Modular operads: Exercises Sheet 4

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Modular envelope

1. Show that $U_\int$ defined in the lecture is symmetric monoidal.

2. Show that π_0 of the Cat-valued modular envelope is the left adjoint of the restriction to cyclic operads.

3. What is π_0 of the modular envelop of the framed E_2 operad?

4. Use the previous exercise to proof that 2-dimensional Vect-valued TQFTs are classified by symmetric commutative Frobenius algebras.

5. ($\star$) The goal of this exercise is to explain the connection between the modular envelope and dihedral homology.

   (a) Denote by $\mathsf{Gr}^{\mathrm{a}}_{\mathrm{conn}}(T)$ the full subcategory of $\mathsf{Gr}_{\mathrm{conn}}(T)$ spanned by graphs containing no vertex of valence one. Show that for any corolla T, the inclusion $\iota : \mathsf{Gr}^{\mathrm{a}}_{\mathrm{conn}}(T) \to \mathsf{Gr}_{\mathrm{conn}}(T)$ is homotopy final.[1]

   (b) For any corolla T and a non-negative integer g, denote by $\mathsf{Gr}^{\mathrm{a},g}_{\mathrm{conn}}(T) \subset \mathsf{Gr}^{\mathrm{a}}_{\mathrm{conn}}(T)$ the full subcategory spanned by graphs whose first Betti number is g. Then $\mathsf{Gr}^{\mathrm{a},g}_{\mathrm{conn}}(T)$ is connected and

$$\mathsf{Gr}^{\mathrm{a}}_{\mathrm{conn}}(T) = \coprod_{g\geq 0} \mathsf{Gr}^{\mathrm{a},g}_{\mathrm{conn}}(T) \ .$$

   As an immediate consequence, we find

$$|U_\int \mathcal{O}|(T) \simeq \coprod_{g\geq 0} |U^g_\int \mathcal{O}|(T)$$
$$\text{with} \quad U^g_\int \mathcal{O}(T) := \int \left(\mathsf{Gr}^{\mathrm{a},g}_{\mathrm{conn}}(T) \subset \mathsf{Gr}_{\mathrm{conn}}(T) \to \mathsf{Forests} \to \mathcal{O}\mathsf{Cat}\right) \ .$$

   We call $U^g_\int \mathcal{O}(T)$ the *genus g contribution* to $U_\int \mathcal{O}(T)$. Let $\bullet$ be the corolla with no legs. Show that the category $\mathsf{Gr}^{\mathrm{a},1}_{\mathrm{conn}}(\bullet)$ is equivalent to the dihedral category without degeneracies defined as follows: Recall that *Connes' cyclic category* Λ is the category with objects $\mathbb{N}_0$; we denote the object corresponding to $n \geq 0$ by $[n]$. A morphism $f : [n] \to [m]$ is given by an equivalence class of functions $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(i+n+1) = f(i)+m+1$ modulo the relation $f \sim g$ if $f-g$ is a constant multiple of $m+1$. The category Λ contains the simplex category Δ as subcategory. As generating morphisms, it has the face and degeneracy maps that we already

[1] A functor $F : \mathcal{C} \to \mathcal{D}$ is called *homotopy final* if the realization $|(d/F)|$ of the nerve of the slice of F under any $d \in \mathcal{D}$ is contractible. This implies that F preserves homotopy colimits.

know from the simplex category Δ and the *cyclic permutations* $\tau_n : [n] \to [n]$ represented by maps $\mathbb{Z} \to \mathbb{Z}$ that shift by one. The cyclic permutations fulfill, besides the obvious relation $\tau_n^{n+1} = \mathrm{id}_{[n]}$, further compatibility relations with the face and degeneracy maps. We denote by $\vec{\Lambda} \subset \Lambda$ the subcategory of the cyclic category without degeneracy maps. The category Λ has a natural action of $\mathbb{Z}_2$ through the *reversal functor* $r : \Lambda \to \Lambda$ which is the identity on objects and sends any morphism $f : [n] \to [m]$ in Λ to $r(f) : [n] \to [m]$ given by $(r(f))(p) := m - f(n-p)$. We denote by $\Lambda \rtimes \mathbb{Z}_2$ the Grothendieck construction of the functor $B\mathbb{Z}_2 \to \mathsf{Cat}$ from the groupoid with one object and automorphism group $\mathbb{Z}_2$ to the category Cat of categories sending $*$ to Λ and the generator $-1 \in \mathbb{Z}_2$ to the reversal functor $r : \Lambda \to \Lambda$ (recall that the Grothendieck construction $\int F$ of a functor $F : \mathcal{C} \to \mathsf{Cat}$ is the category of pairs (c, x) formed by all $c \in \mathcal{C}$ and $x \in F(c)$). The category $\Lambda \rtimes \mathbb{Z}_2$ can be identified with the *dihedral category*. Restriction to $\vec{\Lambda}$ yields a functor $\vec{r} : \vec{\Lambda} \to \vec{\Lambda}$. This allows us to define $\vec{\Lambda} \rtimes \mathbb{Z}_2$, the *semidihedral category*, also via a Grothendieck construction. We can see $(\vec{\Lambda} \rtimes \mathbb{Z}_2)^{\mathrm{op}}$ as the Grothendieck construction of the $\mathbb{Z}_2$-action on $\vec{\Lambda}^{\mathrm{op}}$ through the functor $\vec{r}^{\,\mathrm{op}} : \vec{\Lambda}^{\mathrm{op}} \to \vec{\Lambda}^{\mathrm{op}}$ induced by r. In other words, $(\vec{\Lambda} \rtimes \mathbb{Z}_2)^{\mathrm{op}} = \vec{\Lambda}^{\mathrm{op}} \rtimes \mathbb{Z}_2$. Similarly, $(\Lambda \rtimes \mathbb{Z}_2)^{\mathrm{op}} = \Lambda^{\mathrm{op}} \rtimes \mathbb{Z}_2$.

(c) Functors out of the opposite categories of $\Lambda \rtimes \mathbb{Z}_2$ and $\vec{\Lambda} \rtimes \mathbb{Z}_2$ are called *dihedral objects* and *semidihedral objects*, respectively. Give a more explicit description of dihedral objects. Construct a dihedral object from an algebra with anti-involution. Show that the operations $\mathcal{O}(T_1)$ of a cyclic operad form a semidihedral object. We call it $\mathfrak{U}_{\mathcal{O}}$.

(d) For a semidihedral object $A\colon (\vec{\Lambda} \rtimes \mathbb{Z}_2)^{\mathrm{op}} \to \mathsf{Top}$ in spaces we define the dihedral homology as $\mathrm{DH}(A) = \mathrm{hocolim}_{[n] \in \vec{\Lambda}^{\mathrm{op}} \rtimes \mathbb{Z}_2} A([n])$. Show that $|U_{\int} \mathcal{O}^1(\bullet)| \cong \mathrm{DH}\, |\mathfrak{U}_{\mathcal{O}}|$.

Open 2D field theories

1. Let Z be a Vect valued open 2d TQFT. Compute the value of Z on the following boridsms:
   - (a) $S^1 \times [0,1]$ with no marked boundary intervals
   - (b) $S^1 \times [0,1]$ with two marked boundary intervals (for all possible positions of those)
   - (c) Genus g surfaces with n free boundary components and one interval.
   - (d) Is there a way of building a closed 2-dimensional field theory from an open one?

2. ($\star\star$) Look up/ work out the details missing in the proof $\mathcal{O} \cong |U_{\int} \mathrm{As}\,|$

3. What is the mapping class group of an annulus with a marked interval in both of its boundary components? How can you realize the elements of this group using ribbon graphs?

4. A Frobenius algebra is called Δ-separable if the composition $\Delta \circ \mu$ is the identity. What does this mean for the corresponding open TQFT. Can you use this to define a closed TQFT?

5. ($\star$) What can you say about unoriented open TQFTs?