Fusion categories and TQFT: Problem Set 4

Learning Objectives

- Get experience working with string diagrams representing morphisms in a braided fusion category.
- Reinforce the definitions you were introduced to through previous lectures and exercises, in particular monoidal structures, monoidal functors, and module categories while learning about how they interact with braided structures.

Review

You may need to refer to the other Problem Sets and Lecture Slides quite a bit to complete these exercises so I will keep this bit short today!

Exercises

1. Recall the (skeletal) category Vec_G^ω of G-graded $\mathbb{C}$-vector spaces with associativity constraint given by a 3-cocycle ω from Lecture 1 and Problem Set 1.

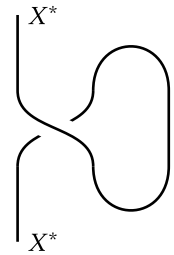
   (a) If Vec_G^ω is braided what must be true of the group G?

   (b) Call such a braiding $b(g,h) : \delta_g \otimes \delta_h \to \delta_h \otimes \delta_g$. Can you write down the hexagon equations that must be satisfied by ω and b?

2. Consider the following string diagram in a strict spherical, braided, rigid monoidal category

   $$(\mathcal{C}, \mathbb{1}, \otimes, \alpha \equiv \mathrm{id}, \mathrm{eval}_X : X^* \otimes X \to \mathbb{1}, \mathrm{coeval}_X : \mathbb{1} \to X \otimes X^*, \psi_X : X \to X^{**}, \beta_{X,Y} : X \otimes Y \to Y \otimes X),$$

   where the pivotal isomorphism has been used but has been suppressed in the picture. (Also we are only going to worry about left duals in the definition of rigidity.)

   Can you translate this picture back into an equation for the morphism in terms of the structure in $\mathcal{C}$? (Your formula should include ψ even though it is not drawn explicitly in the picture.)

3. (a) Recall from Problem Set 1 the definition of a monoidal functor (F, J) between monoidal categories $(\mathcal{C}, \otimes_\mathcal{C}, \mathbb{1}_\mathcal{C}, \alpha^\mathcal{C})$ and $(\mathcal{D}, \otimes_\mathcal{D}, \mathbb{1}_\mathcal{D}, \alpha^\mathcal{D})$. Suppose now that $\mathcal{C}$ and $\mathcal{D}$ are braided with braiding $\beta_\mathcal{C}$ and $\beta_\mathcal{D}$. A *braided* monoidal functor is a monoidal functor (F, J) satisfying an extra commutative diagram. Label the edges of this commutative diagram that expresses this compatibility of the braided structures with the monoidal functor:

$$\begin{tikzcd}
F(X \otimes Y) \arrow[r] \arrow[d] & F(X) \otimes F(Y) F(Y \otimes X) \arrow[d] \\
F(Y \otimes X) \arrow[r] & F(Y) \otimes F(X)
\end{tikzcd}$$

(b) Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha)$ be a monoidal category and recall from Problem Set 1 Exercise 2 the definition of a left $\mathcal{C}$-module category $\mathcal{M}$ with bimodule action functor $\triangleright : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ and natural isomorphisms

$$\mu^L_{X,Y,M} : (X \otimes Y) \triangleright M \to X \triangleright (Y \triangleright M)$$

satisfying the left module pentagon axioms.

A right $\mathcal{C}$-module category $\mathcal{N}$ is defined analogously with respect to a right bimodule action functor $\triangleleft : \mathcal{C} \times \mathcal{N} \to \mathcal{C}$ and natural isomorphisms

$$\mu^R_{N,X,Y} : (N \triangleleft X) \triangleleft Y \to N \triangleleft (X \otimes Y)$$

satisfying right module pentagons axioms.

Show that if $(\mathcal{M}, \triangleright, \mu^L)$ is a left $\mathcal{C}$-module category and $\mathcal{C}$ is *braided* then $\mathcal{M}$ also has the structure of a right $\mathcal{C}$-module category where the right action bifunctor is defined at the level of objects by:

$$M \triangleleft X := X \triangleright M.$$

(Try to write down natural isomorphisms μ^R satisfying the right module pentagons using things you already have lying around.)