Rigid Categories and Hopf Algebras

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1 Summary of Notes

This document provides an introduction to dualizable objects in categories of Hopf modules. The layout is as follows.

1. **Section 2** reviews basic result from category.
2. **Section 3** introduces additional structures a category might possess.
3. **Section 4** explores what it means for a category to have dual objects.
4. **Section 5** defines Hopf algebras, and related topics such as convolution.
5. **Section 6** defines Hopf modules and introduces their categories.
6. **Section 7** provides proofs for well-known results used in this document.

The second section can be safely skipped by those familiar with category theory. The third section is also unnecessary for those familiar with braidings, symmetries, and string diagrams. Sections 4 through to 6 provide the main definitions and results. Whenever possible, all definitions and theorems have been cited. Citations are given to reference texts, and in some cases, original sources are hard to identify. Whenever an example is taken from a third party, the example is also cited. Proofs have been provided for (almost) all theorems, with major steps spelled out explicitly. Note that sections 4 and 5 are independent, though section 6 draws on new concepts from both section 4 and section 5.

Remark 1. *To save time writing these notes, I have given equations rather than commuting diagrams. There is no deeper reason why I made this decision. I encourage anyone studying these notes to draw out the diagrams as well.*

2 Basic Category Theory

I first define some basic concepts from category theory for the unfamiliar reader. I also fix the notation that I use throughout these notes. This section is by no menas comprehense, and only includes the concepts used in later definitions, theorems, and examples.

2.1 Categories, Functors, and Natural Transformations

Definition 1 (Category [3]). A category $\mathcal{C}$ is a tuple $(\mathcal{C}_0, \mathcal{C}_1, \circ)$ subject to the following conditions.

1. $\mathcal{C}_0$ is a collection.
2. For all $A, B \in \mathcal{C}_0$, a collection $\mathcal{C}_1(A, B)$.
3. For all $A, B, C \in \mathcal{C}_0$, a mapping $\circ_{(A,B,C)} : \mathcal{C}_1(B, C) \times \mathcal{C}_1(A, B) \to \mathcal{C}_1(A, C)$.
4. (Associativity). For all $A, B, C, D \in \mathcal{C}_0$, $f \in \mathcal{C}_1(A, B)$, $g \in \mathcal{C}_1(B, C)$, and $h \in \mathcal{C}_1(C, D)$, $h \circ (g \circ f) = (h \circ g) \circ f$.
5. (Identity). For all $A \in \mathcal{C}_0$, there exists an $1_A \in \mathcal{C}_1(A, A)$ such that for all $B \in \mathcal{C}_0$, $f \in \mathcal{C}_1(A, B)$, and $G \in \mathcal{C}_1(B, A)$, both $f = f \circ 1_A$ and $g = 1_A \circ g$.

We call each $A \in \mathcal{C}_0$ an object. For objects $A, B \in \mathcal{C}_0$, we call $f \in \mathcal{C}_1(A, B)$ a morphism from A to B and write $f : A \to B$.

Example 1 (Categories). Categories are ubiquitous in mathematics:

1. (Category of Sets). The category Set has all sets as objects and all functions between sets as morphisms.
2. (Category of Partial Functions). The category **Pfn** has all sets as objects and all partial functions between sets as morphisms. Note that neither **Set** nor **Pfn** are defined by their objects.
3. (Category of Groups). The category **Grp** has all groups as objects and all group homomorphisms as morphisms.
4. (Category of R-Modules). Let R be a commutative ring. Then the category $\mathbf{Mod}(R)$ has all R-modules as objects and all R-module homomorphisms as morphisms.
5. (Category of F-Vector Spaces). Let F be a field. Then the category $\mathbf{Vect}(F)$ has all F-vector spaces as objects and all F-vector space homomorphisms as morphisms.
6. (Category of Finite F-Vector Spaces). Let F be a field. Then the category $\mathbf{Vect}_{\mathsf{fd}}(F)$ has all finite-dimensional F-vector spaces as objects and all homomorphisms between finite-dimensional F-vector spaces as morphisms.

When the context is clear, we refer to morphisms in $\mathbf{Mod}(R)$, $\mathbf{Vect}(R)$, and $\mathbf{Vect}_{\mathsf{fd}}(R)$ as *R-linear maps*. Two excellent sources for additional examples of categories are [3] and [2].

Definition 2 (Functor [3])**.** A *functor from a category $\mathcal{C}$ to a category $\mathcal{D}$* is a mapping $F_0 : \mathcal{C}_0 \to \mathcal{D}_0$, and a mapping $F_{(A,B)} : \mathcal{C}_1(A,B) \to \mathcal{D}_1(F_0(A), F_0(B))$ for each $A, B \in \mathcal{C}_0$, such that the following properties hold.

1. For all $A \in \mathcal{C}_0$, $F_{(A,A)}(1_A) = 1_{F_0(A)}$.

2. For all $A \xrightarrow{f} B \xrightarrow{g} C$, $F_{(A,C)}(g \circ f) = F_{(B,C)}(g) \circ F_{(A,B)}(f)$.

If $\left(F_0, F_{(-,-)}\right)$ is a functor, then we write $F : \mathcal{C} \to \mathcal{D}$.

Example 2 (Functors)**.** Many constructions are functors:

1. (Group Tensors). The tensor product of abelian groups and their homomorphisms is a functor $\otimes : \mathbf{Ab} \otimes \mathbf{Ab} \to \mathbf{Ab}$.

2. (Module Tensors). Let R be a commutative ring. The tensor product of R-modules and their homomorphisms is a functor $\otimes : \mathbf{Mod}(R) \otimes \mathbf{Mod}(R) \to \mathbf{Mod}(R)$. The same is true for $\mathbf{Vect}(R)$ and $\mathbf{Vect}_{\mathsf{fd}}(R)$.

3. (Opposite Category). Let $\mathcal{C}$ be a category. Then $\mathcal{C}^{\mathsf{op}}$ is the category obtained by reversing each arrow in $\mathcal{C}$. The opposite arrow to $f \in \mathcal{C}$ is f^{op}. Define $F : \mathcal{C} \to \mathcal{C}^{\mathsf{op}}$ such that $F_0(A) = A$ and $F(f) = f^{\mathsf{op}}$. It is not hard to show that F is a functor. A functor out of an opposite category is called a *contravariant functor*.

4. (Dualizing Object Functor). Let R be a commutative ring. Consider $F : \mathbf{Vect}(R) \to \mathbf{Vect}^{\mathsf{op}}(R)$ such that $F(V) = V^*$ and $F(f) = (\phi \mapsto \phi \circ f)$ Then F is a contravariant functor.

Definition 3 (Natural Transformation [3])**.** A *natural transformation from a functor $F : \mathcal{C} \to \mathcal{D}$ to a functor $G : \mathcal{C} \to \mathcal{D}$* is a morphism $\alpha_A : F_0(A) \to G_0(B)$ for each $a \in \mathcal{C}_0$ such that $\alpha_B \circ F(f) = G(f) \circ \alpha_A$ for all $A \xrightarrow{f} B$ in $\mathcal{C}$. If α is a natural transformation, then we write $\alpha : F \Rightarrow G$. If α is invertible, then we say α is a *natural isomorphism*.

2.2 Special Objects Characterized by Morphisms

Definition 4 (Terminal Object [11])**.** Let $\mathcal{C}$ be a category. An object $\top \in \mathcal{C}_0$ is *terminal* if for all objects $A \in \mathcal{C}_0$, there exists a unique morphism $A \to \top$.

Example 3 (Terminal Objects)**.** Some categories with terminal objects.

1. (Category of Set). The terminal object in **Set** is any singleton set, say $\{\star\}$. Since singleton sets are isomorphic in set, the choice of set is unimportant. The unique morphism from any set S to $\{\star\}$ is $e_S : x \mapsto \star$. Note that in the category **Pfn**, e_s is no longer unique.

2. (Category of Groups) The terminal object in **Grp** is the trivial group G_1. The unique morphism from any group G to G_1 is $e_G : x \mapsto 1$.

3. (Category of Small Categories). The terminal object in **Cat** is the category **1** with a single object, say $\star$, and a single morphism $i : \star \to \star$ such that $\star$ is the identity. For every category $\mathcal{C}$, there exists a functor $F : \mathcal{C} \to \mathbf{1}$ such that $F_0 : X \mapsto \star$ and $F_{A,B} : f \mapsto i$. Clearly F respects function composition $(i \circ i = i)$ and all identities $(F_{A,A}(1_A) = i = 1_{F_0(A)})$. Since $\star$ is the only object in **1**, and i is the only morphism in **1**, then F is unique.

For other examples, see [11].

2.3 Categorical Products and Product Functors

Definition 5 (Product [3])**.** Let $\mathcal{C}$ be a category. The *binary product* of objects $A_1 \in \mathcal{C}_0$ and $A_2 \in \mathcal{C}_0$ is an object $A_1 \times A_2 \in \mathcal{C}_0$ equipped with a pair of morphisms $\pi_1 : A_1 \times A_2 \to A_1$ and $\pi_2 : A_1 \times A_2 \to A_2$ such that for all objects $C \in \mathcal{C}_0$ morphisms $A_1 \xleftarrow{f_1} C \xrightarrow{f_2} A_2$ in $\mathcal{C}$, there exists a unique morphism $h : C \to A_1 \times A_2$ such that $\pi_1 \circ h = f_1$ and $\pi_2 \circ h = f_2 cat$.

Example 4 (The Cartesian Product is a Product)**.** The Cartesian product in **Set** is a categorical product. Given objects $A \in \mathbf{Set}$ and $B \in \mathbf{Set}$, the morphisms $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ are the Cartesian projections $\pi_1 : (a, b) \mapsto a$ and $\pi_2 : (a, b) \mapsto b$. Given morphisms $A \xrightarrow{f} A'$ and $B \xrightarrow{g} B'$, then morphism $(f \times g) : A \times B \to A' \times B'$ is $(f \times g) : (a, b) \mapsto (f(a), g(b))$. It is left as an exercise to prove this construction is universal.

Theorem 1 ([2])**.** *If $\mathcal{C}$ is a category with all binary products, then there exists a functor $\Pi : \mathcal{C}^2 \to \mathcal{C}$ such that $\Pi_0(A, B) = A \times B$, $\pi_1 \circ \Pi(f, g) = f \circ \pi_1$, and $\pi_2 \circ \Pi(f, g) = g \circ \pi_2$, where π_1 and π_2 are the canonical projections for $A \times B$.*

Proof. See Section 8 □

3 Monoidal Categories and String Diagrams

This section introduces string diagrams for monoidal structures. The diagrams used in this section will be applied to better understand duals and algebras.

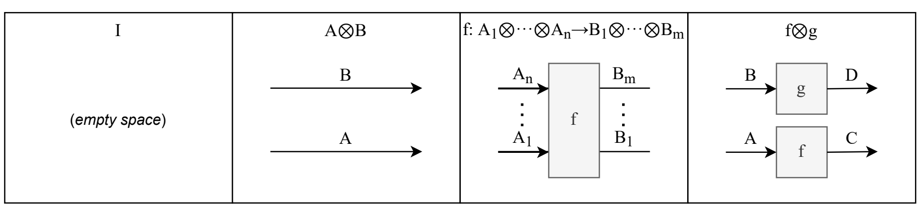
3.1 Monoidal Categories

Definition 6 (Monoidal Category [3])**.** A *monoidal category (alternatively, a tensor category)* is a tuple $(\mathcal{C}, \otimes, I, \lambda, \rho, \alpha)$ subject to the following conditions.

1. $\mathcal{C}$ is a category and $I \in \mathcal{C}_0$.
2. $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor.
3. $\lambda_A : I \otimes A \Rightarrow A$ is a natural isomorphism.
4. $\rho_A : A \otimes I \Rightarrow A$ is a natural isomorphism.
5. $\alpha_{(A,B,C)} : (A \otimes B) \otimes C \Rightarrow A \otimes (B \otimes C)$ is a natural isomorphism.
6. (Triangle Axiom). If $A, B \in \mathcal{C}_0$, then $(\rho_A \otimes 1_B) = (1_A \otimes \lambda_B) \circ \big(\alpha_{(A,B,C)}\big)$.
7. (Pentagon Axiom). If $A, B, C, D \in \mathcal{C}_0$, then $\alpha_{(A,B,C\otimes D)} \circ \alpha_{(A\otimes B,C,D)} = (1_A \otimes \alpha_{(B,C,D)}) \circ \alpha_{A,B\otimes C,D} \circ (\alpha_{(A,B,C)} \otimes 1_D)$

Remark 2. *Note that $(\mathcal{C}, \otimes, I)$ is a monoid where $\otimes$ is the binary relation and I is the unit. The existence of λ, α, and ρ, together with the triangle and pentagon axioms, ensure that $\otimes$ is associative and unital with respect to I.*

Remark 3. *The axioms for monoidal categories are equivalent to planar deformations of monoidal string diagrams [13]. The diagrammatic language for monoidal categories is given below.*



Example 5 (Common Monoidal Categories)**.** All categories mentioned thus far have monoidal structures. Note that a single category can admit many monoidal structures. The following examples are important later in this set of notes.

1. The category **Set** has a monoidal product $\times$ with unit object $\{\star\}$.
2. The category **Grp** has a monoidal product $\otimes$ with unit object G_1.
3. The categories $\mathbf{Mod}(R)$, $\mathbf{Vect}(F)$, and $\mathbf{Vect}_{\mathsf{fd}}(R)$ each have a monoidal product $\otimes_R$ with unit object R.

It is left as an exercise to find the unitors and associators.

Definition 7 (Monoidal Functor [1])**.** A *monoidal functor* from a monoidal category $(\mathcal{C}, \otimes, I, \lambda, \rho, \alpha)$ to a monoidal category $(\mathcal{D}, \oplus, J, l, r, p)$ is a tuple (F, Φ, ϕ) subject to the following conditions.

1. $F : \mathcal{C} \to \mathcal{D}$ is a functor.
2. $\Phi_{(A,B)} : F(A) \oplus F(B) \Rightarrow F(A \otimes B)$ is a natural isomorphism.
3. $\phi : 1_{\mathcal{D}} \to F(1_{\mathcal{C}})$ is an isomorphism.
4. (Associativity). If $A, B, C \in \mathcal{C}_0$, then $\Phi_{(A,B\otimes C)} \circ (1_{F(A)} \oplus \Phi_{(B,C)}) \circ p_{(F(A),F(B),F(C))} = F(\alpha_{(A,B,C)}) \circ \Phi_{(A\otimes B,C)} \circ (\Phi_{(A,B)} \oplus 1_{F(C)})$.
5. (Left Unitor). If $A \in \mathcal{C}_0$, then $F(\lambda_A) \circ \Phi_{(I,A)} \circ (\phi \oplus 1_{F(A)}) = l_A$.
6. (Right Identity). If $A \in \mathcal{C}_0$, then $F(\rho_A) \circ \Phi_{(A,I)} \circ (1_{F(A)} \oplus \phi) = r_A$.

Definition 8 (Monoidal Natural Transformation [1])**.** Let $(\mathcal{C}, \otimes, I, \lambda, \rho, \alpha)$ and $(\mathcal{D}, \oplus, J, l, r, p)$ be monoidal categories with functors $(F, \Phi, \phi) : \mathcal{C} \to \mathcal{D}$ and $(G, \Gamma, \gamma) : \mathcal{C} \to \mathcal{D}$. A *monoidal natural transformation* from (F, Φ, ϕ) to (G, Γ, γ) is a natural transformation $\beta : F \Rightarrow G$ subject to the following conditions.

1. If $A, B \in \mathcal{C}_0$, then $\Gamma_{(A,B)} \circ (\beta_A \oplus \beta_B) = \beta_{A\otimes B} \circ \Phi_{(A,B)}$.
2. $\gamma = \beta_I \circ \alpha$.

If α is invertible, then we say α is a monoidal natural isomorphism.

Example 6 (Monoidal Functors and Transformations)**.** A common example of a monoidal functor is the mapping of a vector space (and its homomorphisms) to its dual space. A common example of a monoidal natural isomorphism is the isomorphism between a finite-dimensional vector space and its double dual.

Definition 9 (Cartesian Category [3])**.** A *Cartesian category* is a monoidal category $(\mathcal{C}, \otimes, I, \lambda, \rho, \alpha)$ such that $\mathcal{C}$ has a terminal object I and $\otimes$ is the categorical product for $\mathcal{C}$.

Example 7 (Set is Cartesian)**.** The monoidal structure $(\mathbf{Set}, \times, \{\star\}, \lambda, \rho, \alpha)$ is Cartesian monoidal since $\times$ is the product for **Set** and $\{\star\}$ represents the isomorphism class of terminal objects in **Set**.

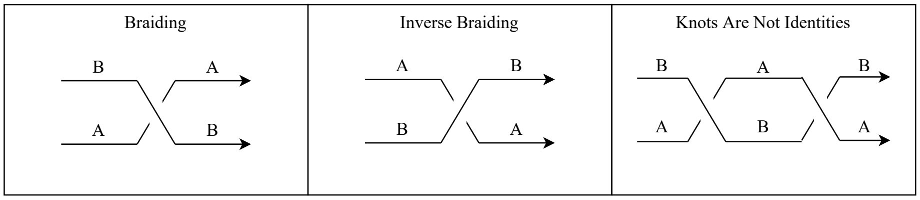
3.2 Braided Monoidal Categories

Definition 10 (Braided Monoidal Category [13])**.** A *braiding* on a monoidal category $(\mathcal{C}, \otimes, I, \lambda, \rho, \alpha, c)$ is a natural isomorphism $c : A \otimes B \Rightarrow B \otimes A$ subject to the following conditions.

1. (Hexagon Axiom 1). If $A, B, C \in \mathcal{C}_0$, then $\alpha_{(B,C,A)} \circ c_{A,B\otimes C)} \circ \alpha_{(A,B,C)} = \left(1_B \otimes c_{(A,C)}\right) \circ \alpha_{(B,A,C)} \circ \left(c_{(A,B)} \otimes 1_C\right)$

2. (Hexagon Axiom 2). If $A, B, C \in \mathcal{C}_0$, then $\alpha_{(B,C,A)} \circ c^{-1}_{A,B\otimes C)} \circ \alpha_{(A,B,C)} = \left(1_B \otimes c^{-1}_{(A,C)}\right) \circ \alpha_{(B,A,C)} \circ \left(c^{-1}_{(A,B)} \otimes 1_C\right)$

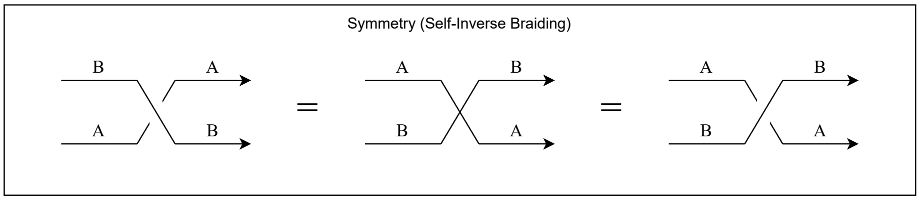
Remark 4. *Note that the braiding axioms do not entail that $c_{(-,-)} = c^{-1}_{(-,-)}$. That is, the order of braiding matters. The notion of braiding is best illustrated through the string diagrams of braided monoidal categories [13].*



3.3 Symmetric Monoidal Categories

Definition 11 (Symmetric Monoidal Category [13])**.** A *symmetry* on a monoidal category $\mathcal{M} = (\mathcal{C}, \otimes, I, \lambda, \rho, \alpha, c)$, is a braiding c on $\mathcal{M}$ such that $c_{A,B} = c^{-1}_{A,B}$ for all $A, B \in \mathcal{C}_0$.

Remark 5. *We obtain string diagrams for symmetric monoidal categories by introducing a symmetry notation for the braiding isomorphism [13].*



Example 8 (Symmetric and Braided Categories)**.** Examples of symmetric categories include.

1. The Cartesian monoidal structure for **Set** admits a symmetry with components $c_{(A,B)} : (a, b) \to (b, a)$.
2. The categories $\mathbf{Mod}(R)$, $\mathbf{Vect}(R)$, and $\mathbf{Vect}_{\mathsf{vd}}(R)$ each admit a symmetry with components $c_{(A,B)} : \sum_{i=1}^{n} k_i(a_i \otimes b_i) \to \sum_{i=1}^{n} k_i(b_i \otimes a_i)$.

By definition, all symmetric monoidal categories are braided monoidal. There exists braided monoidal categories that are not symmetric monoidal. However, these constructions are beyond the scope of these notes. For those interested, the monoidal category of graded R-modules with tensor product admits many braiding that are not self-inverse (see [7]).

Theorem 2. *Every Cartesian category is symmetric.*

Proof. Section 8 □

4 Closed Monoidal Categories

Definition 12 (Adjunct Functors [3])**.** A functor $G : \mathcal{D} \to \mathcal{C}$ is the *right adjoint* to $F : \mathcal{C} \to \mathcal{D}$ if there exist a isomorphism $\mathcal{C}(X, G(Y)) \cong \mathcal{D}(F(X), Y)$ natural in both X and Y. We write that $F \dashv G$.

Definition 13 (Closed Category [8])**.** A monoidal category $(\mathcal{C}, \otimes, I, \lambda, \rho, \alpha)$ is *compact closed* if there exists an *internal hom functor* $[-,-] : \mathcal{C}^{\mathsf{op}} \otimes \mathcal{C} \to \mathcal{C}$ such that for each $B \in \mathcal{C}_0$, the right adjoint to $(-) \otimes B$ is $[B, (-)]$.

5 Duals, Rigid Categories, and String Diagrams

This section introduces the notion of dualizable objects. Finite-dimensional vector spaces over a field F are used as a running example.

5.1 Categories with Dual Objects

Definition 14 (Dualizable [6])**.** Let $(\mathcal{C}, \otimes, I, \lambda, \rho, \alpha)$ be a monoidal category.

1. An object $A \in \mathcal{C}_0$ is *right-dualizable* if there exists an object $A^* \in \mathcal{C}_0$, unit $\eta : 1 \to A \otimes A^*$, and counit $\epsilon : A^* \otimes A \to 1$ such that the following properties are satisfied.
   (a) $\rho_A^{-1} \circ \lambda_A = (1_A \otimes \epsilon) \circ \alpha_{(A,A^*,A)} \circ (\eta \otimes 1_A)$.
   (b) $\lambda_{A^*}^{-1} \circ \rho_{A^*} = (\epsilon \otimes 1_{A^*}) \circ \alpha_{(A^*,A,A^*)}^{-1} \circ (1_{A^*} \otimes \eta)$.
2. An object $A \in \mathcal{C}_0$ is *left-dualizable* if there exists an object ${}^*A \in \mathcal{C}_0$, unit $\eta' : 1 \to {}^*A \otimes A$, and counit $\epsilon' : A \otimes {}^*A \to 1$ such that the following properties are satisfied.
   (a) $\rho_{{}^*A}^{-1} \circ \lambda_{{}^*A} = (1_{{}^*A} \otimes \epsilon') \circ \alpha_{({}^*A,A,{}^*A)} \circ (\eta' \otimes 1_{{}^*A})$.
   (b) $\lambda_A^{-1} \circ \rho_A = (\epsilon' \otimes 1_A) \circ \alpha_{(A,{}^*A,A)}^{-1} \circ (1_A \otimes \eta')$.
3. An object is dualizable if it is right-dualizable and left-dualizable.

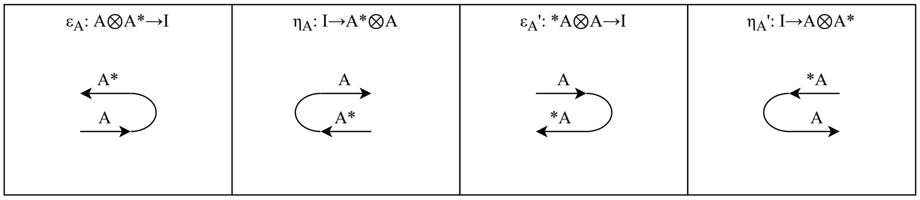
Example 9 (Dualizable Vector Spaces)**.** Let V be a finite-dimensional F-vector space with some basis $\{e_1, \ldots, e_n\}$ and δ_{e_i} denote the Kronecker-delta function $\delta_{e_i} : \sum_{j=1}^n x_j \cdot e_j \to x_i$. Then the right-dual to V is $V^* := \mathrm{Hom}(V, F)$. The unit is $\eta : 1 \to A \otimes A^*$ such that $\eta : 1 \mapsto \sum_{i=1}^n e_i \otimes \delta_{e_i}$ and the counit is $\epsilon : A^* \otimes A \to 1$ such that $\epsilon : (f, v) \mapsto f(v)$. The left-dual to V is also ${}^*V := \mathrm{Hom}(V, F)$. The construction of η' and ϵ' is symmetric.

Example 10 (A Basis-Free Construction[1])**.** Let V be a finite-dimensional F-vector space. Then we can also construct the unit $\eta : F \to V \otimes V^*$ in a basis-free way. First note that $f : F \to \mathrm{Hom}(V, V)$ such that $f : 1 \mapsto 1_V$ defines an injective F-linear mapping. Then define $g : V \times V^* \to \mathrm{Hom}(V, V)$ such that $g : v \otimes \varphi \mapsto (w \mapsto v \cdot \varphi(w))$. Clearly g is bilinear, so g extends to an F-linear map $h : V \otimes V^* \to \mathrm{Hom}(V, V)$. Since every finite-dimension F-linear map is of the form $w \mapsto \sum_{i=1}^n v_i \cdot \varphi_i(w)$, then clearly h is surjective. We can also prove that $V \otimes V^*$ is of the same degree as $\mathrm{Hom}(V, V)$. Then h is an isomorphism. Then define $\eta := h^{-1} \circ f$. This gives a basis-free construction of η.

Definition 15 (Rigid Monoidal Category [6])**.** A *rigid monoidal category* is a monoidal category $(\mathcal{C}, \otimes, I, \lambda, \rho, \alpha)$ such that every object in $\mathcal{C}_0$ is dualizable. For each $A \in \mathcal{C}_0$, we let η_A denote the unit to the right-dual of A, ϵ_A denote the counit to the right-dual of A, η'_A denote the unit to the left-dual of A, and ϵ'_A denote the counit to the left-dual of A.

[1] This construction was based on a discussion with Andre Kornell. The construction isn't new, but we are not sure about a standard reference.

Remark 6. *Rigid monoidal categories also have their string diagrams [13]. Equating rigid monoidal string diagrams rquire a notion of winding numbers, which is beyond the scope of these notes (see [13]).*

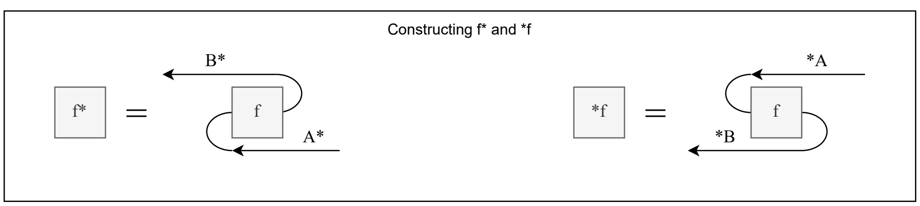


Example 11 (Rigid Vector Spaces [6])**.** By Example 9, the category $\mathbf{Vect}_{\mathsf{fd}}(F)$ is rigid. However, the category $\mathbf{Vect}(F)$ is not rigid. Let V be an infinite-dimensional vector space. Assume that V has a right-dual V^*, whether or not $V^* = \mathrm{Hom}(V, F)$. Then there exists morphisms $\eta : F \to V \otimes V^*$ and $\epsilon : V^* \otimes V \to F$. Since F is finite dimensional, then the image of η is finite dimensional. Call this image $X \otimes Y$. Then $(1_V \otimes \epsilon) \circ \alpha_{(V,X,Y)} \circ (\eta \otimes 1_Y)$ maps V to Y. Since Y is finite dimension, then $(1_V \otimes \epsilon) \circ \alpha_{(V,X,Y)} \circ (\eta \otimes 1_Y)$ is not an isomorphism. By contradiction, V is not dualizable.

5.2 Properties of Rigid Categories

Theorem 3 ([13])**.** *In a rigid monoidal category, the right-dual $(-)^*$ and left-dual ${}^*(-)$ are monoidal endofunctors.*

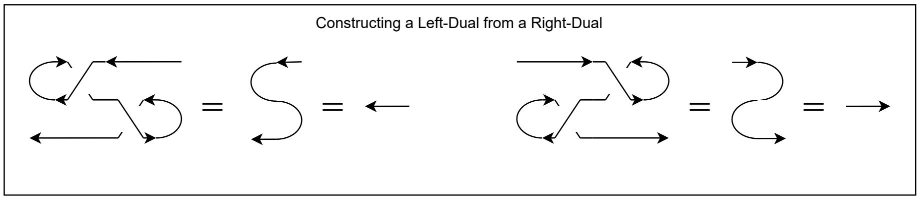
Proof. For each $f : A \to B$, it is necessary to define $f^* : B^* \to A^*$ and ${}^*f : {}^*B \to {}^*A$. This construction is most easily illustrated through string diagrams (see [13]).



It follows immediately that $(1_A)^* = 1_{A^*}$ and ${}^*(1_A) = 1_{{}^*A}$. It is evident from the string diagrams that for all $A \xrightarrow{f} B \xrightarrow{g} C$, both $(g \circ f)^* = (f)^* \circ (g)^*$ and ${}^*(f \circ g) = {}^*(g) \circ {}^*(f)$. Therefore, $(-)^*$ and ${}^*(-)$ are endofunctors. It remains to be shown that $(-)^*$ and ${}^*(-)$ are monoidal, though this proof is omitted. □

Theorem 4. *A braided monoidal category has right-duals if and only if it is a rigid monoidal category.*

Proof. A proof sketch is given by the following equality of string diagrams.



To complete the proof, it must be shown that each equality holds. □

Theorem 5. *A symmetric monoidal category has right-duals if and only if it is monoidal closed.*

Proof. Let $\mathcal{M} = (\mathcal{C}, \otimes, I, \lambda, \rho, \alpha, c)$ be a symmetric monoidal category with right-duals. Assume, without loss of generality, that $\mathcal{M}$ were a strict monoidal category. Since $\mathcal{M}$ is symmetric monoidal, then $\mathcal{M}$ is braided by definition. Then by Theorem 4, $\mathcal{M}$ is rigid monoidal. Let $X, Y \in \mathcal{C}_0$. Since $\mathcal{C}$ is symmetric, then $\mathcal{C}(X \otimes B, Y) \cong \mathcal{C}(B \otimes X, Y)$ and $\mathcal{C}(X, B^* \otimes Y) \cong \mathcal{C}(X, Y \otimes B^*)$. Define $\varphi_{X,Y} : \mathcal{C}(B \otimes X, Y) \to \mathcal{C}(X, Y \otimes B^*)$ such that for $f : B \otimes X \to Y$:

$$X \xrightarrow{\rho^{-1}} X \otimes I \xrightarrow{X \otimes \eta} X \otimes B \otimes B^* \xrightarrow{f \otimes B^*} Y \otimes B^*$$

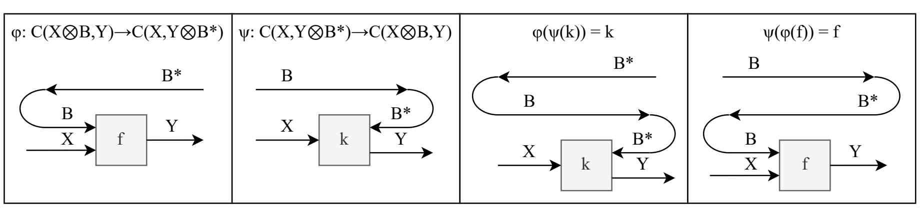
First note that $\varphi_{X,Y}$ is natural. Let $g : Z \to X$ and $h : Y \to W$.

$$\begin{aligned}
(\mathcal{C}(h, g \otimes B^*))(\varphi_{X,Y}(f)) &= (g \otimes B^*) \circ \varphi_{X,Y}(f) \circ \rho^{-1} \circ h \\
&= (g \otimes B^*) \circ ((f \otimes B^*) \circ (X \otimes \eta)) \circ (h \otimes I) \circ \rho^{-1} \\
&= ((g \circ f) \otimes B^*) \circ ((X \otimes \eta) \circ (h \otimes I)) \circ \rho^{-1} \\
&= ((g \circ f) \otimes B^*) \circ ((h \otimes B \otimes B^*) \circ (Z \otimes \eta)) \circ \rho^{-1} \\
&= ((g \circ f \circ (h \otimes B)) \otimes B^*) \circ (Z \otimes \eta) \circ \rho^{-1} \\
&= ((\mathcal{C}(B \otimes h, g))(f) \otimes B^*) \circ (Z \otimes \eta) \circ \rho^{-1} \\
&= \varphi_{Z,W}((\mathcal{C}(B \otimes h, g))(f))
\end{aligned}$$

Now define $\psi_{X,Y} : \mathcal{C}(X, Y \otimes B^*) \to \mathcal{C}(B \otimes X, Y)$ such that for $k : X \to Y \otimes B^*$:

$$X \otimes B \xrightarrow{k \otimes B} Y \otimes B^* \otimes B \xrightarrow{Y \otimes \epsilon} Y \otimes I \xrightarrow{\rho} Y$$

It follows by a similar argument that $\psi_{X,Y}$ is natural. The following string diagrams illustrate why φ and ψ are inverses.



Then $\mathcal{C}(X \otimes B, Y) \cong \mathcal{C}(X, B^* \otimes Y)$. Therefore, $\mathcal{M}$ is monoidal closed. □

Theorem 6 ([4])**.** *If a Cartesian category is rigid for the same monoidal structure, then every object in the category is isomorphic to the monoidal unit.*

Proof. Let $\mathcal{M} = (\mathcal{C}, \otimes, I, \lambda, \rho, \alpha)$ be a rigid Cartesian category. Since $\mathcal{M}$ is Cartesian, then $\mathcal{M}$ has a symmetry c by Theorem 2. Then $\mathcal{M}$ is monoidal closed by Theorem 5. It is left as an exercise to show that for all $X \in \mathcal{C}_0$ and $Y \in \mathcal{C}_0$, $\mathcal{C}(X, Y) \cong \mathcal{C}(I, X^* \otimes Y) \cong \mathcal{C}(Y^*, X^*) \cong \mathcal{C}(X \otimes Y^*, I)$. Since I is the terminal object, then $\mathcal{C}(X, Y)$ is a isomorphic to a singleton set. Then for all $Y \in \mathcal{C}_0$, $\mathcal{C}(X, Y) \cong \{\star\}$. Then for all $X \in \mathcal{C}_0$, $X \cong I$. □

Definition 16 (Pivotal Category)**.** A *pivotal category* is a $(\mathcal{C}, \otimes, I, \lambda, \rho, \alpha)$ with a natural monoidal isomorphism $i : (-) \Rightarrow (-)^{**}$.

Example 12 (A Pivotal Category)**.** Pivotal categories generalize $\mathbf{Vect}_{\mathsf{fd}}(F)$. Let V be a finite dimensional vector space with some basis $\{e_1, \ldots, e_n\}$ and δ_{e_i} denote the Kronecker-delta function $\delta_{e_i} : \sum_{j=1}^{n} x_j \cdot e_j \to x_i$. It is well-known that $V \cong V^{**}$ by $i : v \mapsto (f \mapsto f(v))$ and $i^{-1} : g \mapsto \sum_{i=1}^{n} g(\delta_{e_i})$ (see, e.g., [5]).

Remark 7. *The string diagrams for pivotal categories are simpler than the string diagrams for rigid monoidal categories. Since $A \cong A^{**}$ there is no need for winding numbers. To depict $i : (-) \Rightarrow (-)^{**}$, morphisms in the diagrams are rotated 180-degrees. These rotations are indicated by marking the top-left corner of each morphism box. Details are found in [13].*

6 (Co)algebras, Bialgebras, and Hopf Algebras

Fix a commutative ring R. Let $(\mathbf{Mod}(R), \otimes, R, \lambda, \rho, \alpha, c)$ be the symmetric monoidal structure for $\mathbf{Mod}(R)$. To align with the literature on bialgebras, associators are omitted. I use $\cong$ instead of $=$ where associators should appear.

6.1 Algebras Over Commutative Rings

Definition 17 (Associative Algebra [3])**.** A *associative R-algebra* is a R-module A equipped with R-linear maps $m : A \otimes A \to A$ and $\eta : R \to A$ subject to the following two relations.

1. (Associativity): $m \circ (m \otimes 1_A) = m \circ (1_A \otimes m)$.
2. (Unital): $m \circ (1_A \otimes \eta) \circ \rho^{-1} = 1_A = m \circ (\eta \otimes 1_A) \circ \lambda^{-1}$.

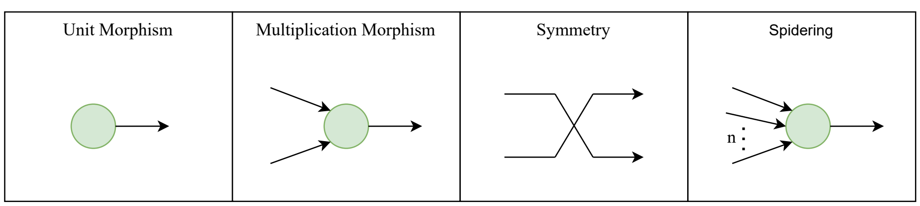
If $m \circ c = m$, then A is a *commutative*.

Definition 18 (Algebra Homomorphism)**.** An *R-algebra homomorphism* from an R-algebra (A, m_1, η_1) to an R-algebra (B, m_2, η_2) is an R-linear map $f : A \to B$ such that the following two relations hold.

1. (Respects Multiplication). $m_2 \circ (f \otimes f) = f \circ m_1$.
2. (Respects Units). $\eta_2 = f \circ \eta_1$.

Remark 8. *Monoid objects in* $\mathbf{Mod}(R)$ *are precisely the R-algebras.*

Remark 9. *Given that R-algebras are simply monoid objects in* $\mathbf{Mod}(R)$*, then we can specialize string diagrams to R-algebras [12]. Recall that* $\mathbf{Mod}(R)$ *is a symmetric tensor category.*



Example 13 (Rings Are Algebras)**.** R is an R-algebra in the obvious way.

1. (Multiplication). Define $f : R \times R \to R$ such that $f : (x, y) \mapsto x \cdot y$. Clearly f is bilinear. Then f extends to an R-linear map $m : R \otimes R \to R$ such that $m\left(\sum_{i=1}^{n} x_i \otimes y_i\right) = \sum_{i=1}^{n} f(x_i, y_i) = \sum_{i=1}^{n} x_i \cdot y_i$.
2. (Unit). Define $\eta = 1_R$.
3. (Associativity). Follows from associativity of multiplication in a ring.
4. (Unital). Since $\eta = 1_R$, then $1_R \otimes \eta = 1_{R \otimes R} = \eta \otimes 1_R$. Let $x \in R$. Then $(m \circ \rho^{-1})(x) = m(x \otimes 1) = x$ and $(m \circ \lambda^{-1})(x) = m(1 \otimes x) = x$. Since x was arbitrary, then m is unital with respect to η.

Then (R, m, η) is an R-algebra. Similarly, $(R \otimes R, m', \eta')$ is an R-algebra where $m' : (x \otimes y) \otimes (u \otimes v) \to (xu) \otimes (yv)$ and $\eta' : k \to k \otimes 1$.

Example 14 (Algebras Generated By Monoids [14]). Let $(M, e, \bullet)$ be a monoid. Then the free R-module $R[M]$ defines is an R-algebra in the obvious way.

1. (Multiplication). Clearly $\mathcal{A} := \{1m \mid m \in M\}$ is a basis for $R[M]$. Then $\mathcal{B} := \{x \otimes y \mid x, y \in \mathcal{B}\}$ is a basis for $R[M] \otimes R[M]$. Define the function $f : \mathcal{B} \to R[M]$ such that $f(1s, 1t) \to 1(s \bullet t)$. Then f extends to an R-linear map $m : R[M] \otimes R[M] \to R[M]$. To illustrate this construction, $m((\alpha s + \beta t) \otimes (\gamma u + \omega v)) = \alpha\gamma(s \bullet u) + \alpha\omega(s \bullet v) + \beta\gamma(t \bullet u) + \beta\omega(t \bullet v)$.
2. (Unit). Define $\eta : R \to R[M]$ such that $\eta(x) = xe$. Clear η is R-linear.
3. (Associativity). *Hint: Monoid and ring multiplication are associative.*
4. (Unital). *Hint:* 1 *is the unit in* R *and* e *is the unit in* $(M, e, \bullet)$.

Then $(R[M], m, \eta)$ is an algebra.

6.2 Coalgebras Over Commutative Rings

Definition 19 (Coassociative Coalgebra [3]). A *coassociative* R*-coalgebra* is a R-module A equipped with R-linear maps $\Delta : A \to A \otimes A$ and $\varepsilon : A \to R$ subject to the following relations.

1. (Coassociativity): $(\Delta \otimes 1_A) \circ \Delta = (1_A \otimes \Delta) \circ \Delta$.
2. (Counital): $\rho \circ (1_A \otimes \varepsilon) \circ \Delta = 1_A = \lambda \circ (\varepsilon \otimes 1_A) \circ \Delta$.

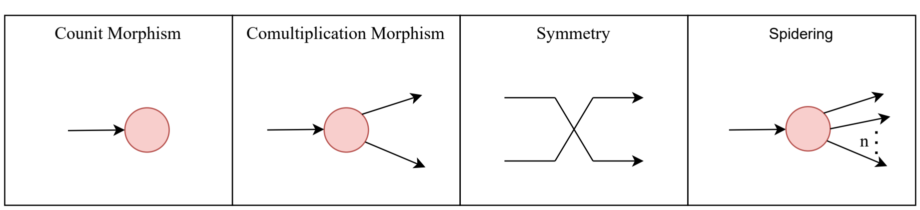
If $c \circ \Delta = \Delta$, then A is *cocommutative.*

Definition 20 (Coalgebra Homomorphism). An R*-coalgebra homomorphism* from an R-colagebra $(A, \Delta_1, \varepsilon_1)$ to an R-coalgebra $(B, \Delta_2, \varepsilon_2)$ is an R-linear map $f : A \to B$ such that the following two relations hold.

1. (Respects Comultiplication). $\Delta_2 \circ f = (f \otimes f) \circ \Delta_1$.
2. (Respects Counits). $\varepsilon_2 \circ f = \varepsilon_1$.

Remark 10. *Comonoid objects in* **Mod**(R) *are precisely the* R*-coalgebras.*

Remark 11. *String diagrams also exist for coalgebras. [12].*



Example 15 (Rings Are Coalgebras)**.** an R is an R-coalgebra in the obvious way. Define $\Delta : R \to R \times R$ such that $\Delta : x \to x \otimes 1$ and $\varepsilon = 1_R$. Clearly Δ and ε are R-linear.

1. (Coassociativity). Follows from the middle linearity of $\otimes$.
2. (Counital). Since $\eta = 1_R$, then $1_R \otimes \varepsilon = 1_{R\otimes R} = \varepsilon \otimes 1_R$. Let $x \in R$. Then $(\rho \circ \Delta)(x) = \rho(x \otimes 1) = 1 \cdot x = x$ and $(\lambda \circ \Delta)(x) = \lambda(x \otimes 1) = x \cdot 1 = x$. Since x was arbitrary, then Δ is counital with respect to ε.

Then (R, Δ, ε) is an R-algebra.

Example 16 (Coalgebra for an Interesting Monoid [14])**.** Let $(M, e, \bullet)$ be the monoid presented by $\langle x, g, g^{-1} \mid gg^{-1} = 1\rangle$. Then the free R-module $R[M]$ defines a unique coalgebra. Clearly $\mathcal{A} := \{1m \mid m \in M\}$ is a basis for $R[M]$.

1. (Comultiplication). Recall the multiplcation $m : R[M] \otimes R[M] \to R[M]$ in Example 14. Define $f : \mathcal{A} \to R[M] \otimes R[M]$ inductively as follows. First define $f(1e) = (e \otimes e)$. Then for $w \in M$, define f as follows:

$$f : 1(x \bullet w) \mapsto m((1 \otimes x) + (x \otimes g), f(w))$$
$$f : 1(g \bullet w) \mapsto m((g \otimes g), f(w))$$

   Either $w = e$ or $|w| > 0$ and the first symbol in w is one of x, g, or the inverse of g, and therefore f is well-defined. Since f is well-defined, then f extends to an R-linear map $\Delta : R[M] \to R[M] \otimes R[M]$.

2. (Counit). Define $h : \{x, g\} \to R$ such that $h : x \mapsto 0$ and $h : g \mapsto 1$. Then h extends to a monoid morphism $\overline{h} : M \to R$. Define $k : \mathcal{A} \to R$ such that $k(1m) = \overline{h}(m)$. Then k extends to an R-linear map $\varepsilon : R[M] \to R$.

3. (Coassociativity). First we show that $((\Delta \otimes 1) \circ \Delta)(g) = ((1 \otimes \Delta) \circ \Delta)(g)$.

$$\begin{aligned}((\Delta \otimes 1) \circ \Delta)(g) &= (\Delta \otimes 1)(g \otimes g) \\ &= (g \otimes g) \otimes g \\ &\cong g \otimes (g \otimes g) \\ &= (1 \otimes \Delta)(g \otimes g) \\ &= ((1 \otimes \Delta) \otimes \Delta)(g)\end{aligned}$$

   Since $\Delta(g^{-1}) = \Delta(g)^{-1}$, then $((\Delta \otimes 1) \circ \Delta)(g^{-1}) = ((1 \otimes \Delta) \circ \Delta)(g^{-1})$. Next we show that $((\Delta \otimes 1) \circ \Delta)(x) = ((1 \otimes \Delta) \circ \Delta)(x)$.

$$\begin{aligned}((\Delta \otimes 1) \circ \Delta)(x) &= (\Delta \otimes 1)((1 \otimes x) + (x \otimes g)) \\ &= ((1 \otimes 1) \otimes x) + ((1 \otimes x) \otimes g) + ((x \otimes g) \otimes g) \\ &\cong (1 \otimes (1 \otimes x)) + (1 \otimes (x \otimes g)) + (x \otimes (g \otimes g)) \\ &= (1 \otimes ((1 \otimes x) + (x \otimes g))) + (x \otimes (g \otimes g)) \\ &= (1 \otimes \Delta)((1 \otimes x) + (x \otimes g)) \\ &= ((1 \otimes \Delta) \circ \Delta)(x)\end{aligned}$$

   Since Δ is defined inductively over the words of M, then it is easy to show that $(\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta$.

4. (Counital). First we show that $(\rho \circ (1 \otimes \varepsilon) \circ \Delta)(x) = x$.

$$(\rho \circ (1 \otimes \varepsilon) \circ \Delta)(x) = (\rho \circ (1 \otimes \varepsilon))((1 \otimes x) + (x \otimes g)) = \rho(0 + (x \otimes 1)) = x$$

Next we show that $(\lambda \circ (\varepsilon \otimes 1) \circ \Delta)(x) = x$.

$$(\lambda \circ (\varepsilon \otimes 1) \circ \Delta)(x) = (\lambda \circ (\varepsilon \otimes 1))((1 \otimes x) + (x \otimes g)) = \lambda((1 \otimes x) + 0) = x$$

Third, we show that $(\rho \circ (1 \otimes \varepsilon) \circ \Delta)(g) = g$. It is clear from symmetry that also $(\lambda \circ (\varepsilon \otimes 1) \circ \Delta)(g) = g$. Since Δ and ε respects g^{-1} as the inverse to g, then this argument also extends to g^{-1}

$$(\lambda \circ (\varepsilon \otimes 1) \circ \Delta)(x) = (\lambda \circ (\varepsilon \otimes 1))(g \otimes g) = \lambda(1 \otimes g) = g$$

Since ϵ and Δ are defined inductively on M, then it is easy to show that $\rho \circ (1 \otimes \varepsilon = 1 = \lambda \circ (\varepsilon \otimes 1) \circ \Delta$. Then Δ is counital with respect to ε.

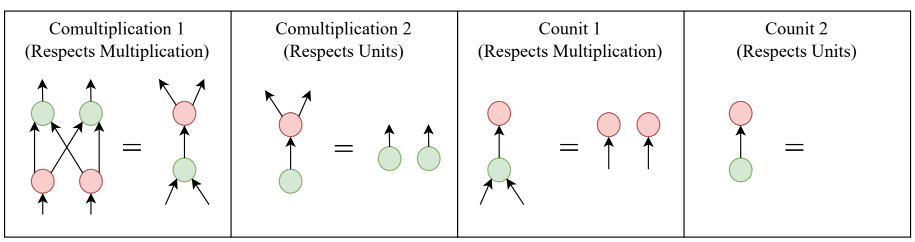
Then $(R[M], \Delta, \varepsilon)$ is a coalgebra.

6.3 Bialgebras and Convolutions

Definition 21 (Bialgebra [3]). An R-*bialgebra* is an R-module A equipped with algebra structure (A, m, η) and coalgebra structure (A, Δ, ε) such that Δ and ε are algebra homomorphisms.

Remark 12. *Bimonoid objects in* $\mathbf{Mod}(R)$ *are precisely the* R*-bialgebras.*

Remark 13. *We can express the compatibility conditions for* R*-bialgebras as string diagrams between the algebraic and coalgebraic structures [12].*



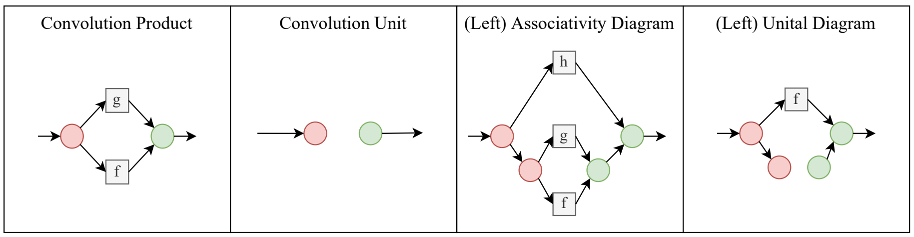
Example 17 (Rings Are Bialgebras). Recall the algebra structure (R, m, η) of Example 13 and the coalgebra structure (R, Δ, ε) of Example 15. Since ε is the identity, then ε is an R-algebra homomorphism. Let $x \in R$ and $y \in R$. Then $m'(\Delta(x) \otimes \Delta(y)) = m'((x \otimes 1) \otimes (y \otimes 1)) = (xy) \otimes 1 = \Delta(xy) = \Delta(m(x \otimes y))$. Similarly, $\Delta(\eta(x)) = \Delta(x) = x \otimes 1 = \eta'(x)$. Since x and y were arbitrary, then Δ is an R-algebra homomorphism.

Example 18 (Bialgebra for an Interesting Monoid [14]). One can show that the algebra structure defined in Example 14 is compatible with the coalgebra structure defined in Example 16.

Definition 22 (Bialgebra Convolution Product [3]). The *convolution product* for an R-bialgebra $(A, m, \eta, \Delta, \varepsilon)$ is the mapping $\star : \mathrm{End}(A) \times \mathrm{End}(A) \to \mathrm{End}(A)$ such that $\star : (f, g) \mapsto m \circ (f \otimes g) \circ \Delta$.

Remark 14. *Let* (A, m, η) *be an* R*-algebra and* (B, Δ, ε) *be an* R*-colagebra. Then the convolution generalizes to* $\mathrm{Hom}(A, B) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B)$ *in the obvious way.*

Remark 15. *Let* A *be an* R*-bialgebra. As stated in Theorem 7, the convolution product for* A *gives rise to an* R*-algebra for* $\mathrm{End}(A)$*. The multiplication and unit are depicted below as string diagrams for* A*. To illustrate the proof of Theorem 7, the composition of the convolution product, both with itself and with the unit morphism, are depicted.*



Example 19 (Understanding Convolution)**.** Convolution is best understood by looking at concrete examples. Let $(R, m, \eta, \Delta, \varepsilon)$ be the bialgebra in Example 17 and $f, g \in \mathrm{End}(R)$. Consider evaluating $f \star g$ at $x \in R$.

$$(f \star g)(x) = (m \circ (f \otimes g) \circ \Delta)(x) = (m \circ (f \otimes g))(x \otimes 1) = f(x) \cdot g(1)$$

Then the convolution of f with g is f scaled by g evaluated at 1. Likewise, the convolution of g with f is g scaled by f evaluated at 1. Clearly convolution algebras are not commutative in general.

Theorem 7 ([9])**.** *If* $(A, m, \eta, \Delta, \varepsilon)$ *is an* R*-bialgebra, then* $(\mathrm{End}(A), \star, \eta \circ \varepsilon)$ *is an* R*-algebra.*

Proof. See Section 8. □

6.4 Hopf Algebras Over Commutative Rings

Definition 23 (Hopf Algebra [3])**.** A *Hopf algebra* is an R-module A equipped with an R-bialgebra structure $(A, m, \eta, \Delta, \varepsilon)$ and an R-linear map $S : A \to A$ such that $1_A \star S = \eta \circ \varepsilon = S \star 1_A$. We say that S is the *antipode*.

Example 20 (Rings are Hopf Algebras)**.** Recall that R-bialgebra structure $(R, m, \eta, \Delta, \varepsilon)$ of Example 17. This is a Hopf R-algebra in the obvious way. Let $x \in R$. Then $(1_R \star 1_R)(x) = (m \circ (1_R \otimes 1_R) \circ \Delta)(x) = (m \circ \Delta)(m) = m(x \otimes 1) = x$. Then $1_R \star 1_R = \eta \circ \varepsilon$. Then 1_R is an antipode. Therefore, $(R, m, \eta, \Delta, \varepsilon)$ is a Hopf R-algebra.

Example 21 (Hopf Algebra for an Interesting Monoid [14])**.** Recall the R-bialgebra structure $(R[M], m, \eta, \Delta, \varepsilon)$ of Example 18. Once again, let $\mathcal{A}$ denote the basis for $R[M]$. Define $f : \mathcal{A} \to R[M]$ inductively as follows. First define $f(1e) = 1e$. Then for $w \in M$, define S as follows:

$$f : 1(x \bullet w) \mapsto -xg^{-1} \cdot f(w) \qquad\qquad f : 1(g \bullet w) \mapsto g^{-1} \cdot f(w)$$

Either $w = e$ or $|w| > 0$ and the first symbol in w is one of x, g, or the inverse of g, and therefore f is well-defined. Since f is well-defined, then f extends to an R-linear map $S : R[M] \to R[M]$. We first show that $(m \circ (1 \otimes S) \circ \Delta)(x) = (\eta \circ \varepsilon)(x)$.

$$\begin{aligned}(m \circ (1 \otimes S) \circ \Delta)(x) &= m((1 \otimes S(x)) + (x \otimes S(g))) \\ &= m(-(1 \otimes xg^{-1}) + (x \otimes g^{-1})) \\ &= -xg^{-1} + xg^{-1} = 0 = \eta(0) = (\eta \circ \varepsilon)(x)\end{aligned}$$

Since $\Delta(g^{-1}) = \Delta(g)^{-1}$, $\varepsilon(g^{-1}) = \varepsilon(g)$, and $S(g^{-1}) = S(g)^{-1}$, then it suffices to show that $(m \circ (1 \otimes S) \circ \Delta)(g) = (\eta \circ \varepsilon)(g)$.

$$(m \circ (1 \otimes S) \circ \Delta)(g) = m(g \otimes S(g)) = m(g \otimes g^{-1}) = gg^{-1} = 1 = \eta(1) = (\eta \circ \varepsilon)(g)$$

Since ϵ and Δ are defined inductively over the words of M, it is easy to show that $1_{R[M]} \star S = \eta \circ \varepsilon$. Through a similar calculation, $S \star 1_{R[M]} = \eta \circ \varepsilon$. Then S is the antipole for the Hopf R-algebra $(R[M], m, \eta, \Delta, \varepsilon)$.

Example 22 (Sweedler's Hopf Algebra H_4 [14])**.** Recall the Hopf R-algebra $(H, m, \eta, \Delta, \varepsilon, S)$ from Example 21. We let H_4 denote the quotient Hopf algebra obtained from the relations $x^2 = 0$, $g^2 = 1$, and $gx = -xg$. The Hopf algebra H_4 is often known as *Sweedler's Hopf Algebra*. H_4 has the following properties.

1. (Finite Dimensional). Since $g^2 = 1$, then $g^{-1} = g$. Since $gx = -xg$, then every basis vector can be written in the form $g^n x^m$ for $n \in \mathbb{N}$ and $m \in \mathbb{N}$. Since $x^2 = 0e$ and $g^2 = 1$, then every basis vector belongs to one of four equivalence classes: e, x, g, xg. Therefore, H_4 has dimension 4.

2. (Non-Commutative). Since $m(x, g) = xg \neq -xg = gx = m(g, x)$, then $m \neq m \circ m$ and H_4 is a non-commutative Hopf algebra.

3. (Non-Cocommutative). Since $(c \circ \Delta)(x) = (x \otimes 1) + (g \otimes x) \neq (1 \otimes x) + (x \otimes g)$, then $\Delta(x) \neq (c \circ \Delta)(x)$ and H_4 is a non-cocommutative Hopf-algebra.

Remark 16. *Another well-known Hopf algebra is the shuffle algebra [3]. This algebra is also constructed from a module over a free monoid. However, the product is now defined by certain permutations of the strings, known as shuffles. This Hopf algebra appears in the representation theory for linear logic.*

7 Hopf Modules and Rigidity

Fix a commutative ring R and some Hopf R-algebra $(H, m, \eta, \Delta, \epsilon, S)$. Let $(\mathbf{Mod}(R), \otimes, R, \lambda, \rho, \alpha, c)$ be the symmetric monoidal structure for $\mathbf{Mod}(R)$.

7.1 The Categories of Hopf Modules

Definition 24 (Hopf Module [3])**.** An H*-module* is an R-module A equipped with an R-linear map $\rho : H \otimes A \to A$ such that the following two relations hold.

1. (Respects Multiplication). $\rho \circ (1_H \circ \rho) = \rho \circ (m \circ 1_A)$.
2. (Respects Units). $\lambda_A = \rho \circ (\eta \circ 1_A)$.

We call ρ an H*-action* on A.

Example 23 (Module for Sweedler's Hopf Algebra)**.** Recall the Hopf R-algebra $(H, m, \eta, \Delta, \varepsilon, S)$ from Example 22. Then H has a basis $\mathcal{B} = \{1, x, g, xg\}$. Define $f_x, f_g : R^2 \to R^2$ such that $f_x : (a, b) \mapsto (b, 0)$ and $f_g : (a, b) \mapsto (a, -b)$. Clearly f_x and f_g are R-linear. Define $h : \mathcal{B} \to \text{Hom}(R^2, R^2)$ such that $h(1) = 1_{R^2}$, $h(x) = f_x$, $h(g) = f_g$, and $h(xg) = f_x \circ f_g$. Then h extends to an R-linear map $\overline{h} : H \to \text{Hom}(R^2, R^2)$. Finally, define $k : H \times R^2 \to R^2$ such that $k : (u, v) \to (\overline{h}(u))(v)$. Clearly k is middle linear, so k extends to an R-linear map $\rho : H \otimes R^2 \to R^2$. Furthermore, (R^2, ρ) is an H-module.

1. (Respects Multiplication). Let $(a, b) \in R^2$. We show that the Hopf-action ρ respects multiplication with the basis elements of H. The case of 1 is ignored, since 1 is the identity of H, and $\rho(1, -) = 1_{R^2}(-)$ is the identity of $\text{Hom}(R^2, R^2)$.

$$\begin{aligned}\rho \circ (1_H \otimes \rho)(x \otimes x \otimes (a, b)) &= \rho(x \otimes (b, 0)) = (0, 0) \\ &= \rho(0 \otimes (a, b)) \\ &= \rho(m(x \otimes x) \otimes (a, b)) \\ &= \rho \circ (m \otimes 1_{R^2})(x \otimes x \otimes (a, b))\end{aligned}$$

$$\begin{aligned}\rho \circ (1_H \otimes \rho)(x \otimes g \otimes (a, b)) &= \rho(x \otimes (a, -b)) = (-b, 0) \\ &= \rho(xg \otimes (a, b)) \\ &= \rho(m(x \otimes g) \otimes (a, b)) \\ &= \rho \circ (m \otimes 1_{R^2})(x \otimes g \otimes (a, b))\end{aligned}$$

$$\begin{aligned}\rho \circ (1_H \otimes \rho)(g \otimes x \otimes (a, b)) &= \rho(g \otimes (b, 0)) = (b, 0) \\ &= -\rho(xg \otimes (a, b)) \\ &= \rho(gx \otimes (a, b)) \\ &= \rho(m(g \otimes x) \otimes (a, b)) \\ &= \rho \circ (m \otimes 1_{R^2})(g \otimes x \otimes (a, b))\end{aligned}$$

$$\begin{aligned}\rho \circ (1_H \otimes \rho)(g \otimes g \otimes (a,b)) &= \rho(g \otimes (a,-b)) = (a,b) \\ &= \rho(1 \otimes (a,b)) \\ &= \rho(m(g \otimes g) \otimes (a,b)) \\ &= \rho \circ (m \otimes 1_{R^2})(g \otimes g \otimes (a,b))\end{aligned}$$

$$\begin{aligned}\rho \circ (1_H \otimes \rho)(g \otimes xg \otimes (a,b)) &= \rho(g \otimes (-b,0)) = (b,0) \\ &= -\rho(x \otimes (a,b)) \\ &= \rho(gxg \otimes (a,b)) \\ &= \rho(m(g \otimes xg) \otimes (a,b)) \\ &= \rho \circ (m \otimes 1_{R^2})(g \otimes xg \otimes (a,b))\end{aligned}$$

$$\begin{aligned}\rho \circ (1_H \otimes \rho)(xg \otimes g \otimes (a,b)) &= \rho(xg \otimes (a,-b)) = (b,0) \\ &= \rho(x \otimes (a,-b)) \\ &= \rho(xgg \otimes (a,-b)) \\ &= \rho(m(xg \otimes g) \otimes (a,-b)) \\ &= \rho \circ (m \otimes 1_{R^2})(xg \otimes g \otimes (a,b))\end{aligned}$$

The cases of $xg \otimes x$, $x \otimes gx$, and $xg \otimes xg$ are identical to $x \otimes x$, and have therefore been omitted. It follows that $\rho \circ (1_H \circ \rho) = \rho \circ (m \circ 1_{R^2})$.

2. (Respects Algebra Units). Let $r \in R$ and $v \in R^2$. Then $\rho \circ (\eta \circ 1_A)(r,v) = \rho(r,v) = r \cdot 1_{R^2}(v) = r \cdot v = \lambda_{R^2}(r,v)$. Since r and v were arbitrary, then $\rho \circ (\eta \circ 1_{R^2}) = \lambda_{R^2}$.

As in the case of group representations, the Hopf-action ρ places an algebraic structure on the automorphisms of R^2.

Remark 17. *Hopf modules are a direct generalization of group actions. One can show that every group action on R*

Definition 25 (Hopf Module Homomorphism [3])**.** An *H-module homomorphism* from an H-module (A,ρ) to an H-module (B,τ) is an R-linear map $f : A \to B$ such that $\tau \circ (1_H \circ f) = f \circ \rho$.

Remark 18. *The modules of H form a category denoted* $\mathbf{Mod}(H)$*.*

7.2 Properties of Hopf Module Categories

Theorem 8 ([3])**.** *Let (A,ρ) and (B,τ) be H-modules, $s : (H \otimes H) \otimes (U \otimes V) \cong H \otimes ((H \otimes U) \otimes V)$, and $t : H \otimes ((U \otimes H) \otimes V) \cong (H \otimes U) \otimes (H \otimes V)$ Then $(A \otimes B, \gamma)$ is an H-module where $\gamma = (\rho \otimes \tau) \circ t \circ \big(1_H \otimes (C_{(H,U)} \otimes 1_V)\big) \circ s \circ (\Delta \otimes 1_{U \otimes V})$.*

Proof. (Sketch). Clearly γ is well-defined (all types check). Furthermore, γ is a composition of tensors of R-linear maps, so γ is R-linear. Since H is a Hopf R-algebra, then Δ is morphism of R-algebras. Since (A,ρ) and (B,τ) are H-modules, then ρ And τ respect the multiplication and unit of H. It follows that $\gamma \circ (1_H \circ \gamma) = \gamma \circ (m \circ 1_A)$ and $\lambda_A = \gamma \circ (\eta \circ 1_A)$. Then $(A \otimes B, \gamma)$ is an H-module. $\square$

Remark 19. *By Theorem 8, the symmetric monoidal structure for* $\mathbf{Mod}(R)$ *induces a monoidal structure for* $\mathbf{Mod}(H)$*.*

Theorem 9 ([3])**.** *If* H *is a cocommutative, then* c *is a symmetry for* $\mathbf{Mod}(H)$*.*

Proof. (Sketch). Since c is a symmetry for $\mathbf{Mod}(R)$, then it suffices to show that the components of c are H-module morphisms. Let (A, ρ) and (B, τ) be H-modules. Then there are H-modules $(A \otimes B, \gamma)$ and $(B \otimes A, \gamma')$. It must be shown that $\gamma' \circ (1_H \otimes c_{A,B}) = c_{A,B} \circ \gamma$. Since H is cocommutative, then $c_{H,H} \circ \Delta = \Delta$. Since ρ and τ are R-linear, then $(\tau \otimes \rho) \circ c_{A,B} = c_{A,b} \circ (\rho \otimes \tau)$. From these relations, it follows that $\gamma' \circ (1_H \otimes c_{A,B}) = c_{A,B} \circ \gamma$. $\square$

Theorem 10 ([10])**.** $\mathbf{Mod}(H)$ *is a rigid monoidal category.*

Remark 20. *I have been told that if* H *is finite-dimension and* R *has characteristic zero, then* $\mathbf{Mod}(H)$ *is pivotal.*

8 Additional Proofs

Several theorems used throughout these notes are well-known results from category theory. The proofs do not provide deeper insight into rigid categories nor Hopf algebras, and have therefore been omitted from the main sections.

8.1 Proof of Theorem 1

Proof. Let $\mathcal{C}$ be a category with all binary products such that the products. Define $\Pi : \mathcal{C}^2 \to \mathcal{C}$ as follows.

1. (Objects). $\Pi_0 : (A, B) \to A \times B$.

2. (Morphisms). Let $A \xrightarrow{f} C$ and $B \xrightarrow{f} D$. Then $f \circ \pi_1 : A \times B \to C$ and $g \circ \pi_2 : A \times B \to D$. Let $h : A \times B \to C \times D$ be unique map such that $\pi_1 \circ h = f \circ \pi_1$ and $\pi_2 \circ h = g \circ \pi_2$. Define $\Pi(f, g) = h$.

It remains to be shown that Π is a functor.

1. (Respects Identities). Let $A \xrightarrow{1_A} A$ and $B \xrightarrow{1_B} B$. Then $\Pi(1_A, 1_B)$ is the unique morphism such that h such that $\pi_1 \circ h = 1_A \circ \pi_1 = \pi_1$ and $\pi_2 \circ h = 1_B \circ \pi_2 = \pi_2$. However, $\pi_1 \circ 1_{\Pi(A,B)} = \pi_1$ and $\pi_2 \circ 1_{\Pi(A,B)} = \pi_2$. Then by uniqueness, $\Pi(1_A, 1_B) = h = 1_{\Pi(A,B)}$. Since (A, B) was arbitrary, then Π respects identities.

2. (Respects Composition). Let $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3$ and $B_1 \xrightarrow{g_1} B_2 \xrightarrow{g_2} B_3$. Then $\Pi(f_2 \circ f_1, g_2 \circ g_1)$ is the unique arrow h such that $\pi_1 \circ h = f_2 \circ f_1 \circ \pi_1$ and $\pi_2 \circ h = g_2 \circ g_1 \circ \pi_1$. Similarly, by definition of $\Pi(f_1, g_1)$ and $\Pi(f_2, g_2)$, the following equaltiites hold.

$$\pi_1 \circ \Pi(f_2, g_2) \circ \Pi(f_1, g_1) = f_2 \circ \pi_1 \circ \Pi(f_1, g_1) = f_2 \circ f_1 \circ \pi_1$$
$$\pi_2 \circ \Pi(f_2, g_2) \circ \Pi(f_1, g_1) = g_2 \circ \pi_2 \circ \Pi(f_1, g_1) = g_2 \circ g_1 \circ \pi_2$$

   By uniqueness, $\Pi(f_2 \circ f_1, g_2 \circ g_1) = \Pi(f_2, g_2) \circ \Pi(f_1, g_1)$. Since (f_1, f_2) and (g_1, g_2) were arbitrary, composable pairs, then Π respects composition.

Therefore, $\Pi : \mathcal{C}^2 \to \mathcal{C}$ is a functor. □

8.2 Proof of Theorem 2

Proof. Let $(\mathcal{C}, \otimes, I, \lambda, \rho, \alpha)$ be a Cartesian monoidal category. Let $A, B \in \mathcal{C}$. Since $\mathcal{C}$ is Cartesian, then $A \otimes B$ is the binary product of A and B. Then there exists projections $\pi_1 : A \otimes B \to A$ and $\pi_2 : A \otimes B \to B$ satisfying the universal property of the product. By the universal property of the product, there exists a unique morphism $c_{A,B} : A \otimes B \to B \otimes A$ such that $\pi_1 \circ h = \pi_2$ and $\pi_2 \circ h = \pi_1$. Now let $A \xrightarrow{f} C$ and $B \xrightarrow{f} D$. It will be shown that $c_{(A,B)}$ is the component of a natural transformation from $A \otimes B$ to $B \otimes A$. By the universal property of products, the following hold.

1. $c_{(C,D)} \circ (f \otimes g) = (\pi_1 \circ c_{(C,D)} \circ (f \otimes g)) \otimes (\pi_2 \circ c_{(C,D)} \circ (f \otimes g))$.
2. $(g \otimes f) \circ c_{(A,B)} = (\pi_1 \circ (g \otimes f) \circ c_{(A,B)}) \otimes (\pi_2 \circ (g \otimes f) \circ c_{(A,B)})$

Then to show that the naturality diagram commutes, is suffices to equate the components of $c_{(C,D)} \circ (f \otimes g)$ with the components of $(g \otimes f) \circ c_{(A,B)}$ By definition of the product functor $\otimes$, $\pi_1 \circ (f \otimes g) = f \circ \pi_1$ and $\pi_2 \circ (f \otimes g) = g \circ \pi_2$. Likewise, $\pi_1 \circ (g \otimes f) = g \circ \pi_1$ and $\pi_2 \circ (g \otimes f) = f \circ \pi_2$. Then the following hold.

1. $\pi_1 \circ c_{(C,D)} \circ (f \otimes g) = \pi_2 \circ (f \otimes g) = g \circ \pi_2 = g \circ \pi_1 \circ c_{(A,B)} = \pi_1 \circ (g \otimes f) \circ c_{(A,B)}$.
2. $\pi_2 \circ c_{(C,D)} \circ (f \otimes g) = \pi_1 \circ (f \otimes g) = f \circ \pi_1 = f \circ \pi_2 \circ c_{(A,B)} = \pi_2 \circ (g \otimes f) \circ c_{(A,B)}$.

Then $c_{(C,D)} \circ (f \otimes g) = (g \otimes f) \circ c_{(A,B)}$. Since f and g were arbitrary, then c is a natural transformation from $A \otimes B$ to $B \otimes A$. It is not hard to show that each component of c is self-inverse, since $\pi_1 \circ 1_{(A,B)} = \pi_1$ and $\pi_2 \circ 1_{(A,B)} = \pi_2$. Then c is a symmetry for the symmetric monoidal category $(\mathcal{C}, \otimes, I, \lambda, \rho, \alpha)$. ☐

8.3 Proof of Theorem 7

Proof. There are four conditions to check.

1. (Multiplication). Let $r \in R$ and $f, g, h \in \mathrm{End}(A)$. Since tensor products are R-modules and m is an R-linear map, then $(rf) \star g = r(f \star g)$ and $f \star (rg) = r(f \star g)$. Since tensor products distribute over addition, with m and Δ both morphims of R-modules, then $(f + g) \star h = (f \star h) + (g \star h)$ and $f \star (g + h) = (f \star h) + (f \star h)$. Then $\star$ is bilinear. Then $\star$ extends to an R-linear map $\star : \mathrm{End}(A) \otimes \mathrm{End}(A) \to \mathrm{End}(A)$ such that:

$$m : \sum_{i=1}^{n} r_i(f_i \otimes g_i) \mapsto \sum_{i=1}^{n} r_i(f_i \star g_i)$$

2. (Unit). Since R-linear maps are closed under composition, then, $\eta \circ \varepsilon$ is an R-linear map.

3. (Associativity). Let $f, g, h \in \mathrm{End}(A)$. Then by the associativity of m and the coassociativity of Δ, the following equalities hold.

$$\begin{aligned}
(f \star g) \star h &= (m \circ (f \otimes g) \circ \Delta) \star h \\
&= m \circ ((m \circ (f \otimes g) \circ \Delta) \otimes h) \circ \Delta \\
&= (m \circ (m \otimes 1)) \circ ((f \otimes g) \otimes h) \circ ((\Delta \otimes 1) \circ \Delta) \\
&= (m \circ (1 \otimes m)) \circ ((f \otimes g) \otimes h) \circ ((1 \otimes \Delta) \circ \Delta) \\
&\cong (m \circ (1 \otimes m)) \circ (f \otimes (g \otimes h)) \circ ((1 \otimes \Delta) \circ \Delta) \\
&= m \circ (f \otimes (m \circ (g \otimes h) \circ \Delta)) \circ \Delta \\
&= f \star (g \star h)
\end{aligned}$$

This extends to tensor sums by R-linearity. Then $\star(1 \otimes \star) = \star(\star \otimes 1)$. Therefore $\star$ is associative.

4. (Unital). Let $f \in \mathrm{End}(A)$. Since η is the unit of m and ε the counit of Δ, then the following equations hold.

$$f \star (\eta \circ \varepsilon) = m \circ (f \otimes (\eta \circ \varepsilon)) \circ \Delta = m \circ (f \otimes \eta) \circ (1_A \otimes \varepsilon) \circ \Delta = f \otimes 1_A$$
$$(\eta \circ \varepsilon) \star f = m \circ ((\eta \circ \varepsilon) \otimes f) \circ \Delta = m \circ (\eta \otimes f) \circ (\varepsilon \otimes 1_A) \circ \Delta = f \otimes 1_A$$

It follows by R-linearity that $\star$ is unital.

Therefore, $(\mathrm{End}(A), \star, \eta \circ \varepsilon)$ is an R-algebra. □

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