Fusion categories and TQFT: Problem Set 1

Learning Objectives

- Begin to appreciate the role of group cohomology in fusion category theory.
- Get comfortable with fusion categories of G-graded vector spaces and the kinds of arguments one can make when the objects under consideration are all invertible up to isomorphism with respect to the tensor product.
- Get used to all the commutative diagrams.
- Get exposed to monoidal functors, module categories, and algebra objects.

Review

Some group cohomology

A 2-cocycle $\nu : G \times G \to \mathbb{k}$ is a function satisfying the 2-cocycle condition

$$\nu(gh,k)\nu(g,h) = \nu(g,hk)\nu(h,k).$$

A 3-cocycle $\omega : G \times G \times G \to \mathbb{k}$ is a function satisfying the 3-cocycle condition

$$\omega(gh,k,l)\omega(g,h,kl) = \omega(g,h,k)\omega(g,hk,l)\omega(h,k,l).$$

Two 3-cocycles ω and $\tilde{\omega}$ represent the same cohomology class if there is a function called a 2-coboundary $\mu : G \times G \to \mathbb{k}$ such that

$$\omega(g,h,k)\mu(gh,k)\mu(g,h) = \mu(g,hk)\mu(h,k)\tilde{\omega}(g,h,k).$$

Functors between fusion categories and symmetries

Definition. *A monoidal functor* (F,J) *between monoidal categories* $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}, \alpha^{\mathcal{C}})$ *and* $(\mathcal{D}, \otimes_{\mathcal{D}}, \mathbb{1}_{\mathcal{D}}, \alpha^{\mathcal{D}})$ *is a functor* $F : \mathcal{C} \to \mathcal{D}$ *such that* $F(\mathbb{1}_{\mathcal{C}}) \cong \mathbb{1}_{\mathcal{D}}$ *together with isomorphisms*

$$J_{X,Y} : F(X \otimes_{\mathcal{C}} Y) \xrightarrow{\cong} F(X) \otimes_{\mathcal{D}} F(Y)$$

satisfying

$$\begin{array}{ccc}
(F(X) \otimes_{\mathcal{D}} F(Y)) \otimes_{\mathcal{D}} F(Z) & \xrightarrow{\alpha^{\mathcal{D}}_{F(X),F(Y),F(Z)}} & F(X) \otimes_{\mathcal{D}} (F(Y) \otimes_{\mathcal{D}} F(Z)) \\
{\scriptstyle J_{X,Y} \otimes_{\mathcal{D}} \mathrm{id}_{F(Z)}} \uparrow & & \uparrow {\scriptstyle \mathrm{id}_{F(X)} \otimes_{\mathcal{D}} J_{Y,Z}} \\
F(X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{D}} \otimes F(Z) & & F(X) \otimes_{\mathcal{D}} (F(Y \otimes_{\mathcal{C}} Z) \\
{\scriptstyle J_{X \otimes_{\mathcal{C}} Y, Z}} \uparrow & & \uparrow {\scriptstyle J_{X, Y \otimes_{\mathcal{C}} Z}} \\
F((X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{C}} F(Z) & \xrightarrow[F(\alpha^{\mathcal{C}}_{X,Y,Z})]{} & F(X \otimes_{\mathcal{C}} (Y \otimes_{\mathcal{C}} Z))
\end{array}$$

A monoidal equivalence of fusion categories is a monoidal functor (F,J) *where* F *is an equivalence of the categories.*

Exercises

1. Suppose ω and $\tilde{\omega}$ are 3-cocycles on G. Show that $\mathrm{Vec}_G^{\omega} \simeq \mathrm{Vec}_G^{\tilde{\omega}}$ if and only if ω and $\tilde{\omega}$ represent the same cohomology class.

2. Take a moment to review the definition of a monoidal category from Lecture 1. Then try to fill in the missing pieces of the following definition based on what seems natural.

   Definition. *Let $(C, \otimes, 1, \alpha)$ be a monoidal category. A (left) C-module category is category $\mathcal{M}$ with an action bifunctor $\triangleright : C \times \mathcal{M} \to \mathcal{M}$ and isomorphisms*

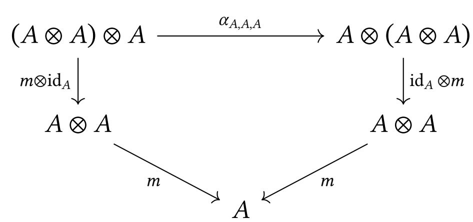
   $$\lambda_M : \mathbb{1} \triangleright M \to M$$
   $$\mu_{X,Y,M} : (X \otimes Y) \triangleright M \to X \otimes (Y \triangleright M)$$

   called the left module unitor and left module associators satisfying

   - *the triangle axiom:*

   - *and (left) module pentagon axiom:*

3. An object A is called an algebra in C or an algebra object in C if there is a morphism $m : A \otimes A \to A$ called multiplication satisfying



   (There is a unit morphism and unit axioms too but let's not worry about those for now.)
   Now consider Vec_G with trivial associativity constraint. Let H be a subgroup of G and consider the object $A = \oplus_{h \in H} \delta_h$. Show that the morphism $m : A \otimes A \to A$ induced by twisting the multiplication on H by a 2-cocycle ψ gives (A, m) the structure of an algebra object in Vec_G.