		$X \cong Y$ & $\mathrm{HOM}_{\mathcal{F}}(X,Y)$ FINITE HILBERT SPACE
BINARY OPERATION $G \times G \to G$	⇝	BIFUNCTOR $\otimes : \mathcal{F} \times \mathcal{F} \longrightarrow \mathcal{F}$
$gh = k$		$X \otimes Y \cong Z_1 \oplus Z_2 \oplus \cdots Z_n$
• IDENTITY		• IDENTITY*
$\exists\, e \in G$ S.T. $eg = g = ge \quad \forall g \in G$		$\exists\, \mathbb{1} \in \mathcal{F}$ S.T. $\mathbb{1} \otimes X \cong X \cong X \otimes \mathbb{1} \quad \forall\, X \in \mathcal{F}$
• INVERSES		• INVERSES* (DUALITY)
$\forall g \in G\ \exists\, g^{-1}$ S.T. $g^{-1}g = e = gg^{-1}$		$\forall X \in \mathcal{F}\ \exists\, X^*, {}^*X$ S.T. $\mathbb{1} \in X^* \otimes X$, etc.
• ASSOCIATIVITY		• ASSOCIATIVITY*
$(gh)k = g(hk) \quad \forall\, g,h,k \in G$		$(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z) \quad \forall\, X,Y,Z \in \mathcal{F}$

* UP TO ISOMORPHISM

AN EXAMPLE TO ILLUSTRATE THE ANALOGY

$S_3 = \{(\,), (12), (13), (23), (123), (132)\}$ $\rightsquigarrow$ IRREDUCIBLE REPRESENTATIONS OF S_3

$\{1, \mathrm{sgn}, \mathrm{std}\}$

	( )	(12)	(13)	(23)	(123)	(132)
( )						
(12)		( )	(123)	(132)	(23)	(13)
(13)			( )			
(23)				( )		
(123)					(132)	( )
(132)					( )	(123)

COMES FROM Vec_{S_3}

	1	sgn	std
1	1	sgn	std
sgn	sgn	1	std
std	std	std	$1 \oplus \mathrm{sgn} \oplus \mathrm{std}$

COMES FROM $\mathrm{Rep}(S_3)$

WE'VE ALREADY MET TWO FAMILIES OF FUSION CATEGORIES: Vec_G & $\mathrm{Rep}(G)$ AND WE HAVEN'T EVEN DEFINED IT YET!

A LARGE PART OF FUSION CATEGORY THEORY IS ONLY CONCERNED WITH THINGS "UP TO ISOMORPHISM"

FUSION RINGS (DEFINITION)

A FUSION RING $(F, \times, +)$ IS A UNITAL, ASSOCIATIVE RING WHICH IS FREE AS A $\mathbb{Z}$-MODULE W.R.T. A BASIS $L = \{a, b, c, \dots\}$ SUCH THAT L IS FINITE

label set

- $1 \in L$
- $a \times b = \sum_{c \in L} N_{ab}^{c}\, c \,, \quad N_{ab}^{c} \in \mathbb{Z}_{\geq 0}$

fusion coefficients

- $\exists$ INVOLUTION $* : L \longrightarrow L$ WHICH EXTENDS TO AN ANTI-INVOLUTION ON F

$(ab)^* = b^* a^*$

EXAMPLES & DIFFERENT WAYS TO PRESENT A FUSION RING

FUSION RULES

FUSION TABLE

TRIVIAL FUSION RING

$L = \{1\}$

$$\{ 1 \times 1 = 1$$

	1
1	1

ISING FUSION RING

$L = \{1, \sigma, \psi\}$

$$\begin{cases} \sigma \times \sigma = 1 + \psi \\ \sigma \times \psi = \psi \times \sigma = \sigma \\ \psi \times \psi = 1 \end{cases}$$

	1	σ	ψ
1	1	σ	ψ
σ	σ	$1+\psi$	σ
ψ	ψ	σ	1

(TORIC CODE)

$\mathbb{Z}_2 \times \mathbb{Z}_2$ FUSION RING

$L = \{1, e, m, f\}$

$$\begin{cases} e^2 = 1 \\ m^2 = 1 \\ f^2 = 1 \\ em = f \end{cases}$$

MORE

"$\frac{1}{2}E_6$" FUSION RING

$$L = \{1, x, y\}$$

$$\begin{cases} x^2 = 1 + 2x + y \\ xy = yx = x \\ y^2 = 1 \end{cases}$$

NON-EXAMPLE BECAUSE $|L| = \infty$

CLEBSCH-GORDON FUSION RULES

$$L = \{1, 2, 3, \dots\}$$

$$\left\{ i \times j = \sum_{\ell=0}^{\min(i,j)} i + j - 2\ell \right.$$

FUSION RINGS (COMBINATORIAL DATA)

$$L = \{1, a, a^*, b, b^*, \ldots\}, \quad N_{ab}^{c} \in \mathbb{Z}_{\geqslant 0} \quad \text{s.T.}$$

- $\sum_x N_{ab}^{x} N_{xc}^{d} = \sum_x N_{bc}^{x} N_{ax}^{d}$ (associativity)
- $N_{1a}^{b} = \delta_{ab} = N_{a1}^{b}$ (unitality)
- $N_{a^*b}^{1} = \delta_{ab} = N_{ba^*}^{1}$ (duality)

  THE ONLY WAY TO GET 1 IN THE FUSION PRODUCT OF a^* AND b IS IF $b = a$
- $N_{ab}^{c} = N_{a^*c}^{b} = N_{cb^*}^{a}$ (Frobenius reciprocity)

PROPERTIES ~~INVARIANTS~~ OF FUSION RINGS

(ILLUSTRATED BY THE EXAMPLE OF TY(A)

ABELIAN GROUP

$L = \{a\}_{a\in A} \cup \{m\}$

$$\begin{cases} a \otimes m = m = m \otimes a \\ m \otimes m = \bigoplus_{a \in A} a \end{cases}$$
)

TAMBARA-YAMAGAMI

LET F BE A FUSION RING W/ LABEL SET L

RANK: $\mathrm{rank}(F) = |L|$

$\mathrm{rank}(TY(A)) = |A| + 1$

MULTIPLICITY: WHETHER $N_{ab}^{c} > 1$ FOR ANY $a, b, c \in L$

MULTIPLICITY-FREE

FROBENIUS-PERRON DIMENSIONS*: $\mathrm{FPdim}(a) =$ FROBENIUS-PERRON EIGENVALUE OF THE MATRIX $\quad a \in L$

$$[N_a]_{bc} := N_{ab}^{c}$$

$\mathrm{FPdim}(a) > 0 \quad \forall\ a \in A$

* FACT: FPdim DEFINES A RING HOMOMORPHISM → $\mathbb{C}^{\times}$

e.g.: BECAUSE FPdim IS A RING HOMOMORPHISM

$$\mathrm{FPdim}(m)^2 = \sum_{a \in A} \mathrm{FPdim}(a) + \mathrm{FPdim}(m)$$

$\Rightarrow$ FPdims ARE ALGEBRAIC #'s

FUSION CATEGORIES (NON-DEFINITION)

A FUSION CATEGORY IS A CATEGORIFICATION OF A FUSION RING

$$\begin{tikzcd}(F, +, \times) \arrow[r, dashed, bend left, "\text{“CATEGORIFICATION”}"] & (\mathcal{F}, \oplus, \otimes) \arrow[l, bend left, "\text{DECATEGORIES}"]\end{tikzcd}$$

GIVEN N_{ab}^{c}, WANT $\mathcal{F}$ WITH

- $|L|$-MANY ISOMORPHISM CLASSES OF INDECOMPOSABLE OBJECTS
- $\dim\left(\mathrm{HOM}_{\mathcal{F}}(A \otimes B, C)\right) = N_{ab}^{c}$
- AND IS NATURAL

THEOREM ("OCNEANU RIGIDITY"):

THERE ARE ONLY FINITELY MANY CATEGORIFICATIONS OF A FIXED FUSION RING TO A FUSION CATEGORY

$$\begin{tikzcd}
(\mathcal{F}_1,\otimes_1) & (\mathcal{F}_2,\otimes_2) & \cdots & (\mathcal{F}_N,\otimes_N) \\
 & (F,\otimes) \arrow[ul, dashed] \arrow[u, dashed] \arrow[urr, dashed] & &
\end{tikzcd}$$

how to tell different categorifications of the same fusion ring apart?

(MOST OF THE INVARIANTS WE USE TO STUDY FUSION CATEGORIES ARE JUST INVARIANTS OF THEIR FUSION RINGS)

ON THE CLASSIFICATION OF FUSION CATEGORIES

" GENERALIZED TANNAKA-KREIN RECONSTRUCTION: "

EVERY FUSION CAT IS Rep(H)

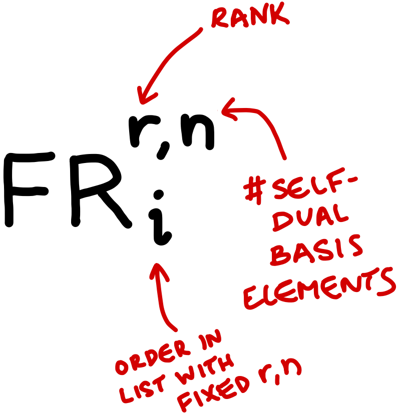
H IS A WEAK HOPF ALGEBRA

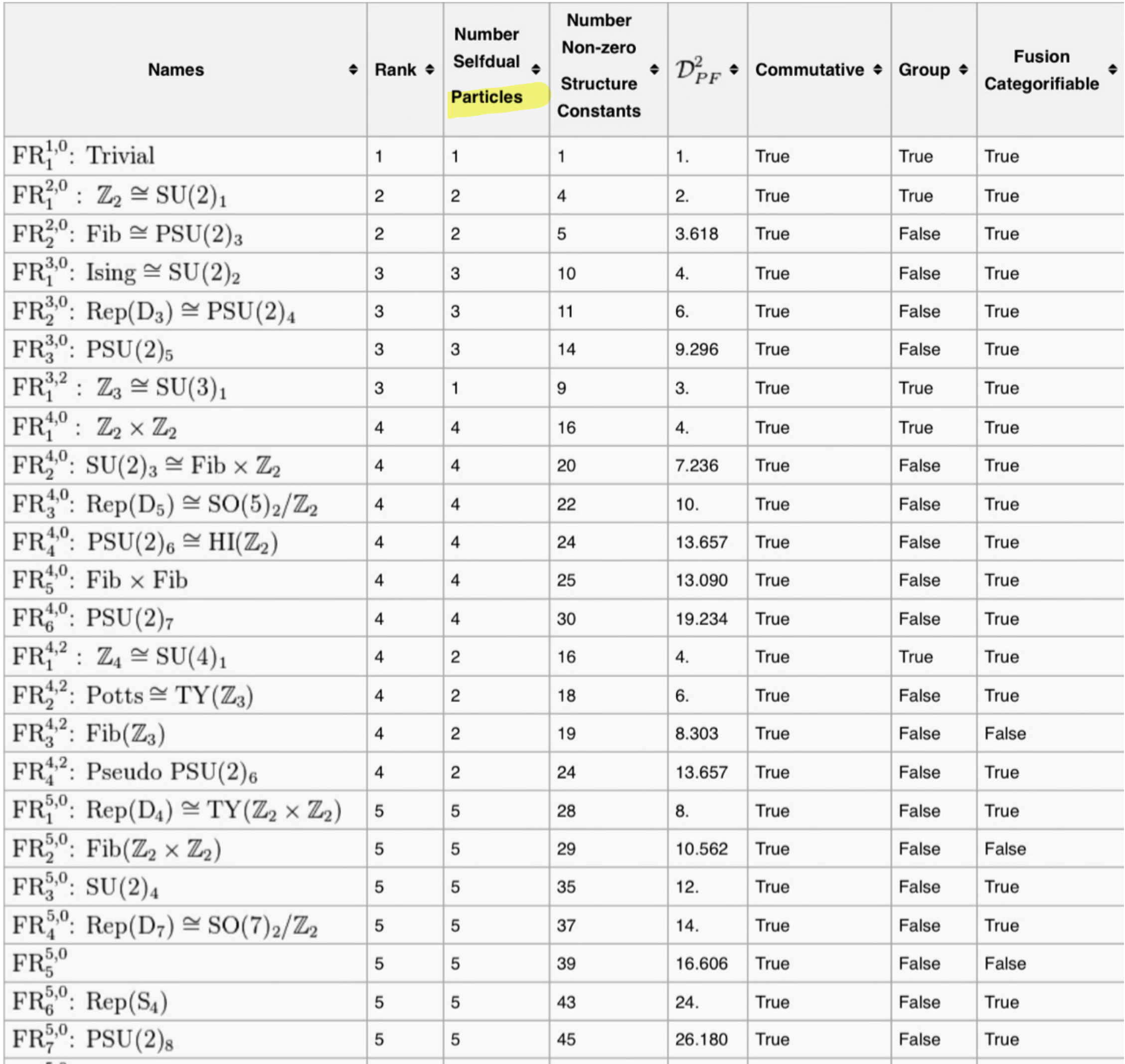
CLASSIFICATION OF MULTIPLICITY-FREE FUSION RINGS UP TO RANK 6

@ ANYONWIKI

http://www.thphys.nuim.ie/AnyonWiki/index.php/List_of_small_multiplicity-free_fusion_rings

[PALCOUX, ...]





Names	Rank	Number Selfdual Particles	Number Non-zero Structure Constants	$\mathcal{D}^2_{PF}$	Commutative	Group	Fusion Categorifiable
$FR^{1,0}_1$: Trivial	1	1	1	1.	True	True	True
$FR^{2,0}_1$: $\mathbb{Z}_2 \cong SU(2)_1$	2	2	4	2.	True	True	True
$FR^{2,0}_2$: $\mathrm{Fib} \cong PSU(2)_3$	2	2	5	3.618	True	False	True
$FR^{3,0}_1$: $\mathrm{Ising} \cong SU(2)_2$	3	3	10	4.	True	False	True
$FR^{3,0}_2$: $\mathrm{Rep}(D_3) \cong PSU(2)_4$	3	3	11	6.	True	False	True
$FR^{3,0}_3$: $PSU(2)_5$	3	3	14	9.296	True	False	True
$FR^{3,2}_1$: $\mathbb{Z}_3 \cong SU(3)_1$	3	1	9	3.	True	True	True
$FR^{4,0}_1$: $\mathbb{Z}_2 \times \mathbb{Z}_2$	4	4	16	4.	True	True	True
$FR^{4,0}_2$: $SU(2)_3 \cong \mathrm{Fib} \times \mathbb{Z}_2$	4	4	20	7.236	True	False	True
$FR^{4,0}_3$: $\mathrm{Rep}(D_5) \cong SO(5)_2/\mathbb{Z}_2$	4	4	22	10.	True	False	True
$FR^{4,0}_4$: $PSU(2)_6 \cong HI(\mathbb{Z}_2)$	4	4	24	13.657	True	False	True
$FR^{4,0}_5$: $\mathrm{Fib} \times \mathrm{Fib}$	4	4	25	13.090	True	False	True
$FR^{4,0}_6$: $PSU(2)_7$	4	4	30	19.234	True	False	True
$FR^{4,2}_1$: $\mathbb{Z}_4 \cong SU(4)_1$	4	2	16	4.	True	True	True
$FR^{4,2}_2$: $\mathrm{Potts} \cong TY(\mathbb{Z}_3)$	4	2	18	6.	True	False	True
$FR^{4,2}_3$: $\mathrm{Fib}(\mathbb{Z}_3)$	4	2	19	8.303	True	False	False
$FR^{4,2}_4$: Pseudo $PSU(2)_6$	4	2	24	13.657	True	False	True
$FR^{5,0}_1$: $\mathrm{Rep}(D_4) \cong TY(\mathbb{Z}_2 \times \mathbb{Z}_2)$	5	5	28	8.	True	False	True
$FR^{5,0}_2$: $\mathrm{Fib}(\mathbb{Z}_2 \times \mathbb{Z}_2)$	5	5	29	10.562	True	False	False
$FR^{5,0}_3$: $SU(2)_4$	5	5	35	12.	True	False	True
$FR^{5,0}_4$: $\mathrm{Rep}(D_7) \cong SO(7)_2/\mathbb{Z}_2$	5	5	37	14.	True	False	True
$FR^{5,0}_5$	5	5	39	16.606	True	False	False
$FR^{5,0}_6$: $\mathrm{Rep}(S_4)$	5	5	43	24.	True	False	True
$FR^{5,0}_7$: $PSU(2)_8$	5	5	45	26.180	True	False	True