SPT AND MANIFOLD INVARIANTS: EXERCISE 1

MENG GUO

1. Prove that bordism is an equivalence relation.
2. Show that diffeomorphic d-manifolds are bordant.
3. Prove that the mod 2 Euler number defines a bordism invariant $\Omega_d(O) \to \mathbb{Z}/2\mathbb{Z}$. (Show it is additive under disjoint union and zero if a manifold is bordant to empty) On the other hand, show that the integer-valued Euler number is not generally a bordism invariant. Give an couterexample by a manifold bordant to empty set but with nonzero Euler number. Euler number is in fact stably almost complex bordism invariant.
4. Let me write down the definition of p_i, the ith Pontryagin class of a real vector bundle V over a manifold M: $p_i(V) = (-1)^i c_{2i}(V \otimes \mathbb{C}) \in H^{4i}(M, \mathbb{Z})$. In fact, Chern classes of $V \otimes \mathbb{C}$ of odd degrees are of 2 torsions.
5. For a $2n$ real vector bundle V over X, there exists $Y \to X$ such that the induced cohomology is injective and the pullback of $V \otimes \mathbb{C}$ can always split as
$$L_1 \oplus \overline{L_1} \oplus L_n \oplus \overline{L_n}$$
L_i are line bundles. You can show that $c(L_i \oplus \overline{L_i}) = (1 + x_i)(1 - x_i)$, $x_i = c_1(L_i) \in H^2(X, \mathbb{Z})$. $p(V) = \prod(1 + x_i^2)$. For a $4k$ manifold, define $p(M) = p(TM)$. Write $\hat{A} = \prod \dfrac{x_i/2}{\sinh(x_i/2)}$.In fact, you can write $\hat{A} = 1 - \frac{p_1}{24} + \cdots \in \mathbb{Q}[p_1, p_2, \cdots]$. In particular, Write $\hat{A}(M)$ or $\hat{A}_{4k}(M) = \langle \hat{A}_{4k}, [M] \rangle \in \mathbb{Q} : \Omega_{4k}(SO) \to \mathbb{Q}$. If $M \in \Omega_{4k}(\mathrm{Spin})$, it lies in $\mathbb{Z}$. Show that $\hat{A}(M)$ is bordism invariant.
6. Here is a simpler question of last one to show that p_1 is bordism invariant. Given a 4-manifold M, let $p_1(M) \in H^4(M)$ denote its first Pontrjagin class. If M is oriented, we can pair that with the fundamental class to obtain $\langle p_1(M), [M] \rangle \in \mathbb{Z}$. It is (oriented) bordism invariant. (Hints: It's additive under disjoint union; if M is boundary of 5-dimensional manifold W, prove that $\langle p_1(M), [M] \rangle = 0$ by consider the exact sequence $H^4(W) \to H^4(M) \to H^5(W, M)$, $p_1(W)$ maps to $p_1(M)$)
7. In fact, all product of $p_i, i \in I$ (I can contains repeated index i) of degree $4n(I) = 4\sum i$ and evaluated on a oriented $4n(I)$ manifold is a bordism invariant.
8. Definition of $\mathrm{Spin_n}$, $\mathrm{Pin_n^+}$, $\mathrm{Pin_n^-}$ and $\mathrm{Spin_n^c}$.