SPT AND MANIFOLD INVARIANTS: EXERCISE 3

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1. Given a Lie group Convince yourself that a G_n structure on a n-dimensional manifold M is equivalent to a choice of a G_n principle bundle P over M such that $P \times_{G_n} O_n$ is isomorphic to the orthonormal frame bundle of tangent bundle $Fr(TM)$.
2. Exhibit a natural homeomorphism $X^{V \oplus \mathbb{R}} \xrightarrow{\cong} \Sigma X^V$.
3. Definition/construction of Pin group:

   (i) 1st definition: For a vector space (over $\mathbb{R}$) with quadratic form $q : V \to \mathbb{R}$, define the Clifford algebra
   $$\mathrm{Cl}(V) = T(V)/(v^2 - q(v) \cdot 1).$$
   $T(E) = \mathbb{R} \oplus V \oplus V \otimes V \oplus \cdots$ be the tensor algebra over V and $(v^2 - q(v) \cdot 1)$ denote the two-sided ideal generated by the elements $v \otimes v - q(v) \cdot 1$ in $T(V)$. Multiplication is given by tensor, we just write $v \cdot w$ or vw to denote the multiplication $v \otimes w \in \mathrm{Cl}(V)$.
   Consider the subset of $\mathrm{Cl}(V)$ consisting of all products $v_1 v_2 \cdots v_k$ of unit vectors $v_i \in \mathbb{R}^n$ where $k \geqslant 0$. (if $k = 0$, it means ± 1). Denote it by $\mathrm{Pin}(V)$. It is a group whose multiplication is just multiplication in $\mathrm{Cl}(V)$. There is a surjection of groups:
   $$\mathrm{Pin}(V) \xrightarrow{\rho} O(V)$$
   sending $v_1 v_2 \cdots v_k$ to the $\gamma_1 \cdots \gamma_k$ where γ_i denotes the reflection in the hyperplane perpendiculular to v_i. <u>Check</u> that the kernel is $\{\pm 1\}$ and it lies in the center of $\mathrm{Pin}(V)$. If regard $x \in V \subset \mathrm{Cl}(V)$, <u>check</u> that $\rho(v_i) = -v_i x v_i$ and
   $$\rho(v_1 \cdots v_k) = (-1)^k v_1 \cdots v_k x v_k \cdots v_1$$
   If take $V = \mathbb{R}^n$ with positive definite quadratic form, it is Pin_n^+. If take $V = \mathbb{R}^n$ with negative definite, it is Pin_n^-.

   (ii) 2nd definition: An central extension of O_n by $\mathbb{Z}/2$ is classified by an element in $H^2(BO_n; \mathbb{Z}/2) = \mathbb{Z}/2[w_1, \cdots, w_n]$. $_1, \cdots, w_n$ are the universal Stiefel Whitney classes. Pin_n^+ corresponds to the element w_2 while Pin_n^- corresponds to the element w_1.
4. This question calculates $\mathrm{Pin}_1^{\pm}$ and $\mathrm{Pin}_2^{\pm}$:

   (i) Use two definition to show that $\mathrm{Pin}_1^+ = \mathbb{Z}/2 \times \mathbb{Z}/2$ while $\mathrm{Pin}_1^- = \mathbb{Z}/4$

   (ii) Notice that $\mathrm{Cl}(V) \cong \mathbb{R}[x]/x^2$ if $V = (\mathbb{R}^2, +)$ and $\mathrm{Cl}(V) \cong \mathbb{C}$ if $V = (\mathbb{R}^2, -)$. Compute $\mathrm{Pin}_2^{\pm}$ as a subgroup of these.

   (iii) Compute the preimage of dihedral group $D_n \subset O_2$. In Pin_2^+, it is D_{2n} while in Pin_2^-, it is D_{2n} dicyclic group.
5. Remark: In Q3, if the number k to be even, then the image of ρ is $SO(V)$. The subset is $\mathrm{Spin(V)}$ as a subgroup of $\mathrm{Pin}(V)$ and an central extension of $SO(V)$ by $\{\pm 1\}$ corresponding to $w_2 \in H^2(BSO_n, \mathbb{Z}/2)$.