LECTURE 1

FUSION CATEGORIES: DEFINITION & EXAMPLES
STRING DIAGRAMS
PIVOTAL, SPHERICAL, ~~UNITARY~~ STRUCTURES — WILL SKIP FOR NOW
MODULE CATEGORIES OVER FUSION CATEGORIES — DEFERRED TO EXERCISES

DEFINITION OF A FUSION CATEGORY (LONG, PAINFUL)

A FUSION CATEGORY $\mathcal{C}$ IS A $\mathbb{k}$- LINEAR ABELIAN CATEGORY WHICH IS

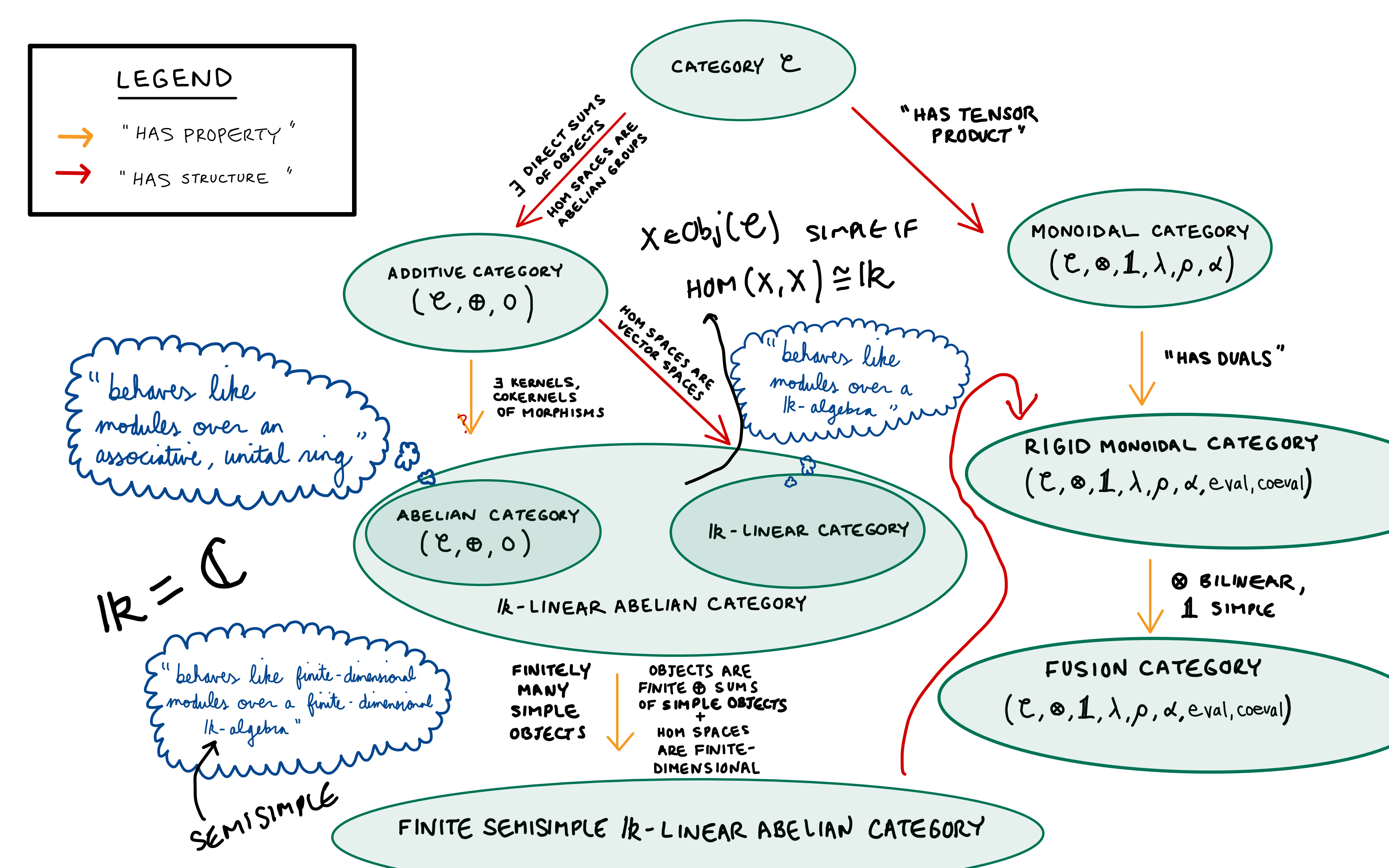
COLOR CODE:

(WILL BE EXPLAINED AWAY)

(WILL BE DEFINED)

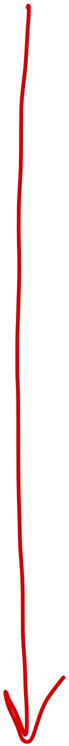
- FINITE
- SEMISIMPLE
- MONOIDAL
- RIGID

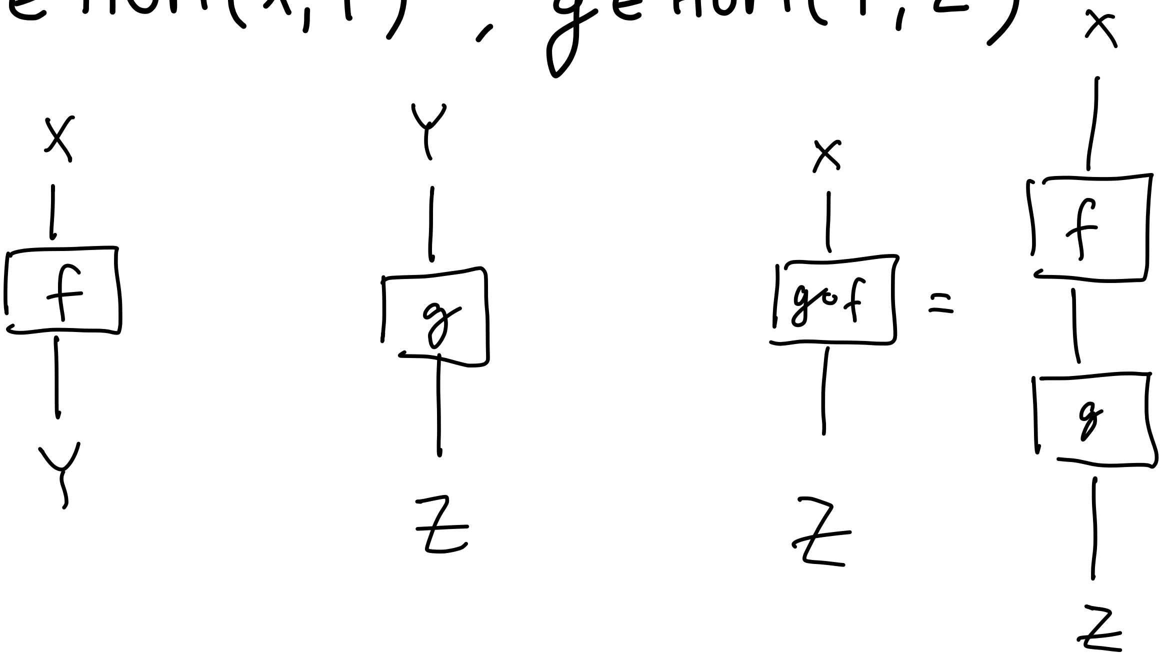
S.T. $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ IS BILINEAR ON MORPHISMS AND THE MONOIDAL UNIT $\mathbb{1}$ IS SIMPLE


LEGEND
" HAS PROPERTY "
" HAS STRUCTURE "
CATEGORY $\mathcal{C}$
∃ DIRECT SUMS OF OBJECTS
HOM SPACES ARE ABELIAN GROUPS
"HAS TENSOR PRODUCT"
ADDITIVE CATEGORY
$(\mathcal{C}, \oplus, 0)$
MONOIDAL CATEGORY
$(\mathcal{C}, \otimes, \mathbb{1}, \lambda, \rho, \alpha)$
$X \in \mathrm{Obj}(\mathcal{C})$ SIMPLE IF
$\mathrm{HOM}(X,X) \cong \mathbb{k}$
HOM SPACES ARE VECTOR SPACES
∃ KERNELS, COKERNELS OF MORPHISMS
"behaves like modules over an associative, unital ring"
"behaves like modules over a $\mathbb{k}$-algebra"
"HAS DUALS"
RIGID MONOIDAL CATEGORY
$(\mathcal{C}, \otimes, \mathbb{1}, \lambda, \rho, \alpha, \mathrm{eval}, \mathrm{coeval})$
ABELIAN CATEGORY
$(\mathcal{C}, \oplus, 0)$
$\mathbb{k}$-LINEAR CATEGORY
$\mathbb{k}$-LINEAR ABELIAN CATEGORY
$\mathbb{k} = \mathbb{C}$
$\otimes$ BILINEAR,
$\mathbb{1}$ SIMPLE
FUSION CATEGORY
$(\mathcal{C}, \otimes, \mathbb{1}, \lambda, \rho, \alpha, \mathrm{eval}, \mathrm{coeval})$
"behaves like finite-dimensional modules over a finite-dimensional $\mathbb{k}$-algebra"
SEMISIMPLE
FINITELY MANY SIMPLE OBJECTS
OBJECTS ARE FINITE $\oplus$ SUMS OF SIMPLE OBJECTS
+
HOM SPACES ARE FINITE-DIMENSIONAL
FINITE SEMISIMPLE $\mathbb{k}$-LINEAR ABELIAN CATEGORY

STRING DIAGRAMS

$f \in \mathrm{HOM}(X, Y) \quad , \quad g \in \mathrm{HOM}(Y, Z)$





VERTICAL STACKING

DEFINITION OF A MONOIDAL CATEGORY

A MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \underbrace{\mathbb{1}}_{\text{MONOIDAL UNIT}}, \rho, \lambda, \alpha)$

IS A BIFUNCTOR $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$

WITH NATURAL ISOMORPHISMS

$$\left.\begin{aligned} \rho_X &: X \otimes \mathbb{1} \xrightarrow{\sim} X \\ \lambda_X &: \mathbb{1} \otimes X \xrightarrow{\sim} X \end{aligned}\right\} \text{UNITORS}$$

$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \ \Big\} \text{ ASSOCIATORS}$$

SATISFYING THE TRIANGLE AND PENTAGON AXIOMS

TRIANGLE AXIOM

$$\begin{tikzcd}
(X \otimes 1) \otimes Y \arrow[rr, "\alpha_{X,1,Y}"] \arrow[dr, "\rho_X"'] & \arrow[d, phantom, "\circlearrowright"] & X \otimes (1 \otimes Y) \arrow[dl, "\lambda_Y"] \\
& X \otimes Y &
\end{tikzcd}$$

CAN ALWAYS ASSUME $X \otimes 1 = X = 1 \otimes X$

PENTAGON AXIOM

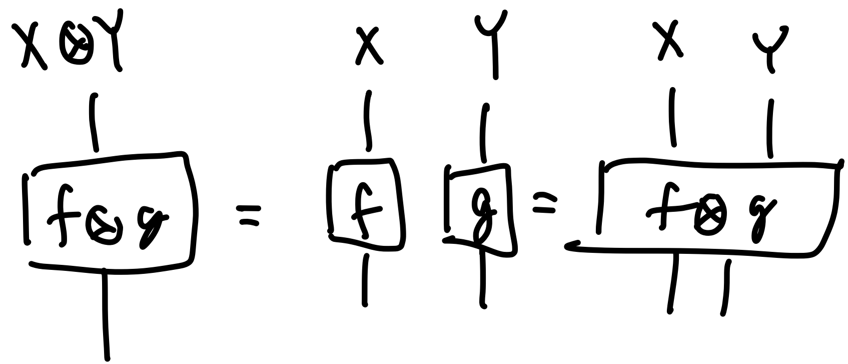
$$\begin{tikzcd}
& (W \otimes X) \otimes (Y \otimes Z) \arrow[dr, "\alpha_{W,X,Y\otimes Z}"] & \\
((W \otimes X) \otimes Y) \otimes Z \arrow[ur, "\alpha_{W\otimes X,Y,Z}"] \arrow[d, "\alpha_{W,X,Y} \otimes Z"'] & \circlearrowright & W \otimes (X \otimes (Y \otimes Z)) \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[rr, "\alpha_{W,X\otimes Y,Z}"'] & & W \otimes ((X \otimes Y) \otimes Z) \arrow[u, "W \otimes \alpha_{X,Y,Z}"']
\end{tikzcd}$$

MACLANE'S COHERENCE THEOREM

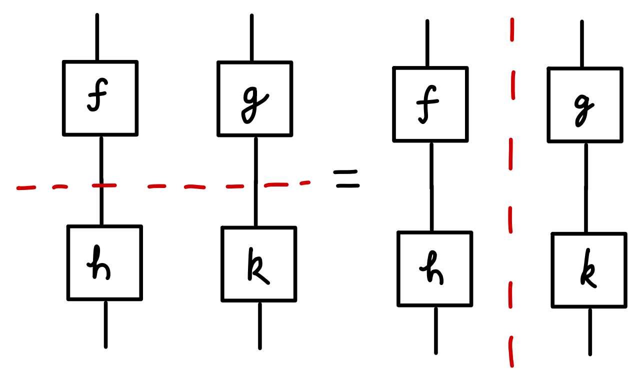
($\Rightarrow$ ANY TWO ISOMORPHISMS REPARENTHESIZING AN N-FOLD $\otimes$-PRODUCT ARE EQUAL)

STRING DIAGRAMS IN A MONOIDAL CATEGORY

$\otimes$ PRODUCT DRAWN AS HORIZONTAL JUXTAPOSITION:



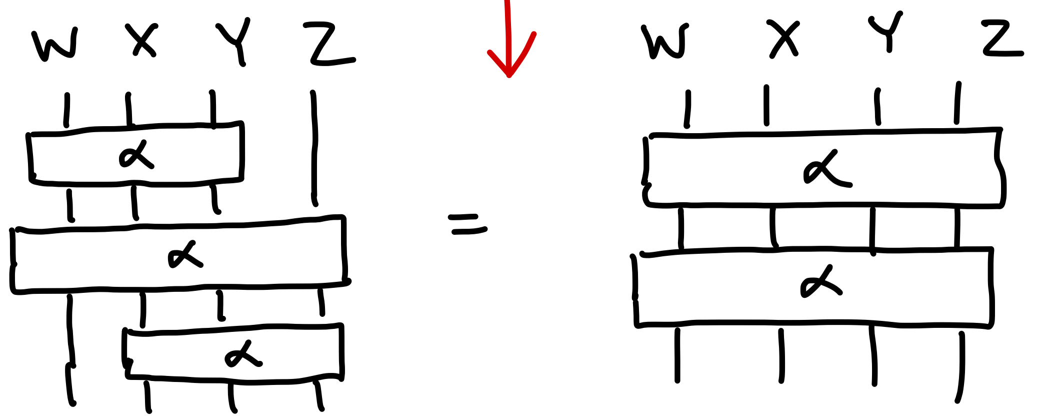
COMPATIBLE WITH COMPOSITION!



TRANSLATING BETWEEN COMMUTATIVE DIAGRAMS AND STRING DIAGRAMS:

PENTAGON AXIOM

$$\begin{tikzcd}
 & & (W\otimes X)\otimes(Y\otimes Z) \arrow[dr, "\alpha"] & \\
((W\otimes X)\otimes Y)\otimes Z \arrow[urr, "\alpha"] \arrow[d, "\alpha"'] & & & W\otimes(X\otimes(Y\otimes Z)) \\
(W\otimes(X\otimes Y))\otimes Z \arrow[rrr, "\alpha"] & & & W\otimes((X\otimes Y)\otimes Z) \arrow[u, "\alpha"']
\end{tikzcd}$$



EXAMPLE: Vec_G^{ω} = CATEGORY OF G-GRADED VECTOR SPACES, $\omega \in Z^3(G, \mathbb{k}^*)$

SIMPLE OBJECTS δ_g, $g \in G$

$$\delta_g \otimes \delta_h = \delta_{gh}$$

$W = \delta_g$
$X = \delta_h$
$Y = \delta_k$
$Z = \delta_l$

$$\begin{tikzcd}
& \delta_{ghkl} \arrow[dr, "\alpha_{w,x,y\otimes z}"] & \\
\delta_{ghkl} \arrow[ur, "\alpha_{w\otimes x,y,z}"] \arrow[d, "\alpha_{w,y,z}"'] & \circlearrowright & \delta_{ghkl} \\
\delta_{ghkl} \arrow[r, "\alpha_{w,x\otimes y,z}"'] & \delta_{ghkl} \arrow[ur, "\alpha_{x,y,z}"'] &
\end{tikzcd}$$

$$\mathrm{HOM}(\delta_{ghkl}, \delta_{ghkl}) \cong \mathbb{k}$$

$\Rightarrow$ $\alpha_{x,y,z}$ SATISFY A 3-COCYCLE CONDITION ON G

EXAMPLE: TAMBARA-YAMAGAMI FUSION CATEGORY

$$TY(A) \rightsquigarrow TY(A, \chi, \tau)$$

NON-DEGENERATE SYMMETRIC BICHARACTER

$\tau \in \left\{ \pm \frac{1}{\sqrt{|A|}} \right\}$

RECALL FROM LECTURE 0

$L = \{a \in A\} \sqcup \{m\}$

$m \otimes m = \bigoplus_{a \in A} a$

$$\alpha_{a,b,c} = \mathrm{id}_{abc}$$

$$\alpha_{a,b,m} = \alpha_{m,b,c} = \mathrm{id}_m$$

$$\alpha_{a,m,c} = \chi(a,c)\, \mathrm{id}_m$$

$$\alpha_{a,m,m} = \alpha_{m,m,c} = \mathrm{id}_{m \otimes m}$$

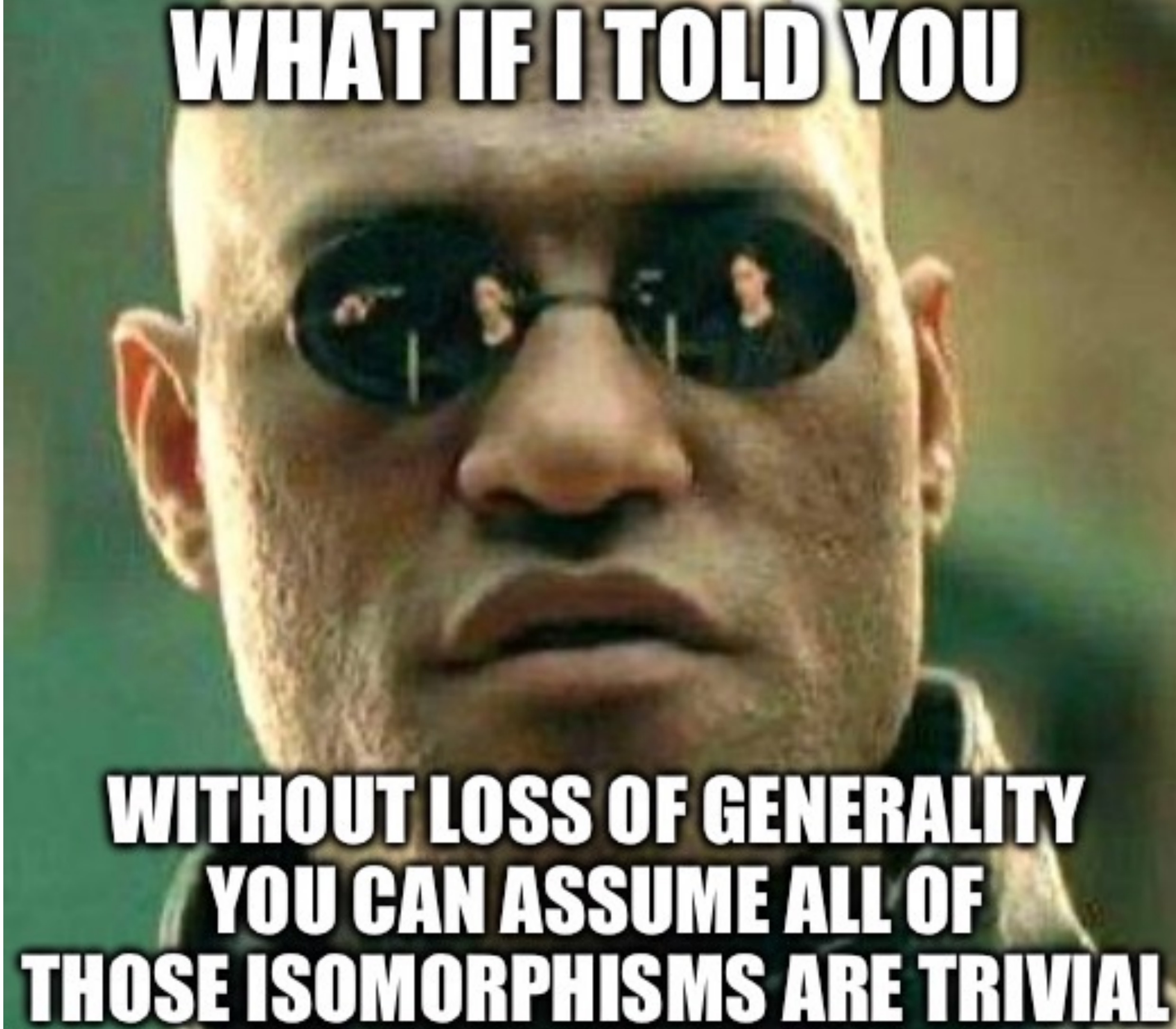
$$\alpha_{m,b,m} = \bigoplus_{a \in A} \chi(a,b)\, \mathrm{id}_a$$

$$\alpha_{m,m,m} = \left(\frac{\tau}{\chi(a,b)} \cdot \mathrm{id}_m \right)_{a,b} : \bigoplus_{a \in A} m \longrightarrow \bigoplus_{b \in A} m$$

STRUCTURE OF A FUSION CATEGORY

$$\left(\mathcal{C}, \otimes, \mathbb{1}, \alpha \right)$$

+ NICE FINITENESS, DUALITY PROPERTIES


WHAT IF I TOLD YOU
WITHOUT LOSS OF GENERALITY
YOU CAN ASSUME ALL OF
THOSE ISOMORPHISMS ARE TRIVIAL

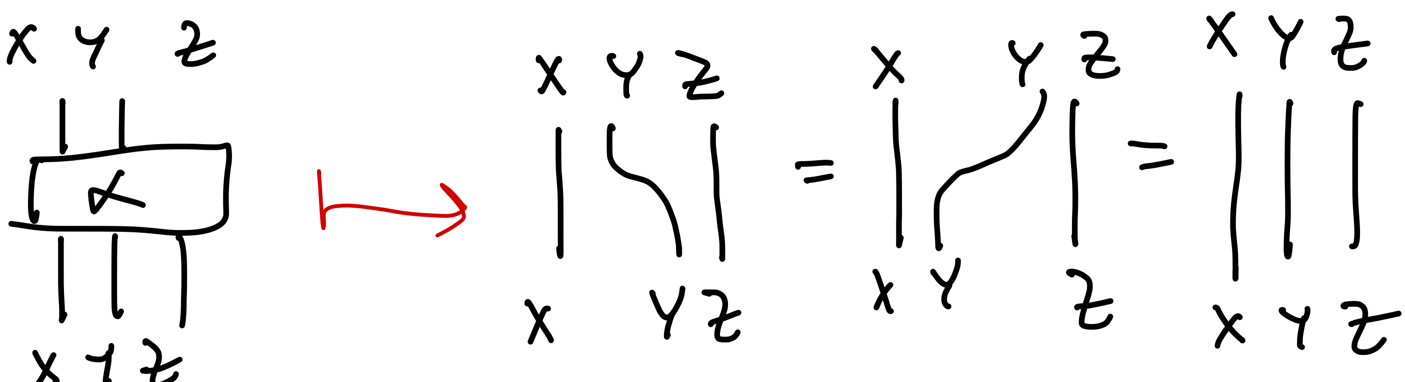
MACLANE'S STRICTNESS THEOREM

EVERY MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, \alpha)$
IS MONOIDALLY EQUIVALENT TO A STRICT MONOIDAL CATEGORY

I.E. ONE IN WHICH

$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \longrightarrow X \otimes (Y \otimes Z)$$

ARE $\alpha_{X,Y,Z} = id_{X \otimes Y \otimes Z}$



DUALS AND RIGIDITY

AN OBJECT IS A LEFT DUAL X^* OF X IF $\exists$ MORPHISMS

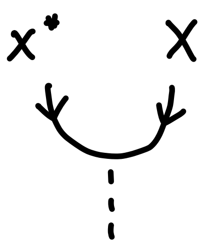
$$\mathrm{eval}_X : X^* \otimes X \longrightarrow \mathbb{1} \quad \& \quad \mathrm{coeval}_X : \mathbb{1} \longrightarrow X \otimes X^*$$

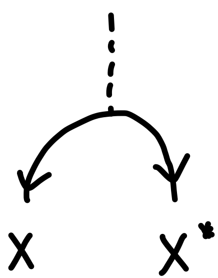
SATISFYING THE "ZIG-ZAG AXIOM"

① $X \xrightarrow{\mathrm{coev}_X} (X \otimes X^*) \otimes X \xrightarrow{\alpha_{X,X^*,X}} X \otimes (X^* \otimes X) \xrightarrow{\mathrm{eval}_X} X = X \xrightarrow{\mathrm{id}_X} X$

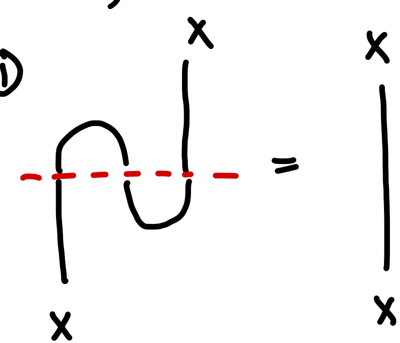
② $X^* \xrightarrow{\mathrm{coev}_X} X^* \otimes (X \otimes X^*) \xrightarrow{\alpha_{X^*,X,X^*}} X^* \otimes (X \otimes X^*) \xrightarrow{\mathrm{eval}_X} X^* = X^* \xrightarrow{\mathrm{id}_{X^*}} X^*$

IN PICTURES WHEN & STRICT





S.T. ①



+ SIMILAR PICTURE FOR ②

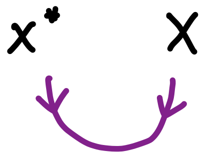
RIGHT DUALS DEFINED ANALOGOUSLY

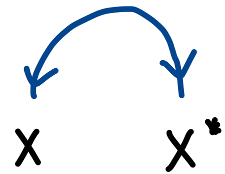
REMARK ON STRING DIAGRAMS IN A FUSION CATEGORY

IF WE HAVE
MORE STRUCTURE,
WE CAN DRAW KNOTS & LINK

(WILL WANT FOR RESHETIKHIN-TURAEV!

IN A RIGID MONOIDAL CATEGORY HAVE

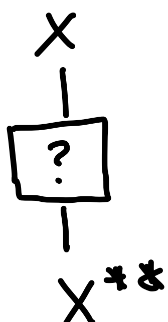


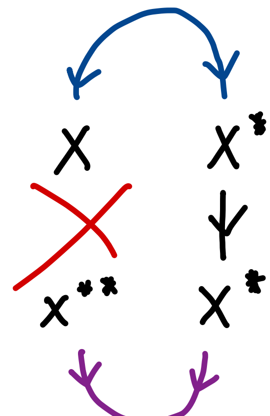


& SIMILARLY FOR RIGHT DUALS

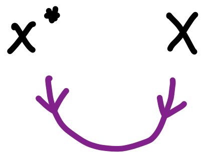
NOTICE SOMETHING ANNOYING THOUGH!

CAN'T COMPOSE THEM TO GET A CIRCLE WITHOUT ISOMORPHISMS

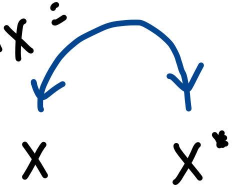




IN A RIGID MONOIDAL CATEGORY HAVE

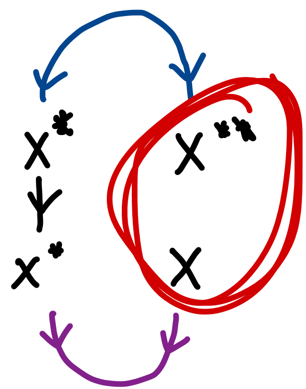


$coeval_X$:

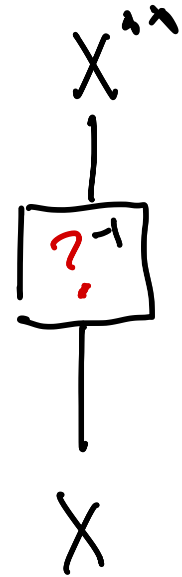


& SIMILARLY FOR RIGHT DUALS

$coeval_{X^*}$



NEED ISOMORPHISM



PIVOTALITY, TRACES, SPHERICALITY, QUANTUM DIMENSION

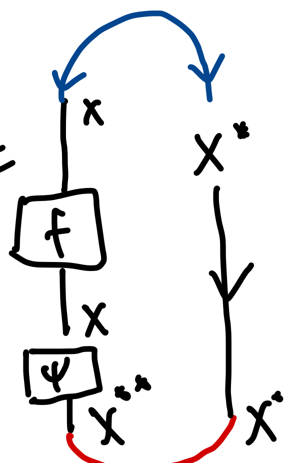
A PIVOTAL STRUCTURE IS

NATURAL ISOMORPHISMS $\psi_X : X \longrightarrow X^{**}$

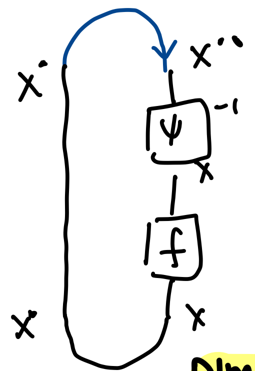
$$\psi_{X \otimes Y} = \psi_X \otimes \psi_Y$$

CAN DEFINE TRACES OF A MORPHISM $f \in \mathrm{Hom}(X, X)$

$Tr^R(f) =$



$Tr^L(f) =$



SPHERICALITY

$Tr^R(f) = Tr^L(f)$

JUST CALL IT $Tr(f)$

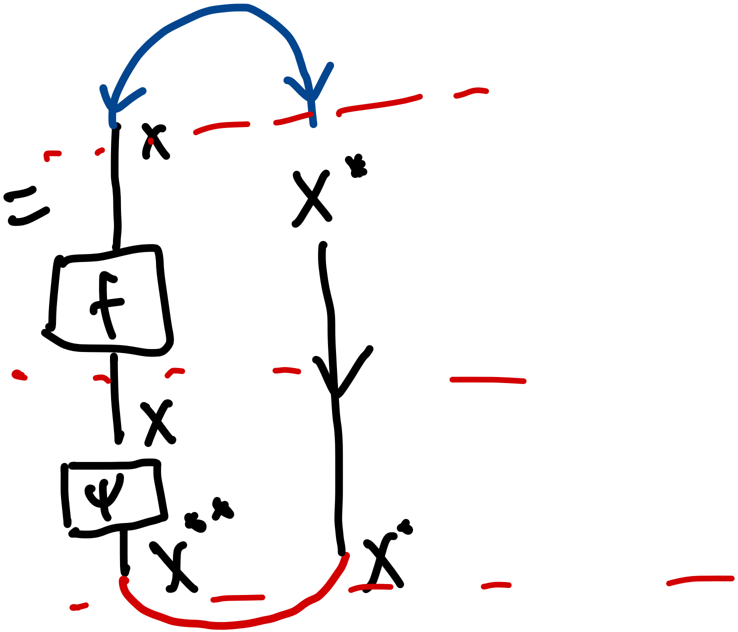
QUANTUM DIMENSION OF SIMPLE OBJECTS

$$d_X = Tr(id_X)$$

DIMENSION $D = \sum_{X \text{ simple}} d_X^2$

CAN ALWAYS TRANSLATE BACK INTO EQUATIONS!

$Tr^R(f) =$



$$Tr^R(f) = (\mathrm{eval}_{X^*}) \circ (\varphi \otimes \mathrm{id}_{X^*}) \circ (f \otimes \mathrm{id}_{X^*}) \circ (\mathrm{coeval}_X)$$