LECTURE 3

BRAIDED FUSION CATEGORIES

DRINFELD CENTERS OF FUSION CATEGORIES

↳ FROM TURAEV-VIRO TO RESHETIKHIN-TURAEV TQFT

BUT FIRST, SOME EXAMPLES OF SKELETAL FUSION CATEGORIES

FIB

$\mathcal{L} = \{1, \tau\}$

$$\begin{cases} \tau \otimes \tau = 1 \oplus \tau \\ \left[F^{\tau\tau\tau}_{\tau}\right] = \begin{array}{c|cc} & 1 & \tau \\ 1 & \phi^{-1} & \phi^{-1/2} \\ \tau & \phi^{-1/2} & \phi^{-1} \end{array} \\ \text{ALL OTHER ADMISSIBLE F-SYMBOLS} = 1 \end{cases}$$

ϕ GOLDEN RATIO

ISING

$\mathcal{L} = \{1, \sigma, \psi\}$

$$\begin{cases} \sigma \otimes \sigma = 1 \oplus \psi \\ \sigma \otimes \psi = \psi \otimes \sigma = \sigma \\ \psi \otimes \psi = 1 \end{cases}$$

$$\begin{cases} \left[F^{\sigma\sigma\sigma}_{\sigma}\right] = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ F^{\psi\sigma\psi}_{\sigma} = F^{\sigma\psi\sigma}_{\psi} = -1 \\ \text{ALL OTHER ADMISSIBLE F-SYMBOLS} = 1 \end{cases}$$

MOTIVATION: FROM FUSION CATEGORIES TO BRAIDED FUSION CATEGORIES

3D TQFT

1-CATEGORICAL VERSION

$$Z : \mathrm{BORD}_3 \longrightarrow \mathrm{Vec}$$

IN PARTICULAR

$$\left\{ \begin{array}{c} \emptyset \\ X^3 \\ \emptyset \end{array} \right. \longmapsto \underbrace{Z_{\mathcal{C}}(X^3, T)}_{\text{TURAEV-VIRO STATE-SUM}} = \underbrace{Z_{Z(\mathcal{C})}(X^3, L)}_{\text{RESHETIKHIN-TURAEV INVARIANT}}$$

TRIANGULATION

SURGERY DIAGRAM ON LINK

DRINFELD CENTER

$(\infty,3)$-CATEGORICAL VERSION

$$Z : \mathrm{BORD}_3 \longrightarrow \mathrm{Alg}_1(\mathrm{Cat})$$

$\vdots \qquad \vdots$

$\bigcirc \longmapsto \mathrm{END}_{\mathcal{C}\text{-bimod}}({}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}}) \simeq Z(\mathcal{C})$

$\bullet\!\!-\!\!\bullet \longmapsto {}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}}$ ($(\mathcal{C},\mathcal{C})$ BIMODULE)

$\bullet \longmapsto \mathcal{C}$

ALGEBRA: BFCs ARE "LIKE" QUANTUM FINITE ABELIAN GROUPS

TOPOLOGY: INVARIANTS (QUANTUM) OF KNOTS/LINKS

BRAIDED MONOIDAL CATEGORIES

A MONOIDAL CAT. $(\mathcal{C}, \otimes, \mathbb{1}, \alpha)$ IS BRAIDED IF $\exists$ NATURAL ISOMORPHISMS

$$\beta_{X,Y} : X \otimes Y \longrightarrow Y \otimes X$$

SATISFYING HEXAGON AXIOMS

1

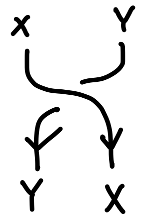
$$\begin{tikzcd}
 & X \otimes (Y \otimes Z) \arrow[r, "\beta_{X, Y \otimes Z}"] & (Y \otimes Z) \otimes X \arrow[dr, "\alpha_{Y,Z,X}"] & \\
(X \otimes Y) \otimes Z \arrow[ur, "\alpha_{X,Y,Z}"] \arrow[dr, "\beta_{X,Y}"'] & & \circlearrowright & Y \otimes (Z \otimes X) \\
 & (Y \otimes X) \otimes Z \arrow[r, "\alpha_{Y,X,Z}"'] & Y \otimes (X \otimes Z) \arrow[ur, "\beta_{X,Z}"'] &
\end{tikzcd}$$

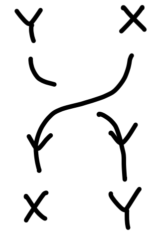
2

$$
\begin{tikzcd}
 & Z\otimes(X\otimes Y) \arrow[r, "\alpha^{-1}"] & (Z\otimes X)\otimes Y \arrow[dr, "\beta_{Z,X}"] & \\
(X\otimes Y)\otimes Z \arrow[ur, "\beta_{X\otimes Y,Z}"] & & & (X\otimes Z)\otimes Y \\
 & X\otimes(Y\otimes Z) \arrow[ul, "\alpha^{-1}"] \arrow[r, "\beta_{Y,Z}"'] & X\otimes(Z\otimes Y) \arrow[ur, "\alpha^{-1}"'] &
\end{tikzcd}
$$

STRING DIAGRAMS IN A (STRICT) BRAIDED MONOIDAL CATEGORY

REPRESENT

$\beta_{X,Y} =:$ 

$\beta^{-1}_{X,Y} =:$ 

NOW TRANSLATE COMMUTATIVE DIAGRAM INTO STRING DIAGRAM

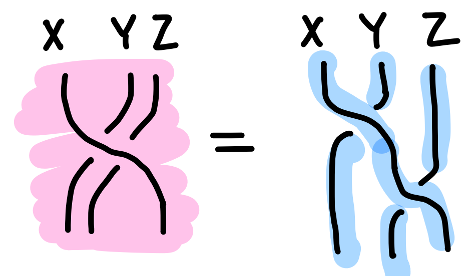
(HEXAGON COLLAPSES TO TRIANGLE)

$$\begin{tikzcd}
 & X\otimes(Y\otimes Z) \arrow[r, "\beta_{X,Y\otimes Z}"] & (Y\otimes Z)\otimes X \arrow[dr, equal] & \\
(X\otimes Y)\otimes Z \arrow[ur, equal] \arrow[dr, "\beta_{X,Y}"'] & & \circlearrowright & Y\otimes(Z\otimes X) \\
 & (Y\otimes X)\otimes Z \arrow[r, equal] & Y\otimes(X\otimes Z) \arrow[ur, "\beta_{X,Z}"'] &
\end{tikzcd}$$

$\rightsquigarrow$

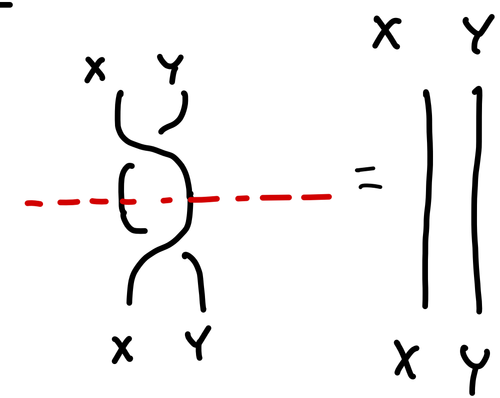
$$\begin{tikzcd}
 & Y\otimes Z\otimes X \\
X\otimes Y\otimes Z \arrow[ur, "\beta_{X,Y\otimes Z}"] \arrow[r, "\beta_{X,Y}"'] & Y\otimes X\otimes Z \arrow[u, "\beta_{X,Z}"'] \arrow[ul, phantom, "\circlearrowright"]
\end{tikzcd}$$

$\rightsquigarrow$



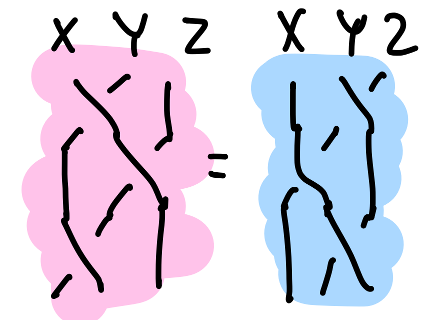
PROPERTIES OF BRAIDED ISOMORPHISMS

NOTICE $\beta_{X,Y}^{-1} \circ \beta_{X,Y} = id_{X \otimes Y}$ $\rightsquigarrow$



THEOREM: $\mathcal{C}$ STRICT BRAIDED MONOIDAL

$$\begin{tikzcd}
 & X \otimes Y \otimes Z \arrow[r, "\beta_{X,Y} \otimes id_Z"] \arrow[dl, "id_X \otimes \beta_{Y,Z}"'] \arrow[dd, "\beta_{X \otimes Y, Z}"] \arrow[ddr, phantom, "\circlearrowright"] & Y \otimes X \otimes Z \arrow[dr, "id_Y \otimes \beta_{X,Z}"] \arrow[dd, "\beta_{Y \otimes X, Z}"'] & \\
X \otimes Z \otimes Y \arrow[dr, "\beta_{X,Z} \otimes id_Y"'] \arrow[r, phantom, "\circlearrowright \ 1"] & {} & {} \arrow[r, phantom, "\circlearrowright \ 2"] & Y \otimes Z \otimes X \arrow[dl, "\beta_{Y,Z} \otimes id_X"] \\
 & Z \otimes X \otimes Y \arrow[r, "id_Z \otimes \beta_{X,Y}"'] & Z \otimes Y \otimes X &
\end{tikzcd}$$

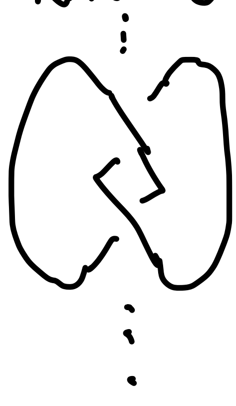


BRAID RELATION

NOW CAN DRAW BRAID DIAGRAMS

IN $(\mathcal{C}, \otimes, \mathbb{1}, \alpha \equiv id, \beta)$

WE CAN ALSO CUP/CAP/TRACE OFF DIAGS TO GET KNOT LINK DIAGRAMS



HOPF-LINK

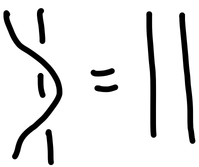
$\in \mathrm{HOM}(\mathbb{1}, \mathbb{1}) \cong \mathbb{C}$

FRAMED REIDEMEISTER THEOREM

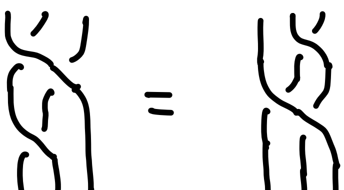
TWO FRAMED LINK DIAGRAMS REPRESENT THE SAME FRAMED LINK IF THEY CAN BE RELATED BY A FINITE # OF REIDEMEISTER MOVES II & III AND ISOTOPY AND RI'

~~RI~~

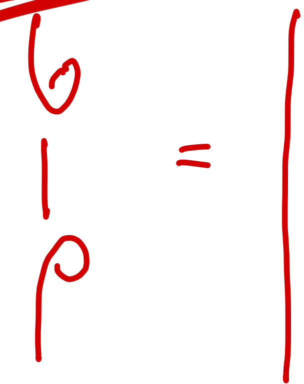
RII



RIII



RI'



FOR FRAMED LINKS

DEGENERACY OF BRAIDED STRUCTURE & EXAMPLES

MOST BORING ⟵⟶ LEAST BORING

SYMMETRIC

$\beta_{X,Y} = \beta_{Y,X}^{-1}$

e.g. Rep(G)

e.g. Rep(G,z) (SUPER REPS)

NON-DEGENERATE

THM: SYMMETRIC BFCS ARE EITHER Rep(G) OR Rep(G,z)

NON-DEGENERACY IN BRAIDED MONOIDAL CATEGORY MEASURED BY $Z_2(\mathcal{C})$

SUBCATEGORY OF $\mathcal{C}$ GENERATED BY OBJECTS X S.T. $\beta_{X,Y} = \beta_{Y,X}^{-1}$ $\forall Y \in X$ — SYMMETRIC CENTER

IN A SPHERICAL BRAIDED FUSION CATEGORY $\mathcal{C}$ CAN MEASURE WITH "S-MATRIX"

$$S_{X,Y} = Tr\left(\beta_{Y,X^*} \circ \beta_{X^*,Y}\right) =$$

WHERE $X, Y \in IRR(\mathcal{C})$

IF $S_{X,Y}$ HAS DETERMINANT 0, SAY $\mathcal{C}$ IS NON-DEGENERATE

THE DRINFELD CENTER OF A MONOIDAL CATEGORY $\mathcal{C}$

$\mathcal{Z}(\mathcal{C})$ IS THE CATEGORY WHOSE OBJECTS ARE PAIRS

$$\Big(Z, \{\beta_{X,Z}\}_{X\in\mathcal{C}}\Big) \qquad \beta_{X,Z} : X \otimes Z \longrightarrow Z \otimes X$$

OBJECT IN $\mathcal{C}$

"HALF-BRAIDING"

SO THAT

$$\begin{tikzcd}
 & X\otimes(Y\otimes Z) \arrow[r, "\mathrm{id}\otimes\beta_{Y,Z}"] & X\otimes(Z\otimes Y) \arrow[dr, "\alpha^{-1}"] & \\
(X\otimes Y)\otimes Z \arrow[ur, "\alpha"] \arrow[dr, "\beta_{X\otimes Y,Z}"'] & & \circlearrowleft & (X\otimes Z)\otimes Y \\
 & Z\otimes(X\otimes Y) \arrow[r, "\alpha^{-1}"'] & (Z\otimes X)\otimes Y \arrow[ur, "\beta_{Z,X}\otimes \mathrm{id}_Y"'] &
\end{tikzcd}$$

AND MORPHISMS $f \in \mathrm{HOM}_{\mathcal{Z}(\mathcal{C})}\Big(\big(Z,\{\beta_{X,Z}\}_{X\in\mathcal{C}}\big), \big(Z', \{\beta'_{X,Z'}\}_{X\in\mathcal{C}}\big)\Big)$

ARE $f \in \mathrm{HOM}_{\mathcal{C}}(Z, Z')$

$$\begin{tikzcd}
X\otimes Z \arrow[r, "\beta_{X,Z}"] \arrow[d, "f"'] & Z\otimes X \arrow[d, "f"] \\
X\otimes Z' \arrow[r, "\beta_{X,Z'}"'] & Z'\otimes X
\end{tikzcd}$$

generally speaking $\mathcal{Z}(\mathcal{C})$ inherits structures from $\mathcal{C}$

$\mathcal{C}$ monoidal $\otimes$ $\Rightarrow$ $\mathcal{Z}(\mathcal{C})$ monoidal $\otimes$

$\mathcal{C}$ pivotal (spherical) $\Rightarrow$ $\mathcal{Z}(\mathcal{C})$ pivotal (spherical)

$\mathcal{C}$ fusion $\Rightarrow$ $\mathcal{Z}(\mathcal{C})$ fusion

but $\mathcal{Z}(\mathcal{C})$ has additional structure, namely a braiding!

$\mathcal{Z}(\mathcal{C})$ IS A WAY TO COOK UP A BRAIDED FUSION CATEGORY FROM ONE WHICH IS NOT

NOTIONS OF EQUIVALENCE OF FUSION CATEGORIES

① MONOIDAL EQUIVALENCE $\quad F: \mathcal{C} \longrightarrow \mathcal{D} \quad J_{X,Y}: F(X \otimes Y) \cong F(X) \otimes F(Y)$

② "MORITA EQUIVALENCE"

IMAGINE I HAD $\mathcal{C}'$ WHOSE $\mathcal{Z}(\mathcal{C}') \simeq \mathcal{Z}(\mathcal{C})$

$$Z_{\mathcal{C}}(X) = Z_{\mathcal{Z}(\mathcal{C})}(X) = Z_{\mathcal{Z}(\mathcal{C}')}(X) = Z_{\mathcal{C}'}(X)$$

TV TQFT $\qquad$ RT TQFT

SO $\mathcal{C}, \mathcal{C}'$ DEFINE SAME 3-MFLD INVARIANT

TECHNICALLY MORITA EQUIVALENCE IF $\underline{\underline{\mathrm{MOD}}}(\mathcal{C}) \simeq \underline{\underline{\mathrm{MOD}}}(\mathcal{D})$ AS 2-CATS, BUT

<u>THM:</u> $\mathcal{C} \underset{\text{MORITA}}{\simeq} \mathcal{D} \Leftrightarrow \mathcal{Z}(\mathcal{C}) \simeq \mathcal{Z}(\mathcal{D})$

AS BRAIDED FUSION CATS

<u>EXAMPLE:</u> $\mathrm{Vec}_G \simeq \mathrm{Rep}(G)$

MORITA EQUIVALENCE IS WEIRD!

e.g. $\mathrm{Vec}_{S_3} \underset{\text{MORITA}}{\simeq} \mathrm{Rep}(S_3)$

RANK 6 RANK 3

SO VERY DIFFERENT THAN MONOIDAL EQUIVALENCE