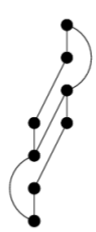
# SPT AND MANIFOLD INVARIANTS: EXERCISE 4

MENG GUO

1. Recall that $H^*(BO_1, \mathbb{F}_2) = H^*(\mathbb{R}P^2, \mathbb{F}_2) = \mathbb{F}_2[x]$, $x \in H^1(\mathbb{R}P^2, \mathbb{F}_2)$. Show that by axiomatic definition of Steenrod squares
$$\mathrm{Sq}^{2^k}\mathrm{Sq}^{2^{k-1}} \cdots \mathrm{Sq}^2\mathrm{Sq}^1 x = x^{k+1}$$
and all others $\mathrm{Sq}^I x = 0$ where $\mathrm{Sq}^I = \mathrm{Sq}^{i_1} \cdots \mathrm{Sq}^{i_n}$ for $I = (i_1, \cdots, i_n) \neq (2^n, \cdots, 1) \in \mathbb{Z}^n_{\geqslant 0}$.
2. Recall that the $BO_n$ classifying rank $n$ vector bundle. Let $f : BO_1 \times \cdots \times BO_1 \to BO_n$ be the map sending the n-tuple of line bundles $(L_1, \cdots, L_n) \mapsto L_1 \oplus \cdots L_n$. You can also think of this map as the induced map of classifying space of the group map $O_1 \cdots O_1 \to O_n$. The induced map on cohomology
$$f^* : H^*(BO_n, \mathbb{F}_2) \to H^*(BO_1 \times BO_n, \mathbb{F}_2) = \otimes H^*(BO_1, \mathbb{F}_2) = \mathbb{F}_2[x_1, x_2, \cdots x_n]$$
$$w_k \mapsto \sigma_k(x_1, \cdots, x_n)$$
where $\sigma_k$ is the elementary symmetric functions on the $n$ variables. (If you don't believe this, consider this map is invariant under the action the symmetric group $\Sigma_n$ permutating copies of line bundles). Use the definition to compute $\mathrm{Sq}^1 w_k$ and $\mathrm{Sq}^2 w_k$.
3. In the computation of $\mathrm{Ext}_{\mathcal{A}(0)}(\mathbb{F}_2, \mathbb{F}_2)$, show that $h_0^i$ is the generator of $\mathrm{Ext}^{i,i}_{\mathcal{A}(0)}(\mathbb{F}_2, \mathbb{F}_2)$, and therefore $\mathrm{Ext}_{\mathcal{A}(0)}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2[h_0]$
4. Show that a more general claim of last question: $\mathrm{Ext}_{\Lambda[x]}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2[y]$, where $\Lambda[x] = \mathbb{F}_2[x]/(x^2)$ and $y \in \mathrm{Ext}^{1,|x|}(\mathbb{F}_2, \mathbb{F}_2)$.
5. Show that $\mathrm{Ext}_{P \times Q}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{Ext}_P(\mathbb{F}_2, \mathbb{F}_2) \otimes \mathrm{Ext}_Q(\mathbb{F}_2, \mathbb{F}_2)$, $P$ and $Q$ are graded $\mathbb{F}_2$-algebras.
6. Use the Adem relations $\mathrm{Sq}^i\mathrm{Sq}^j = \sum_{0 \leqslant k \leqslant \frac{i}{2}} \binom{j-k-1}{i-2k} \mathrm{Sq}^{i+j-k}\mathrm{Sq}^k, i < 2j$ to show that
$$(\mathrm{Sq}^1\mathrm{Sq}^2 + \mathrm{Sq}^2\mathrm{Sq}^1)^2 = 0.$$
Write $Q_0 = \mathrm{Sq}^1$, $Q_1 = \mathrm{Sq}^1\mathrm{Sq}^2 + \mathrm{Sq}^2\mathrm{Sq}^1$. Convince that the subalgebra generated by $Q_0$ and $Q_1$ is $\mathbb{F}_2[Q_0, Q_1]/(Q_0^2, Q_1^2)$
7. Let $\mathcal{A}(1)$ the subalgebra generated by $\mathrm{Sq}^1, \mathrm{Sq}^2$. Show that $\mathcal{A}(1)$ looks like:



Dots stand for a copy $\mathbb{F}_2$ and vertical line means multiple by $\mathrm{Sq}^1$. Curly line mens multiple by $\mathrm{Sq}^2$

8. Try to compute $\mathrm{Ext}_{\mathcal{A}(1)}(\mathbb{F}_2, \mathbb{F}_2)$