Fusion categories and TQFT: Problem Set 2

Learning Objectives

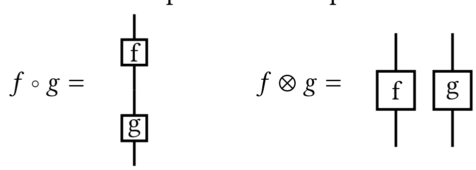
- Get experience working with string diagrams representing morphisms in a monoidal category.
- Get experience working with trivalent graphs representing morphisms in a skeletal fusion category and their associated numerical data.

Review

All of our diagrams are read from the top down unless indicated otherwise by arrows.

String diagrams in a monoidal category

Recall how to compose and take the tensor product of morphisms in terms of string diagrams:

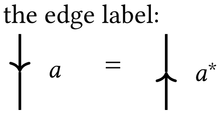


We also have cups and caps representing the evaluations and coevaluation morphisms $\mathrm{eval}_X : X^* \otimes X \to \mathbb{1}$ and $\mathrm{coeval}_X : \mathbb{1} \to X \otimes X^*$, respectively:

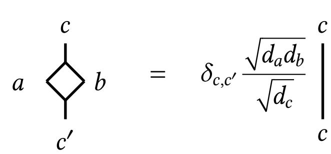
Partial summary of trivalent graphical calculus in a skeletal fusion category

We will assume that our skeletal fusion category is multiplicity-free.

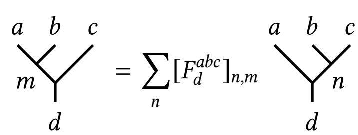
Reversing orientation of edges dualizes the edge label:



"Bubble-popping":



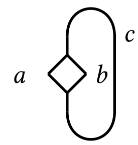
The F-move:



Exercises

1. Let $\mathcal{C}$ be a monoidal category and let $f \in \text{Hom}(X,Y)$. The (left) *dual* $f^* \in \text{Hom}(Y^*, X^*)$ of f is given by
$$f^* = (\text{eval}_Y \otimes_{X^*}) \circ ((\text{id}_{Y^*} \otimes f) \otimes \text{id}_{X^*}) \circ \alpha^{-1}_{Y^*,X,X^*} \circ (\text{id}_{Y^*} \otimes \text{coeval}_X).$$
Draw the string diagram that represents f^* assuming that $\mathcal{C}$ is strict.

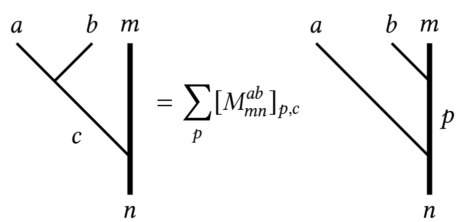
2. Compute the value of the term $\Theta(a,b,c)$ that appears in the Turaev-Viro state sum formula in terms of the quantum dimensions d_a, where

$\Theta(a,b,c) =$ 

3. Recall that the quantum dimension of a simple object in a spherical fusion category $\mathcal{C}$ is given by $d_X = \text{Tr}(\text{id}_X)$. Let's assume that $\mathcal{C}$ is skeletal and moreover that its pivotal coefficients t_a are all trivial. Use the graphical calculus for skeletal spherical fusion categories to find a formula for d_a in terms of the F-symbols.
Hints: Start with the picture for the quantum dimension of a (it looks the same in any spherical fusion category, skeletal or otherwise). You'll need to start by inserting a strand labeled by the identity; think of making your picture look like the picture for $\Theta(a,b,c)$ in Exercise 2 above. Then remember that your trivalent diagrams live on the surface of a sphere and pay close attention to the orientation of edges.

4. Recall the definition of a (left) $\mathcal{C}$-module category, either from Problem Set 1 Exercise 2 or from Definition 7.1.1 in *Tensor Categories*.

   Assume that $\mathcal{C}$ is multiplicity-free and that the structure constants that define the module action on the level of isomorphism classes of objects in $\mathcal{C}$ and $\mathcal{M}$ are multiplicity-free (see Problem Set 0 Exercise 6 or Definition 3.4.1 in *Tensor Categories*). Skeletalizing the (left) module associators
$$\mu_{X,Y,M} : (X \otimes Y) \triangleright M \to X \otimes (Y \triangleright M)$$
gives "M-symbols" where



   Note that $a, b, c \in L$ while $m, n, p \in M$, where L is our chosen basis of the fusion ring of $\mathcal{C}$ and M is the chosen basis of the fusion module of $\mathcal{M}$.

   Derive the (left) module pentagon equations. (It should involve an instance of the F-symbols!)