Character Variety Exercises

1| (GIT vs. Geometric Quotient.)
Suppose that $\mathbb{C}^* \curvearrowright \mathbb{C}^2$ with $\lambda \cdot (x,y) = (\lambda x, \lambda^{-1} y)$.

(a) What are the $\mathbb{C}^*$ orbits? (These are the points of $\mathbb{C}^2 / \mathbb{C}^*$.)

(b) Describe the ring of invariant polynomial functions, $\mathbb{C}[x,y]^{\mathbb{C}^*}$.

(c) Find the points that sit in different orbits but can't be distinguished by $\mathbb{C}^*$-invariant functions. (These points are therefore identified in the GIT quotient.)

2| (Character variety.)

(a) Show that $\left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right) \sim \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$ in $Ch_{SL_2\mathbb{C}}(T^2)$

(b) Let $\Sigma_{1,g}$ be a genus g surface with one puncture. What's $R_G(\Sigma_{1,g})$?

(c) Find a group G such that $R_G(\Sigma) \simeq Ch_G(\Sigma)$ for all Σ.

(d) Show that the Poisson bracket on $Ch_{SL_2\mathbb{C}}(\Sigma)$ is invariant under isotopy, i.e. $\{\tau_\alpha, \tau_\beta\}$ respects: