Modular operads: Exercises Sheet 1

Lukas Müller

Graph categories

1. Define the operations π_0 and ν formally using the definition of graphs from the lecture in terms of $(H \to V, \iota)$. Show that for a rooted forest both π_0 and ν are rooted forests in a canonical way.

2. The goal of this exercise is to familiarize ourselves with the symmetric monoidal categories $\mathsf{RForests}$, $\mathsf{Forests}$, and Graphs introduced in the lecture.

   (a) Compute the automorphisms group of an arbitrary object of $\mathsf{RForests}$, $\mathsf{Forests}$, and Graphs. Are there non-invertible endomorphisms?

   (b) Show that all morphisms in $\mathsf{RForests}$, $\mathsf{Forests}$, and Graphs are generated (under composition and tensor product) by graphs with only one internal edge.

3. Look up the definition of an operad (for example on Wikipedia) and use the previous exercise to show that it is equivalent to symmetric monoidal functors $\mathsf{RForests} \to \mathsf{Set}$.

4. ($\star$) One might think that Graphs is more natural a 2-category. Define such a 2-category and compare it to the 1-categorical definition from the lecture.

5. For a corolla in $T \in \mathsf{Graphs}$ we define the category $\mathsf{Gr}_{\mathrm{conn}}(T)$ as follows: Its objects are connected graphs with an identification of their legs with T. Morphisms are given by collapsing of subtrees and symmetries of the graph. Show that $\mathsf{Gr}_{\mathrm{conn}}(T)$ is equivalent to the silce ℓ/T for $\ell : \mathsf{Forests} \to \mathsf{Graphs}$.

General structure of operads

1. Show that the existence of operadic units is a property, i.e. if an operadic unit exists its unique. Give an example for an operad without operadic unit.

2. Do limits and colimits exist in the category of set valued (modular) operads?

3. What data is needed to extend an ordinary operad to a cyclic one? What about extending from cyclic to modular operads?

4. A symmetric sequence is a symmetric monoidal functor $\mathsf{RForests}^{\times} \to \mathcal{C}$ from the core of $\mathsf{RForests}$ to $\mathcal{C}$. For $\mathcal{C} = \mathsf{Set}$ show that left Kan extension along $\mathsf{RForests}^{\times} \to \mathsf{RForests}$ defines the free operad on a symmetric sequence. In particular, show that it is symmetric monoidal. Does this work for arbitrary $\mathcal{C}$?

5. Can you generalize the previous exercise to cyclic operads?

6. Describe the action of the cyclic group on a composition of two operations in a cyclic operad.