LECTURE 2

SKELETAL FUSION CATEGORIES

↳ COMPUTER-ASSISTED CLASSIFICATION OF FUSION CATEGORIES

↳ TURAEV-VIRO STATE SUM TQFT

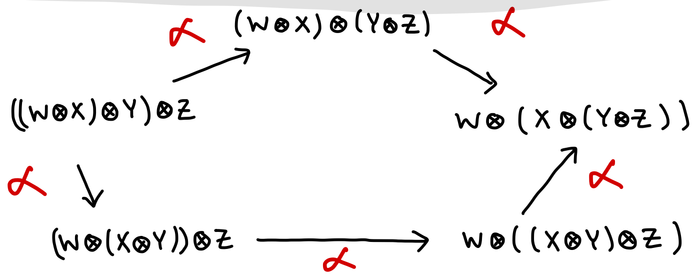
↳ ~~LEVIN-WEN HAMILTONIAN FOR LATTICE TQFT~~

MAYBE LATER

RECAP OF YESTERDAY'S LECTURE & LOOSE ENDS

RECALL THAT A FUSION CATEGORY IS A RIGID $\mathbb{C}$-LINEAR MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, \alpha)$ WITH SOME NICE FINITENESS/SEMISIMPLICITY PROPERTIES

ASSOCIATORS (α)

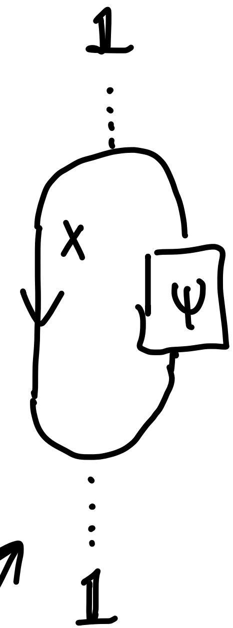


IN PARTICULAR, A FUSION CATEGORY HAS FINITELY MANY SIMPLE OBJECTS UP TO ISOMORPHISM

$IRR(\mathcal{C})$ = SET OF REPRESENTATIVES OF ISO. CLASSES OF SIMPLES $= \{\mathbb{1}, A, A^*, B, B^*, \ldots\}$

EACH SIMPLE (ISO. CLASS) IN A SPHERICAL FUSION CATEGORY HAS AN ASSOCIATED "QUANTUM DIMENSION" $d_X = Tr(id_X)$

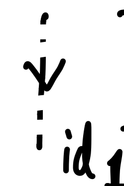
$d_X = Tr(id_X) =$



$\in HOM_{\mathcal{C}}(\mathbb{1}, \mathbb{1}) \cong \mathbb{C}$

$=$

$SPAN_{\mathbb{C}} \{$



$\}$

IS THE SCALAR MULTIPLE OF

TWO POWERFUL METHODS FOR FUSION CATEGORIES

① "STRICTIFICATION"

WLOG $\alpha_{X,Y,Z} = id_{X \otimes Y \otimes Z}$

(DISCUSSED YESTERDAY, MAKES STRING DIAGRAMS EVEN NICER)

② "SKELETONIZATION" OR "SKELETALIZATION" OR "COMBINATORIALIZATION"

TODAY'S TOPIC

CAN'T USE BOTH TOOLS AT SAME TIME THOUGH!

SKELETALIZATION

① PICK ONE OBJECT PER ISOMORPHISM CLASS OF SIMPLE OBJECT

PASS TO $\mathcal{L} = \{ 1, a, a^*, b, b^*, \ldots \}$

(CONFLATE SIMPLE OBJECTS W/ THEIR ISO CLASS)

② PICK A BASIS FOR EVERY "TRIVALENT" HOM SPACE

$$\mathrm{HOM}(a \otimes b, c) \quad \& \quad \mathrm{HOM}(c, a \otimes b)$$

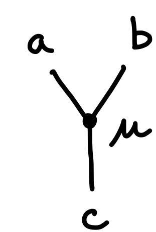
↑ "TRIVALENT FUSION SPACE" ↑ "TRIVALENT SPLITTING SPACE"

BOTH HAVE SAME DIMENSION,

$$\dim(\mathrm{HOM}(a \otimes b, c)) = \dim(\mathrm{HOM}(c, a \otimes b)) = N_{ab}^{c}$$

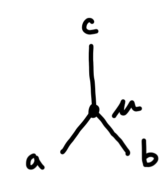
SO NEED TO PICK N_{ab}^{c} BASIS ELEMENTS, SAY

$$\left\{ |a, b, c; \mu\rangle \right\}_{\mu \in 1, 2, \ldots, N_{ab}^{c}} \rightsquigarrow \left\{ \sqrt[1/4]{\frac{d_c}{d_a d_b}} \right.$$



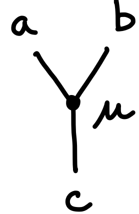
$$\Big\}_{\mu}$$

$$\left\{ \langle a, b, c; \mu| \right\}_{\mu \in 1, 2, \ldots, N_{ab}^{c}} \rightsquigarrow \left\{ \sqrt[1/4]{\frac{d_c}{d_a d_b}} \right.$$



$$\Big\}_{\mu}$$

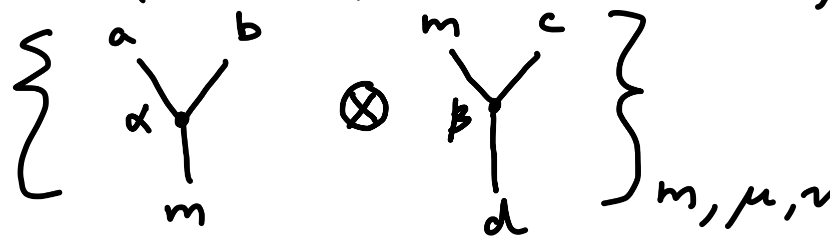
③ THIS CHOICE OF BASIS

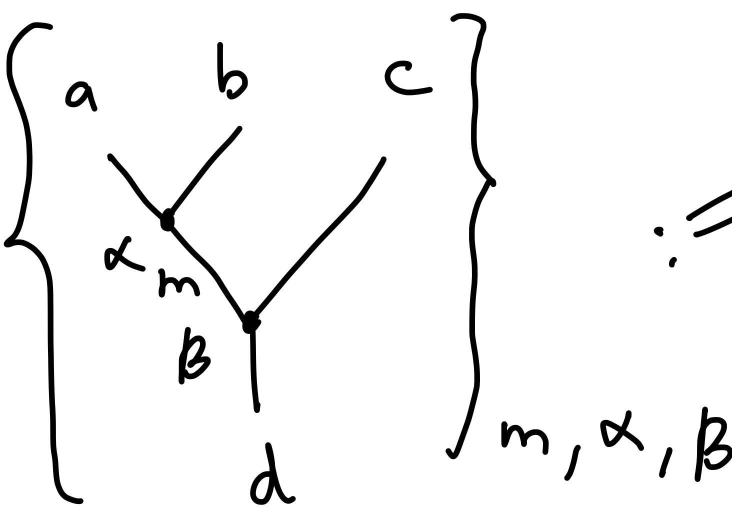


FOR $HOM(a \otimes b, c)$

INDUCES A CHOICE OF BASIS ON ALL HOM SPACES, LIKE SO:

$$HOM((a \otimes b) \otimes c, d) \cong \bigoplus_{m} HOM(a \otimes b, m) \otimes HOM(m \otimes c, d)$$





$:=$

BUT SINCE WE ONLY HAVE ONE OBJECT PER ISOMORPHISM CLASS, $(a \otimes b) \otimes c = a \otimes (b \otimes c)$ AS OBJECTS

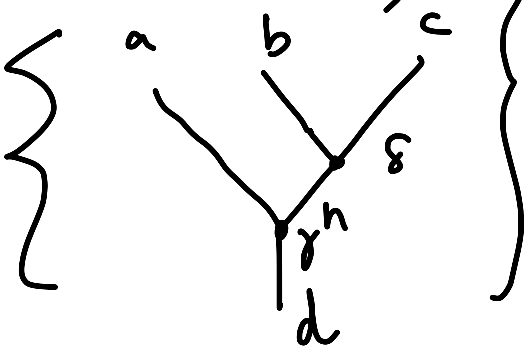
AND $\mathrm{HOM}((a \otimes b) \otimes c, d) = \mathrm{HOM}(a \otimes (b \otimes c), d)$

SO WE EQUALLY WELL COULD HAVE PICKED THE BASIS INDUCED BY

$$\mathrm{HOM}(a \otimes (b \otimes c), d) \cong \bigoplus_n \mathrm{HOM}(a \otimes n, d) \otimes \mathrm{HOM}(n, b \otimes c)$$

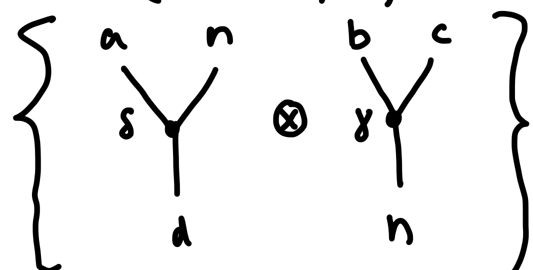
$\|$

$\mathrm{Hom}((a \otimes b) \otimes c, d)$



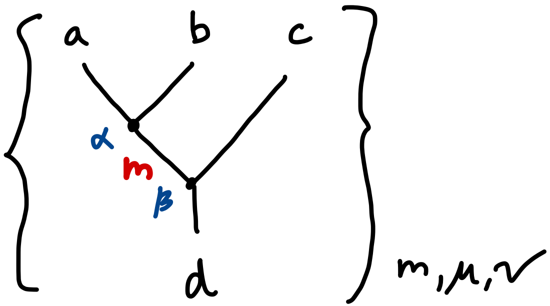
d, δ, γ

$:=$

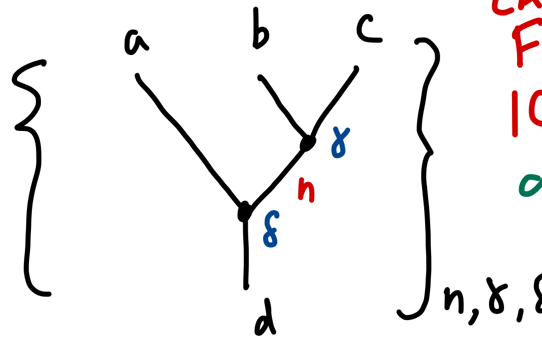


n, γ, δ

SO THERE ARE TWO NATURAL BASES OF $\mathrm{HOM}(a \otimes b \otimes c, d)$



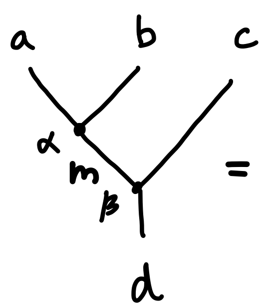
AND



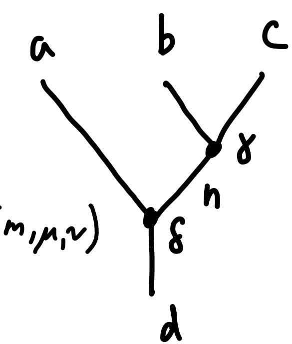
CALLED F-SYMBOLS, 10j-SYMBOLS OR 6j-SYMBOLS

CALL THE CHANGE OF BASIS MATRIX $\left[F^{abc}_{d}\right]_{(n,\gamma,\delta);(m,\mu,\nu)}$

THEN IN PICTURES



$$= \sum_{n,\gamma,\delta} \left[F^{abc}_{d}\right]_{(n,\gamma,\delta);(m,\mu,\nu)}$$

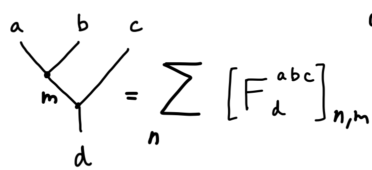


"F-MOVE"

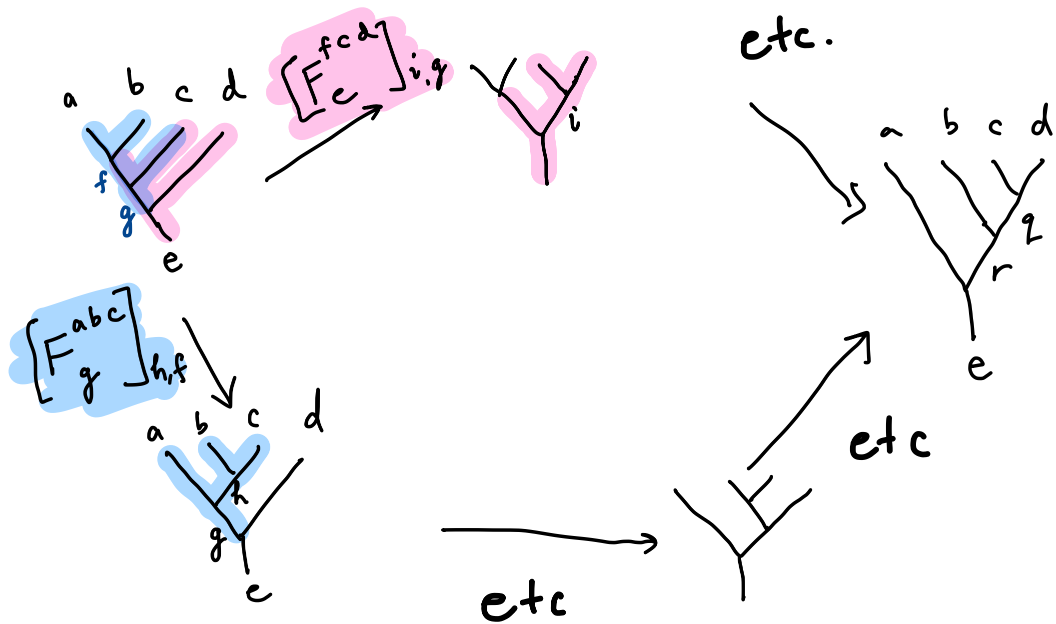
IF $N_{ab}^{c} \in \{0,1\}$ CAN IGNORE GREEK INDICES

PENTAGON EQUATIONS (ASSUME MULTIPLICITY-FREE)

DERIVE BY APPLYING



TO MOVE BETWEEN EDGES



PENTAGON EQUATIONS (IN FULL GENERALITY)

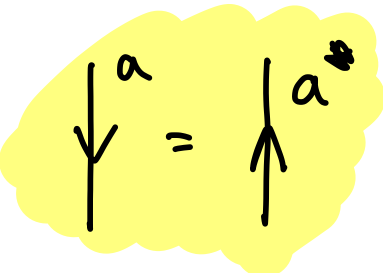
$$\sum_{h,\sigma,\psi,\rho} \left[F^{abc}_{g}\right]_{(f,\alpha,\beta);(h,\psi,\sigma)} \left[F^{ahd}_{e}\right]_{(g,\sigma,\gamma);(k,\rho,\lambda)} \left[F^{bcd}_{k}\right]_{(h,\psi,\rho);(l,\nu,\mu)}$$

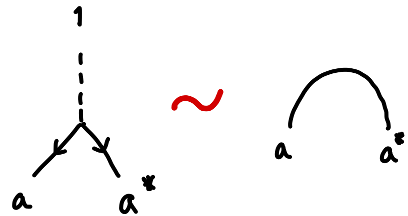
$$= \sum_{\delta} \left[F^{fcd}_{e}\right]_{(g,\beta,\gamma);(l,\nu,\delta)} \left[F^{abl}_{e}\right]_{(f,\alpha,\delta);(k,\mu,\lambda)}$$

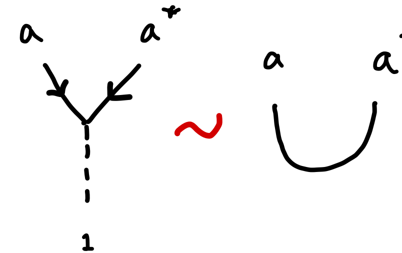
[BONDERSON]

DUALS, RIGIDITY *

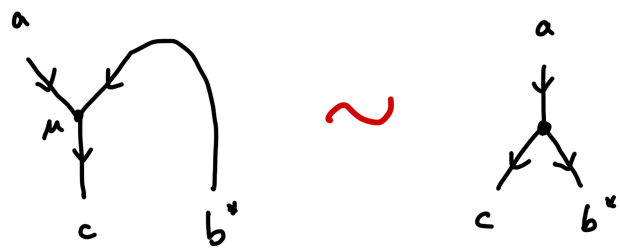
* (SUBTLE, WON'T DESCRIBE RIGOROUSLY HERE)







SUFFICE IT TO SAY THERE ARE ALSO LOCAL MOVES THAT ENCODE DUALITY, RIGIDITY



SPHERICALITY (IN THE MULTIPLICITY-FREE CASE)

WE WON'T DISCUSS THE DERIVATION, BUT YOU CAN "SKELETALIZE" THE SPHERICAL STRUCTURE

$$t_a \in \{\pm 1\} \quad \text{SATISFYING}$$

$$\begin{cases} t_1 = 1 \\ t_a^{-1} = t_{a^*} \\ t_a^{-1} t_b^{-1} t_c = \left[F_1^{abc^*}\right]_{a^*c} \left[F_1^{bc^*a}\right]_{a^*a} \left[F_1^{c^*ab}\right]_{b^*b} \end{cases}$$

WHAT HAVE WE DONE?

$$(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \psi)$$ SPHERICAL FUSION CATEGORY

(ISOMORPHISMS SATISFYING COHERENCE AXIOMS)

"COMBINATORIALIZE"

FUSION RULES

F-SYMBOLS

SPHERICAL CONSTANTS

$$\left\{ N^{ab}_{c}, \left[F^{abc}_{d}\right]_{(n,\gamma,\delta);(m,\alpha,\beta)}, t_a \right\}$$

(COMPLEX NUMBERS SATISFYING EQUATIONS)

"6j FUSION SYSTEM"

APPLICATION: COMPUTER ASSISTED CLASSIFICATION OF FUSION CATEGORIES

LOOK FOR SOLUTIONS

$$N_{ab}^{c} \in \mathbb{Z}_{\geq 0}, \quad \left[F^{abc}_{d}\right]_{(n,\gamma,\delta);(m,\alpha,\beta)} \in \mathbb{C}, \quad t_a \in \{\pm 1\}$$

TO

FUSION RING

$$\begin{cases} \sum_x N_{ab}^{x} N_{xc}^{d} = \sum_x N_{bc}^{x} N_{ax}^{d} \\ N_{1a}^{b} = \delta_{ab} = N_{a1}^{b} \\ N_{a^*b}^{1} = \delta_{ab} = N_{ba^*}^{1} \\ N_{ab}^{c} = N_{a^*c}^{b} = N_{cb^*}^{a} \end{cases}$$

SPHERICAL EQUATIONS

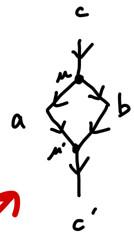
$$\begin{cases} t_1 = 1 \\ t_a^{-1} = t_{a^*} \\ t_a^{-1} t_b^{-1} t_c = \left[F^{abc^*}_{1}\right]_{a^*c} \left[F^{bc^*a}_{1}\right]_{a^*a} \left[F^{c^*ab}_{1}\right]_{b^*b} \end{cases}$$

PENTAGON EQUATIONS

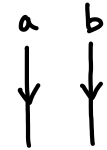
$$\begin{cases} \sum_{h,\sigma,\psi,\rho} \left[F^{abc}_{g}\right]_{(f,\alpha,\beta);(h,\psi,\sigma)} \left[F^{ahd}_{e}\right]_{(g,\sigma,\gamma);(k,\rho,\lambda)} \left[F^{bcd}_{k}\right]_{(h,\psi,\rho);(l,\nu,\mu)} \\ = \sum_{\delta} \left[F^{fcd}_{e}\right]_{(g,\beta,\gamma);(l,\nu,\delta)} \left[F^{abl}_{e}\right]_{(f,\alpha,\delta);(k,\mu,\lambda)} \end{cases}$$

(INCOMPLETE)

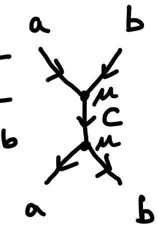
SUMMARY OF TRIVALENT GRAPHICAL CALCULUS



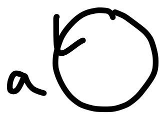
$$= \delta_{c,c'}\,\delta_{\mu,\mu'}\,\frac{\sqrt{d_a d_b}}{\sqrt{d_c}}$$



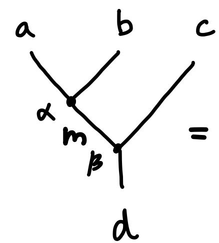
$$= \sum_{c,\mu} \sqrt{\frac{d_c}{d_a d_b}}$$



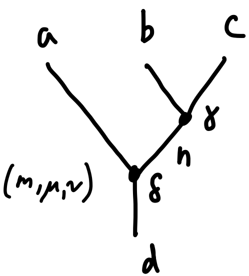
"BUBBLE POPPING"



$$= d_a$$



$$= \sum_{n,\gamma,\delta} \left[F^{abc}_{d}\right]_{(n,\gamma,\delta);(m,\mu,\nu)}$$



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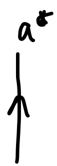


APPLICATION: TURAEV-VIRO STATE SUM TQFT

($\mathcal{C}$ SKELETAL FUSION CAT & MULTIPLICITY-FREE)

INPUT

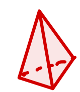
$\{N_c^{ab}, [F_d^{abc}]_{nm}, t_a\}$ → STATE SUM → $Z_{\mathcal{C}}(X^3) \in \mathbb{C}$

(X^3, T) → STATE SUM

OUTPUT

$Z_{\mathcal{C}}(X^3) \in \mathbb{C}$ WHICH IS AN INVARIANT OF THE MANIFOLD

X^3: 3-MANIFOLD

T: $\bigsqcup_i \Delta_i / \sim$ aka TETRAHEDRA $\Delta_i :=$  MODULO GLUING RELATIONS

CALL

$T^{(0)}$ VERTICES $\quad T^{(2)}$ FACES

$T^{(1)}$ EDGES $\quad T^{(3)}$ TETRAHEDRA

OF T

APPLICATION: TURAEV-VIRO STATE SUM TQFT

STATE SUM FORMULA

$$\hookrightarrow \quad Z_e(X^3) = \sum_S \frac{\prod_{T^{(3)}} \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} \prod_{T^{(1)}} d_a}{\prod_{T^{(2)}} \Theta(a,b,c) \prod_{T^{(0)}} D}$$

WHERE A STATE IS A MAP FROM $S: T^{(1)} \longrightarrow \mathcal{L} = \{a, b, c, \ldots\}$

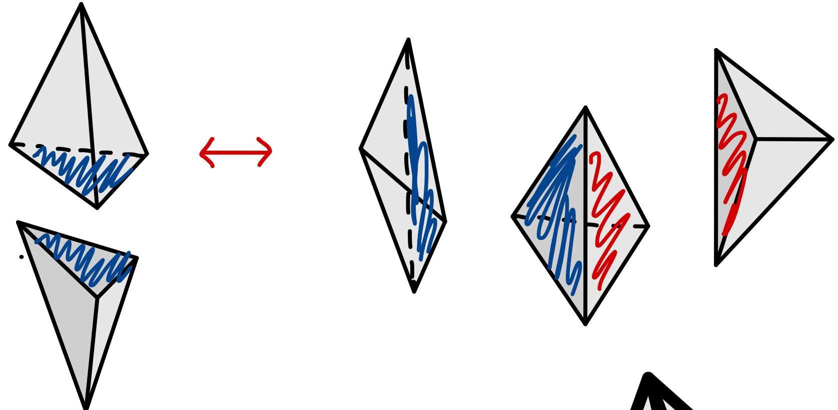
(edge) (labels)

& WHERE $\Theta(a,b,c) \sim \sqrt{d_a d_b d_c}$ $\quad \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} \sim \left[F^{abc}_{d}\right]_{ef}$

APPLICATION: TURAEV-VIRO STATE SUM TQFT

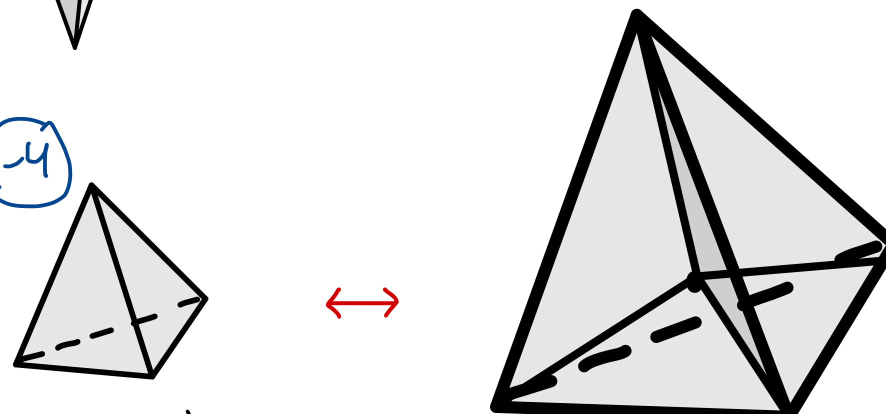
ANY TWO TRIANGULATIONS T, T' OF X^3 ARE RELATED BY 2-3 & 1-4 PACHNER MOVES

2-3



SO IT SUFFICES TO SHOW THE STATE SUM FORMULA IS INVARIANT UNDER THESE MOVES

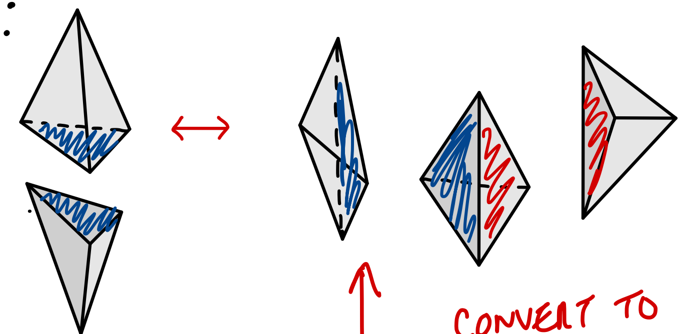
1-4



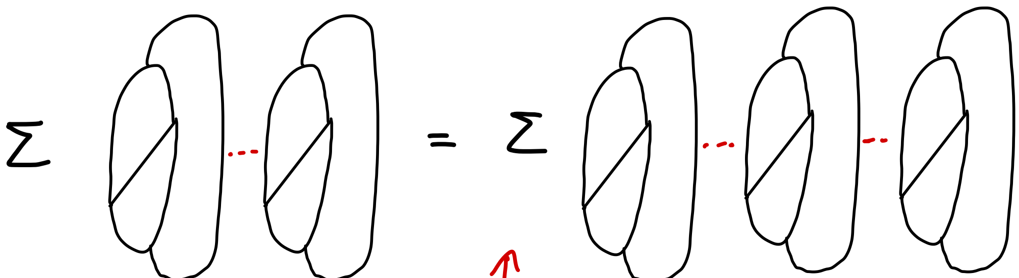
WANT TO SHOW

$$Z_e(X^3, T) = Z(X^3, T')$$

IDEA OF PROOF:



CONVERT TO TRIVALENT GRAPHS



diagrams labeled by data from $\mathcal{C}$

$$\sum F^2 = \sum F^3$$

PENTAGON EQUATIONS

STRICTIFICATION VS SKELETALIZATION