# Quantization of Hitchin's Moduli Spaces and Liouville Theory

Jorg Teschner
DESY, Hamburg

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## Contents



Notes by Theo Johnson-Freyd UC Berkeley, Fall 2011
theojf@math.berkeley.edu

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## 0 Introduction

As the title says, the original title is "Quantization of moduli spaces and Louiville theory." These are a priori unrelated, so we begin with a schematic overview.

The aims are to explain the relations between the following four (interesting, extensively studied in their own right) subjects:

1. Hitchin's integrable systems (classical)
2. Quantized Hitchin's instegrable systems
3. Moduli spaces of flat connections
4. Liouville theory

There is a parameter $\epsilon = \hbar$ introduced in the quantization, and a natural arrow 2→1 setting $\epsilon = 0$. Number 4 is a qft which has attracted much attention, for example because it is a main building block in 2d quantum gravity. There is another parameter which connects 4→2, so that we have a square of quantizations.

One of the motivations for returning to a subject like Liouville theory, which I spent some time on but then got bored with, is that there are relations with gauge theories. This is a recent active subject of research in mathematical physics, and if we have time later we will go into it, but just to mention right now what we have in mind: the works of Alday–Gaiotto–Techikawa (AGT) and of Nekrasov–(Rosly)–Shatashvili. These people have discovered some amazing relations between partition functions in gauge theory and correlation functions in Liouville theory. But they are discussing the level at items 2,3, where one of the two parameters has been turned off. More generally, there is an amazing and nontrivial duality between items 2,3 in the "magic diamond".

There are also interesting relations with noncompact Chern–Simons theory. In particular, here one might mention the recent work by Dimofte, Gukov, and others.

So these are the motivations. To give a plan, we will work with three main parts. First, we will talk about moduli spaces of flat connections, and then to talk about the quantization of them (without identifying this as Louiville theory). In the second part of the lectures, we will talk about Liouville theory, and recognize it as what we got in the first part. In the third part, we will talk about quantization of Hitchin's integrable systems, and thereby make the story into a coherent whole. (Of course, experience shows that we do not always get as far as we hope in lectures.)

**Mina:** In the diagram, going up involves turning off a parameter, but the two parameters are very different. **Jorg:** Yes. In fact, it is a very surprising result that there is a duality between 2,3 above. The punchline will be that there is a symmetry in Liouville theory that switches the two parameters. So when we turn off one parameter, we can interpret it on either side. To use some big words, this relationship is related to "Geometric Langlands". The quantized Hitchin's system is the "D-module side" of Langlands, and the moduli space side is the "local systems" side.

# 1 Quantization of moduli spaces of flat connections

## 1.1 Moduli spaces of flat connections

We are considering flat $\mathrm{SL}(2,\mathbb{C})$ connections on a Riemann surface $C$ (on a complex vector bundle $E$ on $C$). So the connection is locally $\nabla = \mathrm{d}+A$, and we consider the connections up to $\mathrm{SL}(2,\mathbb{C})$ gauge equivalence. The space of such connections is closely related to $\mathrm{Hom}(\pi_1(C),\mathrm{SL}(2,\mathbb{C}))/\,\mathrm{SL}(2,\mathbb{C})$ acting by conjugation. Let's briefly recall the correspondence. For any flat connection $\nabla$, we can associate to it a representation $\rho : \pi_1(c) \to \mathrm{SL}(2,\mathbb{C})$, which is the monodromy representation of this flat connection: to each closed curve $\gamma$ we associate some matrix $M_\gamma = \mathrm{Pexp}\int_\gamma \mathrm{d}s A \in \mathrm{SL}(2,\mathbb{C})$. For "almost all" $\rho : \pi_1(C) \to \mathrm{SL}(2,\mathbb{C})$ (not all — you need an "irreducibility" condition, and if there are punctures then you need to add a condition about the monodromy around the punctures), the Riemann–Hilbert correspondence says that there exists a connection $\nabla$ which has $\rho$ as its monodromy. **Harold:** So you're saying that if it is not irreducible, then you cannot find an operator? **Jorg:** There are counterexamples to the fully general statement of Riemann–Hilbert, which were only recently found. For us, asking for irreducibility is good enough.

So, $\mathcal{M}$ = the moduli space of flat $\mathrm{SL}(2,\mathbb{C})$ connections. This space is a symplectic space — for us physicists, it will be the "phase space" we want to quantize. Let's recall where this natural symplectic structure comes from. The construction that is in some sense most convenient starts with the infinite-dimensional space of fields, and one writes down a form on this connection, check that it is gauge-invariant, and so descends to the moduli space. The form is:

$$\Omega(\delta A_1,\delta A_2) = \frac{1}{2}\int \mathrm{tr}(\delta A_1 \wedge \delta A_2)$$

Here $\delta A_i$ is a tangent vector at a flat connection. Then one checks that this 2-form is closed and gauge-invariant. This way of introducing symplectic forms on moduli space of flat connections was introduced by Atiyah and Bott.

So far, this description of $(\mathcal{M}, \Omega)$ is *independent* of any choice of complex structure on $C$. Most of the time, we really want a *Riemann surface*, which is a surface with a complex structure, and this complex structure will be important. But here it has not yet entered.

However, using the complex structure, there exists other interesting and useful models of the space $\mathcal{M}$:

1. The moduli space of *local systems*. This is a complex holomorphic vector bundle $\mathcal{E}$ with holomorphic connection $\nabla'$ (or rather an $\mathrm{SL}(2,\mathbb{C})$ bundle), and the holomorphicity locally means that we're talking about matrix differential operators of the form $\nabla' = \bigl(\frac{\partial}{\partial y} + \mathcal{M}(y)\bigr)\mathrm{d}y$. **Question from the audience:** This doesn't really need the complex structure either: we can say what a local system is in terms of quasicoherent sheaves and so on. **Jorg:** There is some incoherence in the literature. Some use this definition.

   We used the complex structure to split $\nabla = (\partial_y + A_y)\mathrm{d}y + (\bar\partial_{\bar y} + A_{\bar y})\mathrm{d}\bar y$, and the second part is the $\bar\partial_A$ operator that defines the holomorphicity of the vector bundle.

2. As a space of meromophic *opers*. In our case, we don't actually need the more general terminology of "opers", and can work with projective connections, but the terminology is handy. We recall the definition:

   These are second-order differential operators of the form $\partial_y^2 + t(y)$, so that under change of coordinates $y \rightsquigarrow w$ the function $t(y)$ transforms as

   $$t(y) \rightsquigarrow (y'(w))^2 t(y(w)) - \frac{1}{2}\{y, w\}$$

   where $\{y, w\} = (\frac{y''}{y'})' - \frac{1}{2}(\frac{y''}{y})^2$.

   Here $y, w$ are local coordinates, and $'$ means "derivative of $y$ with respect to $w$."

   I should elaborate a bit more. We're talking about meromorphic opers, but the poles turn out to be somewhat restricted, because we have given a description of a space that is canonically equivalent to the earlier description of $\mathcal{M}$.

   **Harold:** These are differential operators on? **Jorg:** Ah, you mean on what they act? They typically act on the square root of the canonical line bundle. **Harold:** It's the transformation rule that tells you that they act on that line bundle? **Jorg:** Yes, I believe so.

   For me the more familiar way comes from: $\nabla' = \partial_y + \left(\begin{smallmatrix} 0 & t(y) \\ 1 & 0 \end{smallmatrix}\right)$. Locally by gauge transformations you can always achieve this form, and that's how you get from 1 to 2. If you now wander about the transformation law of these guys, you have to take into account that the derivative transforms, and this gives some terms in the transformation law that you have to compensate for with an additional transformation law, and that's the explanation for the somewhat unusual term.

In both cases, the singular behavior is rather restricted.

1. The only poles are at punctures $P_1, \dots, P_n$, and $M(y) \sim M_r/(y - z_r)$ near the pole.

2. At the punctures we see behavior $t(y) \sim \delta_r/(y-z_r)^2 + H_r/(y-z_r) + \dots$. These are the first terms in the Laurent expansion that we allow at the punctures. This is a standard thing: when you go from a first-order differential operator to a second-order one, then singularities go from first- to second-order.

   There may exist also $d < \infty$ *apparent* singularities away from the punctures. At the punctures, the numbers $\delta_r$, $H_r$ are arbitrary parameters. At the apparent singularities, located at $w_1, \dots, w_d$, the Laurent expansion is always of the form:

   $$t(y) \sim -\frac{3}{4(y-w_r)^2} + \frac{\kappa_r}{y-w_r} + \eta_r + O(y-w_r)$$

   and moreover the parameters satisfy $\kappa_r^2 + \eta_r = 0$.

   Why these strange conditions? We are talking about flat connections, so there can be monodromy around handles, and around punctures, but we do not want monodromy anywhere else. The statement is: these conditions are necessary and sufficient for the triviality (in $\mathrm{PSL}(2,\mathbb{C})$) of the monodromy around $w_k$. Actually, the monodromy will be $-1 \in \mathrm{SL}(2,\mathbb{C})$, but we'll work with $\mathrm{PSL}(2,\mathbb{C})$ because we don't want to be bothered by this minus sign.

   How to see this? You can ask for solutions to the differential equation $(\partial_y^2 + t(y))\psi(y) = 0$. There is always a solution of the form $\psi(y) \sim y^\lambda$ for $\lambda(\lambda-1) = \delta_r$. Well, then there are usually two solutions $\lambda_\pm$. Something interesting happens where $\lambda_+ - \lambda_- \in \mathbb{Z}$. For example, this happens when $\delta_r = -\frac{3}{4}$. In this integer case, you still have one solution $\psi_+ \sim y^{\lambda_+}$, but when you look for the second solution, you find that the second solution does not have a pure power law at the singularity, but rather is of the form $\psi_- \sim y^{\lambda_-}\log y$. Then you find out that the monodromy can be conjugated to $\left(\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$. Anyway, iff $\kappa_k^2 + \eta_k = 0$, then you don't need the logarithm, and the monodromy is just $-1$.

We now mention a few comments to help see the close connection between opers and the other models of $\mathcal{M}$.

- The component $\psi_2$ of solutions to $\big(\partial_y + M(y)\big)\left(\begin{smallmatrix} \psi_1 \\ \psi_2 \end{smallmatrix}\right) = 0$ — we can combine the two differential equations into one for $\psi_2$ — satisfies to ODE:

  $$\left(\partial_y^2 + \frac{R'}{R}\partial_y + U\right)\psi_2(y) = 0$$

  where $M = \left(\begin{smallmatrix} P & Q \\ R & S \end{smallmatrix}\right)$.

  The denominator $R$ is the origin of these apparent singularities that arise when we move from a first-order to second-order equation. Away from the zeros $w_1, \dots, w_d$ of $R$, we may write $\psi_2 = \sqrt{R}\phi_2$, and then $\phi_2$ satisfies an ODE of the form:

  $$(\partial_y^2 + t(y))\phi_2 = 0$$

  The $\sqrt{R}$ is what's needed to remove the linear part in the differential equation.

More generally, when you try to implement this, you see that $t(y)$ has an apparent singularity at $w_1, \dots, w_d$. Then you can also understand why the monodromy around these points is trivial. Basically, the monodromy of the solutions of the $\phi$ differential equation is, by construction, equal to the monodromy of the solution $\psi$ to the matrix differential equation.

- It is also worth noting that we can discuss the Riemann–Hilbert correspondence directly in terms of these opers:

**Theorem (Yoshida):** *For any $\rho \in \mathrm{Hom}(\pi_1(C), \mathrm{PSL}(2,\mathbb{C}))$ that is irreducible and semisimple around each puncture $P_r$, there exists a meromorphic oper with monodromy $\rho$. And there exist $d \leq 3g-3+n$ apparent singularities.*

Here $g$ is the genus of the surface and $n$ is the number of punctures.

So I'd like to draw your attention to the fact that we don't need infinitely many apparent singularities, but only at most $3g-3+n$. This is intriguingly similar to the fact that $\dim_{\mathbb{C}} \mathcal{M} = 6g-6+2n$. What is our overall goal? Quantization, and for this we'd like some explicit control over $\mathcal{M}$, like good coordinates on it. So it is intriguing to think that the parameters of the differential operators at the $3g-3+n$ singularities may provide such coordinates.

What is clear, now, is that $\mathcal{M}$ is stratified by the minimal number of apparent singularities necessary to represent the representation as the monodromy of a differential operator.

There exist two extreme cases:

(a) When $d = 3g-3+n$, i.e. the oper really does need all of its apparent singularities. There are two parameters at the $r$th singularity: the position $w_r$ and the residue $\kappa_r$. So then $(w_r, \kappa_r)$ for $r$ ranging from 1 to $3g-3+n$ is a (local) system of coordinates for $\mathcal{M}$. It is only local because it depends on the coordinates on the curve.

(b) In general, though, life is not that simple. It is possible that $d = 0$, and we have a holomorphic oper. Still we want to follow our idea that the coordinates on $\mathcal{M}$ should depend on parameters of the oper. If you look two different opers, with functions $t, t'$, then $t - t'$ is a quadratic differential, because the funny transformation law drops out. Let us fix once and for all a reference oper. For example, uniformize the Riemann surface, and on the uniformization there is a distinguished reference oper $t_0$. Then we can expand the quadratic differential $t - t_0$ in a basis of quadratic differential. The space of quadratic differentials is known to be $(3g-3+n)$-dimensional. We pick a basis $\vartheta_r(y)$, and define $t - t_0 = \sum \vartheta_r(y)\, H_r$. So these $3g-3+n$ coordinates $H_r$, which we will call "Hamiltonians", aren't enough to get coordinates for $\mathcal{M}$.

**Harold:** The dimensions of the strata are $3g-3+d$? **Jorg:** No. All of the strata, it turns out, have the same dimension $6g-6+2n$. The strata are the *quasi-Fuchsian components*, and are related to the hyperbolic geometry.

So, variations of $H$ sweep out a $(3g-3+n)$-dimensional submanifold $\mathcal{B}$ of $\mathcal{M}$. In the physics literature, $\mathcal{B}$ is called the *brane of opers*.

So to move off of $\mathcal{B}$ into this component of $\mathcal{M}$, what I can do is to vary the complex structure.

**Harold:** $\mathcal{M}$ has connected components indexed by $d$? **Jorg:** I believe that saying "strata" is better. These are connected components if you restrict to a real slice — if you ask for representations to $\mathrm{PSL}(2,\mathbb{R})$. But I have not found a clear statement for the complex case. Perhaps they somehow connect at $\infty$ or for another reason aren't honestly connected components.

So the statement is that variations of complex structure on $C$ give transverse directions to $\mathcal{B}$. So this means that I may take coordinates on $\mathcal{M}$ to be $H_1, \dots, H_{3g-3+n}$ and $q_1, \dots, q_{3g-3+n}$ — the latter are the coordinates for Teichmüller space $\mathcal{T}_{g,n}$.

**Example:** When $g = 0$, the only opers oare of the form $t(y) = \sum \frac{\delta_r}{(y-z_r)^2} + \frac{H_r}{y-z_r}$. The coordinates on $\mathcal{T}_{0,n}$ are the cross-ratios of the positions of the singularities. The $\delta$s we do not think of as interesting variables: we fix them once and for all. ♢

**Kolya:** There are two very different real cases: $\mathrm{PSL}(2,\mathbb{R})$ and the compact one $\mathrm{SO}(3)$. You mentioned something about the representations for $\mathrm{PSL}(2,\mathbb{R})$ — what about the other one? **Jorg:** I haven't come across a statement for this case. **Harold:** In the $\mathrm{SU}(2)$ case, the connection goes to zero. **Jorg:** Ah, yes, so all of the information is in the holomorphic vector bundle, and this is certainly the space of flat $\mathrm{SU}(2)$ connections.

**Kolya:** Another question: you do not consider surfaces with boundary. **Jorg:** No. It is a very interesting question, but I do not want to deal with it now. Between punctures are boundaries, you can consider at punctures higher-order singularities.

## 1.2 Integrability

Let us return to the maximal case $d = 3g - 3 + n$. In the minimal case, it was necessary to consider the dependence on the complex structure. What happens when we consider this in the maximal case? We consider the *extended moduli space* $\hat{\mathcal{M}} = \mathcal{M}_{\mathrm{loc}} \times \mathcal{T}_{g,n}$, where $\mathcal{M}_{\mathrm{loc}}$ is the moduli space of local systems — it is basically the full space of parameters of the holomorphic differential operators.

Now we may consider the monodromy map:

$$m\big((\mathcal{E}, \nabla'), \mu\big) = \rho \in \mathrm{Hom}\big(\pi_1(C), \mathrm{SL}(2,\mathbb{C})\big)$$

where $\mu$ is a generic point in Teichmüller space. Then let's require that $\rho$ is kept constant: we consider all points in the extended moduli space for some fixed $\rho$. Then generically this defines a $(3g - 3 + n)$-dimensional submanifold inside $\hat{\mathcal{M}}$.

This shows that we can combine variations $\delta_\mu$ of the complex structure with variations $\delta_{\mathcal{E},\nabla'}$ on $\mathcal{M}_{\mathrm{loc}}$ such that $\delta\rho = 0$. Then we get a family of commuting flows on $\mathcal{M}_{\mathrm{loc}}$, called the *isomonodromic deformations*.

Explicitly for $g = 0$, then $M = \sum_{r=1}^{n-1} \frac{M_r}{y-z_r}$, and we have moved the $n$th puncture to $\infty$ for convenience and choosing $M_n = \left(\begin{smallmatrix} j_\infty & 0 \\ 0 & -k_\infty \end{smallmatrix}\right)$ and $\sum_{r=1}^{n} M_r = 0$. Then the statement is that we get


theojf@math.berkeley.edu

the equations:

$$\frac{\partial}{\partial z_s} M_r = \frac{[M_r, M_s]}{z_r - z_s}, \qquad r \neq s$$
$$\frac{\partial}{\partial z_r} M_r = -\sum_{s \neq r} \frac{[M_r, M_s]}{z_r - z_s}$$

This is a famous and rather classical system of integrable partial differential equations, called the *Schlesinger equations.*

In terms of opers, we consider

$$\partial_y^2 + \sum_{r=1}^{n} \left( \frac{\delta_r}{(y - z_r)^2} + \frac{H_r}{y - z_r} \right) + \sum_{k=1}^{n-3} \left( \frac{-3}{4(y - w_k)^2} + \frac{\kappa_k}{y - w_k} \right)$$

Recall that $\kappa_k^2 + \eta_k = 0$. Then you can calculate what is the constant part of the Laurent expansion:

$$\eta_k = \sum_r \left( \frac{\delta_r}{(w_k - z_r)^2} + \frac{H_r}{w_k - z_r} \right) + \sum_{k' \neq k}^{n-3} \left( \frac{-3}{4(w_k - w_{k'})^2} + \frac{\kappa_k}{w_k - w_{k'}} \right)$$

so what you see is something very interesting. What we have here are relations between these parameters $H$ and the rest of the parameters. If we further add the regularity of $t(y)$ at $y = \infty$, then we get a further equations for $\ell = -1, 0, 1$:

$$0 = \sum_r z_r^\ell \left( z_r \partial_r + (\ell + 1) \delta_r \right) + \sum_k w_k^\ell \left( w_k \partial_{y_k} - (\ell + 1) \frac{3}{4} \right)$$

Now you count equations and unknowns, and discover that you get $H_r = H_r(\kappa, w)$, and that

$$\frac{\partial w_k}{\partial z_r} = \frac{\partial H_r}{\partial \kappa_k}, \qquad \frac{\partial \kappa_k}{\partial z_r} = -\frac{\partial H_r}{\partial w_k}$$

**Kolya:** What is $\delta$? **Jorg:** It is some parameter. We had the relation

$$M = \sum \frac{M_r}{y - z_r} = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

and then $\delta_r = -\det M_r + \frac{1}{2}$, and conversely

$$R = \sum_{r=1}^{n-1} \frac{R_r}{y - z_r} = u \frac{\prod_{k=1}^{n-3} (y - w_k)}{\prod_{r=1}^{n-1} (z - z_r)}$$

This is a standard application of separation of variables: you find variables that make things work. In any case, we have $\kappa_k = P(w_k)$.

# Tuesday, October 18

Let's begin by reviewing some of yesterday. We had a magic diamond with classical Hitchin at the top, flat connections on the left, quantum Hitchin on the right, and Liouville on the bottom. We will mostly talk about flat connections and Liouville, and return to the whole diamond at the end.

Yesterday we started talking about the moduli space $\mathcal{M}$ of flat connections, and many of its standard incarnations.

1. Locally a flat connection is of the form $d + A$.
2. Each flat connection defines something in $\mathrm{Hom}(\pi_1(C), \mathrm{SL}(2,\mathbb{C}))$, and gauge equivalence is modding out by conjugation — sometimes I will work with $\mathrm{PSL}(2,\mathbb{C})$ instead.
3. $(\mathcal{E}, \nabla')$, locally of the form $\big(\partial_y + M(y)\big)\mathrm{d}y$. Using gauge freedom, we can locally get $M$ into the form $M = \left(\begin{smallmatrix} 0 & t(y) \\ 1 & 0 \end{smallmatrix}\right)$.
4. and then the equation for parallel transport with respect to this connection reduces to a second-order differential equation of the form $\big(\partial_y^2 + t(y)\big)\psi = 0$, which is an *oper*. **Kolya:** This is the definition of *oper*? In $\mathrm{SL}(n)$? **Jorg:** I can use gauge to get $M$ into the form $\left(\begin{smallmatrix} 0 & t_1 & \dots \\ 1 & 0 & \dots \\ 0 & 1 & \dots \end{smallmatrix}\right)$ and then I'd get an $n$th order oper. An *oper* is one of these with specific transformation laws for $t$.

   There is a Riemann–Hilbert correspondence for opers, and it says that to essentially anything in $\mathrm{Hom}(\pi_1(C), \mathrm{SL}(2,\mathbb{C}))$, I get an oper, but I have to allow *apparent singularities*, which are when $t(y) \sim -\frac{3}{4(y-w_k)^2} + \frac{\kappa_k}{y-w_k} + \eta_k + \dots$ where $\eta_k + \kappa_k^2 = 0$. We need at most $d \leq 3g-3+n$ such singularities, and this number $d$ breaks the space into different strata.

   The most interesting strata are $d = 0$ and $d = 3g-3+n$. In the maximal case, I can take $\kappa_k$ and $w_k$ together as coordinates.

So, we then begin discussing an extended moduli space $\hat{\mathcal{M}} = \mathcal{M} \times \mathcal{T}_{g,n}$, or rather it is a fibration over $\mathcal{T}_{g,n}$ with fiber $\mathcal{M}$. Then we get a flow on these by asking: how do I have to vary my local system if I want to keep the monodromy representation constant under variations in the symplectic structure?

### 1.3 Systems of Darboux coordinates on $\mathcal{M}$

I had already introduced the *Atiyah–Bott* symplectic form on $\mathcal{M}$, given by $\Omega = \int \mathrm{tr}(\delta A \wedge \delta A)$. We'd like to now describe this symplectic form in terms of the coordinates coming from opers.

To review terminology, when we have a symplectic space $(\mathcal{N}, \omega)$, with $\dim \mathcal{N} = 2d$, then a collection of local coordinates $(p_1, \dots, p_d, q_1, \dots, q_d)$ on $\mathcal{N}$ are *Darboux coordinates* if $\omega = \sum_{k=1}^d \mathrm{d}p_k \wedge \mathrm{d}q_k$.

From the point of view of opers, it is particularly easy to find Darboux coordinates. Indeed, on the $d = 3g - 3 + n$ stratum, we can simply observe that $(w, \kappa)$ are Darboux coordiantes on $\mathcal{M}$. Recall that $\dim \mathcal{M} = 6g - 6 + 2n$.

We had yesterday in this direction something very enticing. Recall that we defined a flow and it was given by Hamiltonian flow: $\frac{\partial w_k}{\partial z_r} = \frac{\partial H_r}{\partial \kappa_k}$ and another one. We would like to recognize this as Hamilton's equations $\frac{\partial \omega_k}{\partial z_r} = \{H_r, \kappa_k\}$. And sure enough, the coordinates $(\omega, \kappa)$ satisfy the *canonical form* relation for the Poisson bracket: $\{\omega_k, \kappa_{k'}\} = \delta_{k,k'}$ and the rest are 0. See the theorems of Iwasaki and Pinchbeck.

So in the $d = 0$ case, recall that we have coordinates $H_1, \dots, H_{3g-3+n}, q_1, \dots, q_{3g-3+n}$. When $d = 0$, we have a brane $B$, and it is a standard fact from complex analysis that in order to have many meromorphic functions, we need to allow many poles. So the $d = 0$ case restrains us.

In any case, the statement about the symplectic structures is as follows. First of all, there are many ways to define coordinates on Teichmüller space. Having fixed the $H$s by $t - t_0 = \sum \vartheta_r H_r$, the statement is that there exist coordinates $q_1, \dots, q_{3g-3+n}$ on $\mathcal{T}_{g,n}$ such that locally $\Omega = \sum_{r=1}^{3g-3+n} \mathrm{d}H_r \wedge \mathrm{d}q_r$. **Kolya:** Do I understand correctly that these $H$s you can always introduce? **Jorg:** Yes, but the difference is: in the $d = 0$ case, I need the $H$s as independent coordinates, whereas in the $d = 3g - 3 + n$ then I can write the $H$s as functions of the coordinates $w, \kappa$.

**Question from the audience:** The statement that $t - t_0$ can be expanded in a basis of quadratic differentials — how does this interact with the singularities? **Jorg:** The Schwarzians cancel under changes of coordinates, so $t - t_0$ is a quadratic differential. Of course, $t$ is allowed to have singularities at any punctures on my Riemann surface. **Question from the audience:** What about the apparent singularities? **Jorg:** If you move the apparent singularities, that's like changing the coordinates on the momentum variables. See, we have $t \sim \frac{\delta_r}{(y - z_r)^2} + \frac{H_r}{y - z_r}$. If we shift to some $t'$, that shifts $H'_r = H_r - H_{r,0}$. It's just a constant.

**Kolya:** So in both cases, the $H$s are $3g - 3 + n$ independent functions on the Moduli space. **Harold:** When there are apparent singularities, I don't know how to define the $H$s? You take the difference of two things, but they have different apparent singularities, so I don't know how to expand in a basis of quadratic differentials. **Jorg:** Last time we said it completely explicitly in $g = 0$. In general, you solve the condition that the $\mathrm{PSL}(2, \mathbb{C})$-monodromy around the apparent singularities is $\pm 1$, and this is equivalent to $\kappa_k^2 + \eta_k = 0$, and then this gives you some formulas $H_r = H_r(w, \kappa) = \sum_k \kappa_k^2 + \dots$. In general, near the $r$th puncture I have $t \sim \frac{\delta_r}{(y - z_r)^2} + \frac{H_r}{y - z_r}$, and $\delta_r = \det M_r - \frac{1}{2}$ or also as some trace of monodromy around the puncture. What I'm doing is considering that part of the moduli space with fixed conjugacy class (coadjoint orbit) around each puncture. If I let the $\delta$s be variables, I would have $6g - 6 + 3n$ dimensions, and I'm working with the $6g - 6 + 2n$ version.

Anyway, the statement when $d = 0$ is that there exist coordinates on $\mathcal{T}_{g,n}$ so that $\Omega = \sum \mathrm{d}H_r \wedge \mathrm{d}q_r$. The names are Kawai and Pinchbeck.

**Kolya:** In the maximal case, you have Darboux coordinates, and also a system of Hamiltonians, so you are in the situation of integrable systems. In the minimal case you just have the Darboux coordinates? **Jorg:** Yes. As you reduce $d$, you constrain the system, and lose the integrability. **Kolya:** $\mathcal{M}$ is symplectic. On $\hat{M}$ do you have some natural Poisson structure? **Jorg:** I think the statement is yes. What you can do is to produce an $\hat{\Omega}$. Here $\Omega = \sum \mathrm{d}\kappa \wedge \mathrm{d}w$, and you can extend to $\hat{\Omega} = \Omega + \sum \mathrm{d}H \wedge \mathrm{d}q$. **Kolya:** Then the minimal case is the Hamiltonian reduction of this? **Jorg:** I think so, but haven't thought it through.

### 1.4 Darboux coordinates for $\mathrm{Hom}(\pi_1(C), \mathrm{SL}(2,\mathbb{C}))$

We follow Nekrasov–Rosly–Shatashvili, although the part we will present isn't really their invention, but they take it up well and use it in a new way.

So for this, what we are going to use is a pants decomposition of the Riemann surface, so that we have building blocks out of which to build the Darboux coordinates.

Recall that I cut $C$ along a system of nonintersecting closed curves, and produce pants. Consider a fixed curve $\gamma$. Then around each curve I find either a once-punctured torus $C_{1,1}$ or a four-times-punctured sphere $C_{0,4}$. So what will happen is that the pants decomposition breaks up the problem of finding Darboux coordinates into a problem of finding them on each of these two pieces.

**Kolya:** Do you really mean surfaces with boundary, or do you want to pinch the boundaries into punctures? **Jorg:** When we cut, we of course produce an honest boundary. In order to put it onto equal footing with what we've already done, on the building blocks we identify the puncture with a hole, perhaps with a normal vector there.

The second bit of preparation: associated to a $C_{1,1}$ or a $C_{0,4}$ with a curve $\gamma$, then there are two other curves canonically associated to it. One of them is $\check{\gamma}$, which is just the dual cycle — if $\gamma$ is an A-cycle, then $\check{\gamma}$ is the B-cycle. The third curve is $\hat{\gamma} = \gamma \circ \check{\gamma}$.

So the idea is to define coordiante functions. We take $x_\gamma = -\mathrm{tr}(M_\gamma)$, where $M_\gamma = \rho(\gamma)$ is the monodromy of going around $\gamma$. Then I also define $y_\gamma = -\mathrm{tr}(M_{\check{\gamma}})$ and $z_\gamma = -\mathrm{tr})M_{\hat{\gamma}}$. These are all functions on the moduli space, and I'd like to use them to define coordinate functions.

What we will see is that these functions know some of the algebraic structure of the Moduli space. First, let's do some counting. We have $3g-3+n$ curves in our pants decomposition, so now we have $3(3g-3+n)$ functions, and this is too many. But we can write down some relations of these functions, coming from the fundamental group and, et, the fact that $\mathrm{tr}\, g \,\mathrm{tr}\, h = \mathrm{tr}(gh) + \mathrm{tr}(gh^{-1})$.

So from these you can derive some relations. For example:

$$
\begin{aligned}
C_{1,1}: &\quad x_\gamma^2 + y_\gamma^2 + z_\gamma^2 - x_\gamma y_\gamma z_\gamma - M - 2 = 0 \\
C_{0,4}: &\quad xyz = x^2 + y^2 + z^2 - 4 + x(M_1M_2 + M_3M_4) + y(MM + MM) + Z(MM + MM) + \\
&\qquad + M_1^2 + M_2^2 + M_3^2 + M_4^2 + M_1M_2M_3M_4
\end{aligned}
$$

where the $M$s are the monodromies around the punctures.

So we fix the symplectic structure $\Omega$. In the end, everything we're building can be directly built from the path-ordered exponential of the gauge field. So in principle you can compute the Poisson bracket of these $x, y, z$ functions. This takes some work, so I just want to flash the results. The non-trivial Poisson brackets are:

$$\begin{aligned} C_{1,1}: &\quad \{x_\gamma, y_\gamma\} = z_\gamma - \tfrac{1}{2} x_\gamma y_\gamma \\ C_{0,4}: &\quad \{x_\gamma, y_\gamma\} = 2z_\gamma - x_\gamma y_\gamma + M_1 M_3 + M_2 M_4 \end{aligned}$$

And moreover $\{x_\gamma, x_{\gamma'}\} = 0$ if $\gamma \circ \gamma' = \emptyset$.

**Kolya:** There are other coordinates on this space, for example cluster coordinates. You do not use this? **Jorg:** There are advantages and disadvantages to each. They are associated to different ways to cut a surface into pieces: we use pants, the other uses triangles. I want to eventually connect this with conformal field theory, and I don't know how to treat triangles in CFT. **Kolya:** I want to emphasize this for the students. CFT is well-developed for surfaces with punctures, but not with true boundary. **Jorg:** Well, there is CFT with true holes, and this corresponds to allowing arbitrary descendents — arbitrary Virasoro structure — at the boundary. But open edges are out of reach.

Ok, so we want to introduce Darboux coordinates for $\mathrm{Hom}(\pi_1(C), \mathrm{SL}(2,\mathbb{C}))/\mathrm{SL}(2,\mathbb{C})$. I choose a set $\mathcal{C}$ of cutting curves. Then I introduce for each $\gamma \in \mathcal{C}$ coordinates $(\lambda_\gamma, \tau_\gamma)$ with $\{\lambda_\gamma, \tau_{\gamma'}\} = \delta_{\gamma,\gamma'}$ and zero otherwise. We choose these to be related to our basic functions by:

$$x_\gamma = 2\cosh \lambda_\gamma \qquad y_\gamma = \begin{cases} 2\cosh \frac{\tau_\gamma}{2} \sqrt{\frac{x_\gamma^2 - M - 2}{x_\gamma^2 - 4}} & C_{1,1} \\ \frac{1}{x_\gamma^2 - 4}\,(\text{mess}) & C_{0,4} \end{cases} \qquad z_\gamma = \dots$$

**Harold:** So these define $\lambda, \tau$? **Jorg:** We are after the $\lambda, \tau$. We can take the above formulas as the definitions, and then you can check that all the formulas before follow from these definitions. **Harold:** So these restrict to the Fenchel–Nielsen coordinates on Teichmüller space? **Jorg:** These are precisely those coordinates.

We should emphasize here one of the nice things about using pants to get the Darboux coordinates. We see that $x_\gamma$s Poisson commute when the $\gamma$s don't intersect, so we have the natural locality. If you use triangulations, the locality is much less obvious.

By the way, you can eliminate the square roots, but you have to modify the coordinates a bit to something less natural. By the way, these are related to Fenchel–Nielsen coordinates, where to the curve $\gamma$ I measure its geodesic length $\ell_\gamma = 2x_\gamma$, and then I can also measure the "twist angle" for the other coordinate. These are naturally geometrically defined, and they lead to the above expressions.

**Kolya:** For $\mathrm{SL}(3,\mathbb{C})$, there is no hope for such coordinates? **Jorg:** For $\mathrm{SL}(3)$, we need to add something extra in the pairs of pants. If one knew how to do this, one would make a large step forward in proving some major conjectures in gauge theory and CFT, and in the AGT conjectures.

### 1.5 Generating function for change of coordaintes

We have now described various different coordinates, one from the Hom point of view, and one that depended on the complex structure. A key step now is to understand the relationship between these.

We will describe it for $d = 0$, and go between $(q, H)$ and $(\lambda, \tau)$.

Recall: it is a basic fact of life that when you go between two different systems of Darboux coordinates, there is a *generating function* that relates them. A *canonical change of coordinates* is simply a change between two sets of Darboux coordinates. The standard fact is that a change of variables $(g, H) \leftrightarrow (\lambda, \tau)$ is canonical iff there exists a function $S(q, \lambda)$ such that

$$H_r = -\frac{\partial S}{\partial q_r}, \qquad \tau_r = \frac{\partial S}{\partial \lambda_r}.$$

This will end up being a major player in some applications. For example, it is the Yang potential, and we will identify it also with the classical conformal blocks of Liouville theory.

The proof of the above standard statement, is that locally $\Omega = \mathrm{d}(\sum_r \tau_r \mathrm{d}\lambda_r)$, and if there exists such $S$, then $\Omega = \mathrm{d}(\sum_r \frac{\partial S}{\partial \lambda} \mathrm{d}\lambda_r) = \mathrm{d}^2 S - \mathrm{d} \sum \frac{\partial S}{\partial q_r} \mathrm{d}q_r = \mathrm{d}H \wedge \mathrm{d}q$. In the other direction, $0 = \mathrm{d}(\sum \tau \mathrm{d}\lambda - \sum H \mathrm{d}q)$, so locally there exists $S$ such that $\sum \tau \mathrm{d}\lambda - \sum H \mathrm{d}q = \mathrm{d}S$.

So in some sense this is totally trivial, but in fact $S$ is very nontrivial, because for our particular coordinates $S$ encodes the monodromy map. In this sense, $S$ is very transcendental. We have two completely different worlds: a topological one, with variables $\lambda, \tau$, and a holomorphic one with variables $q, H$. Some variables make some structures completely obvious and other structures completely obscured. For example, in the $\lambda, \tau$ variables it's clear that $\mathcal{M}$ is an algebraic variety, whereas this is not clear in the holomorphic side. On the other hand (restricting to the real slice), the $q, H$ are complex analytic functions on the Teichmüller space, and so they describe Teichmüller space as a complex analytic manifold. Our space has different faces, and these look simple from different angles.

So it is particularly interesting to quantize this $S$.

**Kolya:** But these are just local functions. So what will the global quantization mean? **Theo:** $\lambda, \tau$ are almost global: their coshs are globally defined functions. **Jorg:** There is a global story to tell, but I will so far remain local.

### 1.6 Remarks on quantization of $\mathcal{M}$

As a preliminary, I want to make a few general remarks about quantization. First: the problem of quantization of $\mathcal{M}$ can be formulated/addressed on two levels.

1. **Algebraic**

Construct a noncommutative algebra $\mathcal{A}_\hbar$ together with a map $\mathrm{Fun}(M) \to \mathcal{A}_\hbar$, where $\mathrm{Fun}(M)$ is the space of algebraic functions on $M$ — say in the $x_\gamma, y_\gamma, z_\gamma$ variables — mapping $f \mapsto \mathcal{O}_f$, such that $[\mathcal{O}_f, \mathcal{O}_g] = \hbar \mathcal{O}_{\{f,g\}} + O(\hbar^2)$.

We'd also like to ask that the symmetries of our classical space act on the associative structure. Which symmetries? There is the mapping class group, which is the diffeomorphisms that are not connected to the identity, and this moves closed curves to other closed curves, so it acts on $\mathcal{M}$. I will write $\Gamma(C)$ for the mapping class group. So to $\Gamma(C)$ we'd like to associated automorphisms of $\mathcal{A}_\hbar$.

2. **Analytic**

   This is closer to what a physicist understands as quantization. It amounts to constructing a representation of $\mathcal{A}_\hbar$ as operators on a topological vector space $\mathcal{S}_C \subseteq \mathcal{H}$ a Hilbert space. People usually say to act on the Hilbert space, but in general one gets $\mathcal{O}_x$ acting as unbounded operators on the Hilbert space, and only really acting on the subspace of Schwarz functions. Notation: the operator corresponding to $\mathcal{O}_f$ I write as $\pi(\mathcal{O}_f)$.

   Furthermore, we'd like there to exist a representation of $\Gamma(C)$ as unitary operators on $\mathcal{H}_C$. Notation: to $\mu \in \Gamma(C)$ I associate $U_\mu$ a bounded operator.

**Theo:** Will I expect the mapping class group to preserve $\mathcal{S}_C$, or will it move it? **Jorg:** It act on $\mathcal{S}_C \subseteq \mathcal{H}_C$. **Theo:** Often when I've seen such problems stated, in order to lift classical symmetries to quantum symmetries, I can do this at the cost of reparameterizing $\hbar$, say $\hbar' = \hbar + O(\hbar^2)$. Will I need this here? **Jorg:** No, I don't need to reparameterize $\hbar$.

I should warn that for the second step we will need to restrict to real slices within $\mathcal{M}$, defined by $x, y, z \in \mathbb{R}$. As far the algebraic level is concerned, we do not need to make this restriction, and we will actually do everything algebraically with $q = e^{i\hbar}$ and so on.

# Wednesday, October 19

We have been discussing the moduli space of flat connections, and various systems of Darboux coordinates. One system of Darboux coordinates came from a pair-of-pants decomposition of the surface. The other system depends on representing the moduli space in terms of opers, and we only discussed two strata: when the necessary number of apparent singularities is either minimal or maximal.

At the end we introduced in the minimal case the *Yang potential*, which is the generating function of the change of coordinates between these two systems.

**Kolya:** Before you continue, what happens in the intermediate strat? **Jorg:** This is very work in progress, and includes a number of research projects. There does not exist in the literature a good description of coordinates of the intermediate strata. The ultimate (quantum) Liouville theory suggests and answer, but there's no literature.

### 1.7 The fun part: quantization of $\omega, \kappa$ coordiantes

Let's consider $g = 0$. Then we're in the maximal strata, so $d = n - 3$, and we're working on the Riemann sphere with $n$ punctures. We will define a map $w_r \mapsto \hat{w}_r$ and $\kappa_r \mapsto \hat{\kappa}_r$, and we'd like the commutation relations for the algebra to be $[\hat{\kappa}_r, \hat{\omega}_s] = b^2 \delta_{r,s}$ and other zeroa. This is just *canonical quantization* of these coordinates. Physicists may miss the $i$s, but we are dealing with complex variables, so you can absorb them.

These new $\hat{w}$s and $\hat{\kappa}$s are defined in terms of their actions somewhere. Namely, we work with *wave functions* $\phi(w)$, and the actions will be

$$\hat{w}_r \, \phi(w) = w_r \, \phi(w) \qquad \hat{\kappa}_r \, \phi(w) = b^2 \frac{\partial}{\partial w_r} \phi(w)$$

So far we will leave open what the regularity should be of the allowed wave functions.

**Theo:** Am I correct that so far, this is just quantizing a space of functions on a patch of the moduli space, not on the whole thing? **Jorg:** Yes. Eventually (in an hour) we will discuss global things.

Now we'd like to quantize the constraints $\kappa_r^2 + \eta_r = 0$. Recall those were:

$$\eta_k = \sum_s \left( \frac{\delta_s}{(w_k - z_s)^2} + \frac{1}{w_k - z_s} H_s \right) + \sum_{k' \neq k} \left( \frac{-3}{4(w_k - w_{k'})^2} + \frac{\kappa_{k'}}{w_k - w_{k'}} \right)$$

The quantization gives:

$$b^4 \frac{\partial^2}{\partial w_k^2} + \sum_{r=1}^n \left( \frac{\delta_r}{(w_k - z_r)^2} + \frac{1}{w_k - z_r} \hat{H}_r \right) + \sum_{k' \neq k} \left( -\frac{3 + b^2 \nu}{4(w_k - w_{k'})^2} + \frac{1}{w_k - w_{k'}} b^2 \frac{\partial}{\partial w_{k'}} \right) \qquad (\star)$$

Here $\nu$ is an allowed correction that goes away in the classical limit. See, we have replaced the phase-space variables by elements of the noncommutative algebra.

The logic is: $w$ and $\kappa$ are actually global coordinates on the $d = n - 3$ stratum of the moduli space. The equations above can be taken as a definition of $\eta$ and $H$ functions. We would like to find a quantum counterpart of the $H$ function, and the expression $H = H(\kappa, w)$ is horrible, and so instead we quantize the defining relation. The rôle of the $\nu$ is that, of course, there are ordering issues — $H(\kappa, w)$ does not determine $\hat{H}$ precisely.

**Kolya:** And we want the $\hat{H}$s to commute. **Jorg:** Yes, of course.

We should mention that we also demanded regularity of $t(y)$ at $\infty$, and this gives some more equations, enough to define $\hat{H}_r = \hat{H}_r(w, \kappa)$.


theojf@math.berkeley.edu

Recall now that the $z_r$ variables play the rôle of *times* in the classical dynamics of the isomonodromic deformations. In the Schrödinger representation of dynamics in quantum mechanics, the way functions are allowed to depend on "time": $\Psi = \Psi(w,z)$. (We will sometimes identify $q$s and $z$s, because we are working on the $n$-times punctured Riemann sphere, and so three of the $z$s can be set to $0, 1, \infty$, and the $q$s are the other cross-ratios.) Then the *Schrödinger equation* is

$$b^2 \frac{\partial}{\partial z_r} \Psi(w,z) = \hat{H}_r \Psi(w,z).$$

Let's set $\Delta_r = b^{-2}\delta_r$. Then $(\star)$ implies:

$$\left( \sum_{r=1}^{n} \left( \frac{\Delta_r}{(w_k - z_r)^2} + \frac{1}{w_k - z_r} \frac{\partial}{\partial z_r} \right) + b^2 \frac{\partial^2}{\partial w^2} - \sum_{k'=k} \left( \frac{2 + 3b^{-2}}{4(w_k - w_{k'})^2} - \frac{1}{w_k - w_{k'}} \frac{\partial}{\partial w_{k'}} \right) \right) \Psi(w,z) = 0$$

For reasons that I shall not now explain, we have set $\nu = 2$. The reason is that these are well-known equations, the *BPZ equations*. One reason these are good is that the equations are analytic (with respect to both $w, z$) away from the poles. You can count parameters, and these are not *holonomic*, because we have more parameters than equations. But you can nevertheless classify solutions to equations by their asymptotics when $z$s collide — which is to say, at the boundary of Teichmüller space (in genus zero). I.e. by specifying the behavior of the equations and the boundary, it determines the solutions uniquely. We won't develop this here, because there are other approaches that are more convenient, but you could take this as a starting point. Kolya should appreciate this story: he did the same thing with KZ equations as the quantization of Schlesinger. And what we have written down is some second-order version of KZ.

**Kolya:** There must be some secret behavior when the $z$s collide. **Jorg:** Yes. I can show you calculations in private — they are obscured here. The statement is that near $z_1 \to z_2$, if I prescribe boundary conditions of the power-law form $(z_1 - z_2)^\delta$, then there exists a unique solution. So the whole space of solutions depends on these continuous parameters $\delta$.

## 1.8 Quantization of FN coordinates

We also had coordinates $\lambda_\gamma, \tau_\gamma$. We now look for an algebra with $[\hat{\lambda}_\gamma, \hat{\tau}_{\gamma'}] = 2\pi i b^2 \delta_{\gamma,\gamma'}$. Then the work is to quantize the functions $x_\gamma, y_\gamma, z_\gamma$ to operators $\hat{x}_\gamma, \hat{y}_\gamma, \hat{z}_\gamma$. This is a very nontrivial story, but we will give the flavor. Here are the answers:

$$\begin{aligned} \hat{x}_\gamma &= 2\cosh \hat{\lambda}_\gamma \\ \hat{y}_\gamma &= \frac{1}{\sqrt{\sinh \hat{\lambda}}} e^{\hat{\tau}/4} \big(\cosh(\hat{\lambda} + m/2) \cosh(\hat{\lambda} - m/2)\big)^{1/2} e^{\hat{\tau}/4} \frac{1}{\sqrt{\sinh \hat{\lambda}}} + \\ &\quad + \frac{1}{\sqrt{\sinh \hat{\lambda}}} e^{-\hat{\tau}/4} \big(\cosh(\hat{\lambda} + m/2) \cosh(\hat{\lambda} - m/2)\big)^{1/2} e^{-\hat{\tau}/4} \frac{1}{\sqrt{\sinh \hat{\lambda}}} \end{aligned}$$

In gauge-theory jargon, $\hat{x}$ is the *Wilson loop*, and $\hat{y}$ is the *'t Hooft loop*.

**Kolya:** There is an approach to quantization of moduli spaces that uses quantum groups. Is this the same? **Jorg:** In principal, it is equivalent. Indeed, I am currently working this out with a student: you can nicely reconstruct the quantum group from the framework. There is yet another approach, called *quantum Teichmüller theory*, worked out by Kashaev and collaborators, and it is also equivalent, but no one has written that down. I plan to start writing up the technicalities at the beginning of next year. **Kolya:** Since you mentioned these *'t Hooft operators*, is it difficult to explain what these are? **Jorg:** We will postpone it. **Kolya:** To wine and cheese.

In any case, the above equations are for the $C_{1,1}$ case. There are also expressions for $\hat{z}$, and also for the $C_{0,4}$ case. Classically, you can just see that these reduce to the correct expressions in the classical limit. What I have not explained is why the above ordering is the correct one. But I'll now explain what's behind the scene: I claim that the above are the unique expression satisfying all natural requirements.

To support that this is the correct choice, we have defined explicitly an algebra $\mathcal{A}_\hbar$, with $\hbar = b^2$. Forgive me for omitting the $\gamma$s and restricting to the once-punctured torus. Then $\mathcal{A}_\hbar$ is defined by the relations:

$$q^{\frac{1}{2}}\hat{x}\hat{y} - q^{-1/2}\hat{y}\hat{x} = -(q - q^{-1})\hat{z}$$
$$q\hat{x}^2 + q^{-1}\hat{y}^2 + q\hat{z}^2 + q^{1/2}\hat{x}\hat{y}\hat{z} = qe^m + q^{-1}e^{-m} + q + q^{-1}$$

Here $\operatorname{tr} M = -2\cosh m$, so that $m$ parameterizes the holonomy around the puncture — it is just the length of the loop from the Teichmüller space point of view. The parameter $q = e^{\pi i b^2}$.

There are similar formula for $C_{0,4}$.

A philosophical remark: you should view this noncommutative algebra as encoding the *quantized algebraic geometry* of $\operatorname{Hom}(\pi_1(C) \to \mathrm{SL}(2,\mathbb{C}))$.

So much for the "algebraic level" of quantization in these coordinates. (Well, we should also discuss the action of the Mapping Class Group, but we skip it.) We restrict to the *real slice* $\mathcal{M}_\mathbb{R}$, which classically is given by the conditions $\lambda_\gamma = \lambda_\gamma^*$ and $\tau_\gamma = \tau_\gamma^*$, and we replace these by Hermiticity conditions $\hat{\lambda}_\gamma = \hat{\lambda}_\gamma^\dagger$ and $\hat{\tau}_\gamma = \hat{\tau}_\gamma^\dagger$.

We can realize these as operators. We must choose a polarization, and use the Schwarz space $\mathcal{S} \in \mathrm{L}^2(\mathbb{R}^{3g-3+n})$, with the actions by:

$$\hat{\lambda}_\gamma \Psi(\gamma) = \lambda_\gamma(\psi(\lambda)) \qquad \hat{\tau}_\gamma \Psi(\lambda) = -2\pi i b^2 \frac{\partial}{\partial \lambda_\gamma} \Psi(\lambda)$$

**Theo:** In the classical moduli space, $x, y, z$ are globally defined functions, but $\lambda, \tau$ are not global functions, but $\tau = \tau + 2\pi i$. How does this play with the quantization? **Jorg:** This is part of taking the real slice. More interesting is to take $\lambda \to \lambda + \pi i$, which introduces minus signs, and I think these signs play a crucial rôle in the description of the other components, but that's a wide open research project.

The main issue now is the dependence on the pants decomposition. Let $\sigma$ denote some choice of pants decomposition. When the dependence on the pants decomposition is important, we will put a subscript: $\Psi \rightsquigarrow \Psi_\sigma$, $\hat{x}_\gamma \rightsquigarrow \hat{x}_{\gamma,\sigma}$. Of course, different pants give different coordinates, but these are all just functions on the same space, so there must be changes-of-coordinates. On the quantum side, we'd really like to associate a quantum theory to the Riemann surface, not to the Riemann surface along with a pants decomposition. So we need to show that the dependence on the pants decomposition is "irrelevant" in the appropriate sense.

Namely, this dependence can be shown to be irrelevant if there exist *unitary* operators $U_{\sigma_2\sigma_1} : \mathcal{H}_{\sigma_1} \to \mathcal{H}_{\sigma_2}$ such that

$$\hat{L}_{\gamma(\sigma_2)} U_{\sigma_2\sigma_1} = U_{\sigma_2\sigma_1} \hat{L}_{\gamma(\sigma_1)} \tag{I}$$

Here the letter $L$ stands for any of $x, y, z$. I call it "$L$" for "loop operator" or "length operator".

**Harold:** So we should have a loop operator for any loop, not just the basic ones from the pants decomposition. **Theo:** So when the loop is one of the basic ones, $L$ is $x, y, z$. **Jorg:** I will check this for a generating set, and this suffices. **Theo:** No, I don't believe you. In the classical limit, I know how to write a complicated loop out of the basic ones. But when I quantize, even if the complicated loop *is* generated by the basic ones, I don't know how to write it as such, because of ordering issues. **Jorg:** Ok, so there is something to check. I believe it to be true, but it needs proving.

It turns out that we may construct $U_{\sigma_2\sigma_1}$ as an integral operator:

$$\Psi_{\sigma_2}(\lambda_2) = \big(U_{\sigma_2\sigma_1}\Psi_{\sigma_1}\big)(\lambda_2) = \int \mathrm{d}\lambda_1\, K_{\sigma_2\sigma_1}(\lambda_2,\lambda_1)\, \Psi_{\sigma_1}(\lambda_1)$$

Then the condition (I) implies a difference equation for the integral kernels $K_{\sigma_2\sigma_1}$

**Example:** We consider the once-punctured torus. Denote the A-cycle by $\gamma$ and the B-cycle by $\check{\gamma}$. Define $\sigma_1$ to be "cut along $\gamma$" and $\sigma_2$ to be "cut along $\check{\gamma}$".

Then we have $\hat{L}_{\gamma,\sigma_1} = \hat{x}_{\gamma,\sigma_1}$, the representative of the curve in the representation we wrote down above. On the other hand, $\hat{L}_{\gamma,\sigma_2} = \hat{y}_{\check{\gamma},\sigma_2}$, because $\gamma \leftrightarrow \check{\gamma}$ by S-duality.

Then equation (I) gives:

$$2\cosh\lambda_1\, K_{\sigma_2\sigma_1}(\lambda_2,\lambda_1) = \mathcal{D}_{\lambda_2} K_{\sigma_2\sigma_1}(\lambda_2,\lambda_1)$$

where $\mathcal{D}$ is the difference operator obtained from $\hat{y}_\gamma$ by replacing $\hat{\lambda}$ with $\lambda_2$ and $e^{\pm\hat{\tau}/4}$ by $e^{\mp\pi i\frac{b^2}{2}\frac{\partial}{\partial\lambda_2}}$, i.e. $e^{\mp\pi i\frac{b^2}{2}\frac{\partial}{\partial\lambda_2}} f(\lambda_2) = f(\lambda_2 - \pi i\frac{b^2}{2})$.

So, you can exhibit the solution, with some work, in terms of "noncompact quantum dilogarithm". The point is: the solution exists, and it is nontrivial to show, but it is possible, that the solution is unique. $\Diamond$

Now we remark: any two pants decompositions can be related by a sequence of two elementary moves. One is the *S-transformation* which replaces $C_{1,1}$ with cutting $\gamma$ by the same once-punctured torus with the cutting $\check{\gamma}$. The other is the *F-transformation* or *A-transformation* which is $\gamma \to \check{\gamma}$ on the $C_{0,4}$ case. In fact, we have explicit formulas for these two cases, and so completely determined the quantization. The point: the general $U_{\sigma_2\sigma_1}$ can be decomposed into F and S. The corresponding kernels $S(\lambda_2, \lambda_1)$ and $F(\lambda_2, \lambda_1)$ have been found.

**Harold:** The S move, is it really well-defined? Can I distinguish between it and the move that replaces $\gamma$ by the curve that in homology is $\check{\gamma} + \gamma$? **Jorg:** Thanks. For saving time, I was oversimplfying. Pants decomposition is not enough to determine the coordinates. You have to add extra data, namely a 3-valent graph that in each pair of pants looks like the standard form. Using this graph you can resolve the possibility to add Dehn twists. So when I say $\sigma$, I really mean the pair $(\mathcal{C}, \Gamma)$ where $\mathcal{C}$ is the cut system and $\Gamma$ is the graph. In fact, I should have added one more move, which is the braiding, but it just introduces a phase.

**Theo:** Returning to the question of the long curves — I can take these as the definition of the action by the long curve, by chosing a pants that includes the long curves, and then moving to some other decomposition. But to check this, I need to prove the well-definedness. **Jorg:** Right. There is a groupoid, whose vertices are decompositions, and the basic edges are these basic moves. Then there are 2-cells, and it is a theorem that these are generated by some finite list of basic 2-cells: a pentagon, a hexagon, something for the twice-punctured torus (which assures that the modular group acts). I can assert to you that the equations corresponding to these 2-cells hold for these formulas. But it would be better to have a more direct proof: what I do uses some connections with quantum groups.

### 1.9 Quantization of $(q, H)$ coordinates ($d = 0$, $\mathcal{M}_{\mathbb{R}}$)

First, we should remark that this is not quite "canonical", or at least less conventional than the others. There are two reasons.

- On the real slice, we don't need restrictions on the $q$s: $q_r$ are complex analytic coordinates on $\mathcal{M}_{\mathbb{R}}$.
- So we have $H_r(q, \bar{q})$ on $\mathcal{M}_{\mathbb{R}}$

This should be familiar from quantum mechanics. Compare to the state representation: we have $\mathbb{R}^2 = \mathbb{C}$, and we set $a = \frac{1}{\sqrt{2}}(q + ip)$ and $a^\dagger = \frac{1}{\sqrt{2}}(q - ip)$. So by fixing one, we have in some sense fixed our location in $\mathbb{C}$: on the real slice, we can take $a$ as the independent variable, and $a^\dagger$ is then determined by complex conjugation, which is of course not a complex-analytic operation.

In terms of opers, we have $\partial^2 - \sum \left( \frac{\delta_r}{(y - z_r)^2} + \frac{H_r}{y - z_r} \right)$. The restriction to the real slice is the request that the monodromy of the oper be conjugate to $\mathrm{PSL}(2, \mathbb{R})$. Then not every choice of $H$ works, rather only one choice $H = H(q, \bar{q})$.

So we want now to quantize in this holomorphic context. In physics, we know how to quantize in terms of $a, a^\dagger$. It has a generalization, which in mathematics is called the *Kähler quantization*. The first part, which is purely algebraic, is essentially trivial: we replace Poisson brackets $\{H_r, q_s\} = \delta_{r,s}$ by commutation relations $[\hat{H}_r, \hat{q}_s] = b^2\delta_{r,s}$, and this defines $\mathcal{A}_\hbar$. But in this context, it is most natural to consired representations of this algebra in terms of holomorphic wave functions. So we consider wave functions $\phi(q)$ and realize the variables by $\hat{q}_r\phi(q) = q_r\phi(q)$ and $\hat{H}_r\phi(q) = b^2\frac{\partial}{\partial q_r}\phi(q)$.

Then what is important is to realize the symmetries, which is to say the action of the mapping class group $\Gamma(C)$. It is a theorem that the mapping class group action on $\mathcal{T}_{g,n}$ is complex analytic. In particular, if I have a point $q \in \mathcal{T}_{g,n}$, it maps under $\mu \in \Gamma(C)$ to $\mu(q)$ and this mapping is complex analytic.

So in particular, if we have wave functions on the Teichmüller space, then there is only one reasonable way to define the action of $\Gamma(C)$ on these wave functions. It must be:

$$(U_\mu\phi)(q) = \phi(\mu(q))$$

In other words, I can appeal to the fact that a holomorphic function is determined by its values in some neighborhood, and since Teichmüller space is simply connected I can always analytically continue. In any case, this is the only reasonable way to define the action.

# Thursday, October 20

We are in the progress of describing the quantization of the moduli space of flat connections. Recall that we have two different sets of Darboux coordinates $(q, H)$ and $(\lambda, \tau)$. We explained how to quantize both, at least to some level. From the $(q, H)$ side, we chose wave functions $\phi(q)$, and we should keep in mind that the natural symmetries of the system are the mapping class group acting on the moduli space, and also on Teichmüller space, and we posit that the symmetries act on wave functions by $(U_\mu\phi)(q) = \phi(\mu(q))$.

On the $(\lambda, \tau)$ side, we let $\sigma_2, \sigma_1$ denotes pants decompositions (decorated with trivalent graphs), and we described an integral operator to get between different quantizations: $\psi_{\sigma_2}(\lambda_2) = \int \mathrm{d}\lambda_2\, K_{\sigma_2\sigma_1}(\lambda_2, \lambda_1)\psi_{\sigma_1}(\lambda_1)$. But this gives us for free an action by the mapping class group. If we have $\mu \in \Gamma(C)$, then it maps $\sigma$ to some $\mu(\sigma)$, and we posit that

$$(U_\mu\psi_\sigma)(\lambda_2) = \int \mathrm{d}\lambda_1\, K_{\mu(\sigma),\sigma}(\lambda_2, \lambda_1)\, \psi_\sigma(\lambda_1)$$

We can compute this very explicitly, because we can break up any element of the mapping class group into basic transformations of the pair of pants decomposition.

**Mina:** The $\lambda, \tau$, they are somewhat complicated. They are defined out of holonomies $x, y, z$, and $\lambda$ is fairly simple. Is there any reasonably geometric description of $\tau$? **Jorg:** No, not really. You work out what it is to be conjugate to $\lambda$, and it's complicated.


theojf@math.berkeley.edu

### 1.10 Relation between the quantum theories between $\phi(q)$ and $\psi(\lambda)$?

In quantum mechanics, if we have two different Hilbert space representations of the same abstract algebra, then there should be a unitary transformation between the two Hilbert spaces. We can moreover hope that it is implemented by a unitary operator:

$$\phi(q) = \int \mathrm{d}\lambda\, \Theta_\sigma(q,\lambda)\, \psi_\sigma(\lambda) \tag{$*$}$$

What physicists expect is that in the classical limit $b \to 0$,

$$\Theta_\sigma(q,\lambda) \sim e^{-\frac{1}{b^2}S(q,\lambda)}(\cdots).$$

I will leave the following as an amusing exercise for those who don't remember their quantum mechanics formulas: since this is a transformation between two canonical representations in Darboux coordinates, then $S$ satisfies $\frac{\partial S}{\partial q_r} = -H_r$ and $\frac{\partial S}{\partial \lambda_r} = \tau_r$, i.e. it is a *generating function of a canonical transformation.*

In other words, $\Theta_\sigma$ is a *quantization of the generating function for the brane of opers.* It is the main player in these lectures. It will ultimately become the conformal block of Liouville theory.

Then $\Theta_\sigma(q,\lambda)$ must satisfy:

1. It intertwines the two $\Gamma(C)$-actions, i.e.:

$$\Theta_\sigma(\mu(q),\lambda_2) = \int \mathrm{d}\lambda_1\, \Theta_\sigma(q,\lambda_1)\, K_{\mu(\sigma),\sigma}(\lambda_1,\lambda_2)$$

2. We know the asymptotic behavior:

$$\Theta_\sigma(q,\lambda) \sim q_r^{(\frac{\lambda_r}{4\pi})^2+\chi_c}\big(\mathrm{const} + O(q)\big), \qquad \text{when } q_r \to 0$$

Here we have chosen variables $q$ on Teichmüller space so that $q \to 0$ is the boundary.

The second condition follows form the quantization of the following classical formula:

$$q_r H_r \sim \left(\frac{\lambda_r}{4\pi}\right)^2 - \frac{1}{2}, \qquad \text{when } q_r \to 0$$

At this point I should define more precisely what the coordinates $q$ are. I have a gluing construction of Riemann surfaces. Pick a puncture $p$, and some annulus around it, on one of your surface. On the other, pick another annulus around a puncture. Then pick coordinates $y_1, y_2$ on each annulus, and to "glue the punctures", I really identify the two annuli, via $y_1 y_2 = q$. Then $q \to 0$ is the pinching of the tube that we create when gluing.

The formula is based on results from Wolpert and Zograf. What happens in the quantization? The left-hand side becomes $q_r \frac{\partial}{\partial q_r}$ up to a constant, and the eigenvectors are determined, and quantization of this formula gives the second condition above.

So, this is a Riemann–Hilbert type problem. The mapping class group — it rearranges the monodromies. So we can think of $\Theta$ as simply a monodromy representation, and the only new feature is that it's an infinite-dimensional monodromy representation, which is why we integrate over the variable $\lambda$ and not sum.

So the problem is to find a multi-valued analytic function on the moduli space whose linear monodromy representation is controlled by $K$. And then as always, this doesn't quite determine the solution, but usually you expect that the asymptotics precisely pin down the solution. And I claim that this is the same.

**Ed:** What are the $H$s? What is the moduli space? **Jorg:** I fix a Riemann surface, except I allow to vary the complex structure, and I fix the monodromy around each puncture. Then $\mathcal{M}_0$ is the space of flat connections that can be written as an oper with no apparent singularities. Then for a fixed complex structure, the space of opers then is a Lagrangian within the space of connections. But I now allow to vary the complex structure, and this moves this *brane of opers* off itself. **Ed:** So the statement is that the complex variables and the oper variables are Darboux. **Jorg:** Yes.

Then the claim is that the conditions 1,2 above define a Riemann–Hilbert problem, and there exists a unique solution. Half of this is clear: the asymptotics are certainly enough to cut down the solution space to at most one. The existence is much less trivial, because there aren't a lot of ways to construct infinite-dimensional representations.

See, $\Theta$ is a function of $q$, and $\lambda$ is a label for a basis of the space of functions. Then the monodromy, which is an element of the mapping class group, rearranges the elements of the basis by an infinite matrix $K$. It is the infinite-dimensional version of the formula $\psi_i(\mu(q)) = \sum_j M_{ij}\psi_j(q)$.

Ok, so I won't go into the construction, but say some words. There is a construction of the solution using vertex operators. So give an idea, $\Theta = \langle p_4 | h_{s_3}^{\alpha_3}(z_3) h_{s_2}^{\alpha_2}(z_2) | p_1 \rangle$, where $h_s^\alpha = e^{2\alpha\phi} Q^s$, and the point is that we can vary the $\alpha_i$ and the main difficulty to overcome is that we need to allow the power of the screening charge $Q$ to be continuous, not integer. **Ed:** This is the special case for four points on $\mathbb{P}^1$. **Jorg:** Yes, but everything boils down to this.

**Mina:** Can you say a little more about the brane of opers? What is its classical definition? **Jorg:** The definition that I need is just $\frac{\partial S}{\partial q} = H$ and $\frac{\partial S}{\partial \lambda} = \tau$, and I have already defined $(q, H)$ and $(\lambda, \tau)$. If you consider $q$ fixed, and just look at the submanifold $q =$constant, then you get a Lagrangian submanifold in the moduli space of flat connections. This is what's called the "brane of opers" by Nekrasov and Witten and so on.

## 2 Crash course in Liouville Theory

Liouvill theory is defined by writing down an action

$$S_L = \int_C \frac{\mathrm{d}^2 z}{4\pi} \left( (\partial\phi)^2 + 4\pi\mu e^{2b\phi} \right)$$

where $\phi : C \to \mathbb{R}$ is a field. As a classical theory, the extrema of this action are what we care about. They are precisely the solutions to

$$\partial_z \partial_{\bar{z}} + \pi \mu b e^{2b\phi} = 0.$$

These define metrics $\mathrm{d}s^2 = e^{2b\phi}\mathrm{d}z\mathrm{d}\bar{z}$ of constant negative curvature on $C$. So classical Liouville Theory is just the theory of regularization of Riemann surfaces.

In quantum theory, formally we are interested in path integrals and expectation values, defined by:

$$\left\langle \prod_{r=1}^{n} e^{2\alpha_r \phi(z_r, \bar{z}_r)} \right\rangle = \int_{\phi : C \to \mathbb{R}} [\mathcal{D}\phi]\, e^{-S_L[\phi]} \prod_{r=1}^{n} e^{2\alpha_r \phi(z_r, \bar{z}_r)}$$

Many physicists would stop here: we have "defined" the theory.

What I'd like to outline is a procedure for rigorously constructing the theory. We begin with the physically motivated assumption of *Holomorphic factorization*. It states that expectation values have the following shape:

$$\left\langle \prod_{r=1}^{n} e^{2\alpha_r \phi(z_r, \bar{z}_r)} \right\rangle = \int_{\mathbb{R}_+^{3g-3+n}} \mathrm{d}\mu(P) \left| \mathcal{F}(P, b, \alpha, q) \right|^2$$

Here $q = (q_1, \dots, q_{3g-3+n})$ are defined by the gluing construction, and to each $q$ we define a $P : \gamma \to P_\gamma$.

The point is: if you cut out a neighborhood of $C$ along a $\gamma$, by conformal invariance what you see is a cylinder. And again by conformal invariance, the theory on a cylinder is controlled by a Hilber space of the form $\mathcal{H} = \int \mathrm{d}P\, \mathcal{V}_p \otimes \mathcal{V}_p$, where the first is a representation of Vir and the second of $\overline{\mathrm{Vir}}$. This comes from a general theory, but for now we take it as an ansatz. We should mention that the term $\left| \mathcal{F}(P, b, \alpha, q) \right|^2$ is a *conformal block*.

**Harold:** What is the meaning of this integral? **Jorg:** It is a direct integral of Hilbert spaces. **Harold:** And the $P$s are? **Jorg:** It suffices to let them range over $\mathbb{R}_+$. We take $\mathcal{V}_p$ the Vir representation defined by $L_n|p\rangle = 0$ for $n > 0$ and $L_0|p\rangle = (p^2 + \frac{Q^2}{4})|p\rangle$, where $c = 1 + 6Q^2$ and $Q = b + b^{-1}$.

This function $\mathcal{F}$, called the *conformal block*, is defined completely (almost) by representation theory of Vir.

Given a representation $\mathcal{V}_r$ for $r = 1, \dots, n$ of Vir, and the curve $C$ with $n$ punctures, then the *conformal block* is

$$F_C : \bigotimes_r \mathcal{V}_r \to \mathbb{C}$$

which satisfies some invariance under a subalgebra of Vir defined by the curve. Namely, we demand:

$$F(T[\chi] \cdot v) = 0 \quad \forall \chi \in \mathrm{Vir}_{\mathrm{out}} \tag{CWI}$$

where $\mathrm{Vir}_{\mathrm{out}}$ is the algebra of meromorphic vector fields on $C$ with poles at $z_1, \dots, z_n$. (The word "CWI" is for *conformal ward identity.*) We define

$$T[\chi] = \sum_r 1 \otimes \cdots \otimes T[\chi_r] \otimes \cdots \otimes 1$$

where the $T$ is in the $r$th spot, and

$$T[\chi_r] = \sum_k L_k \chi_{k,r}, \quad \text{if } \chi(t_r) = \sum_k \chi_{k,r} t_r^{k+1} \partial_{t_r}$$

and $t_r$ is a coordinate near the $r$th puncture.

So this is a punch of linear conditions on the vectors in a tensor product of representations, or in other words it is an element of the dual to the tensor product, and there is some linear space of solutions, and that's what a mathematician calls a *conformal block*.

The claim is that in our case, there is a Hilbert space of solutions to (CWI), which has a basis $F_{C,P,\sigma}$ for pants decomposition $\sigma$ of $C$, and $P$ as above. What is the relation between $F$ and $\mathcal{F}$? It is:

$$\mathcal{F}(P, b, \alpha, q) = F_{C(q),P}(\mathcal{V}_{\alpha_1} \otimes \cdots \otimes V_{\alpha_n})$$

So we are distinguishing between the physicists' and mathematicians' conformal blocks by using different fonts for F.

More precisely, the claim is that I can introduce a family of physicists' conformal blocks indexed by $P = (P_1, \dots, P_{3g-3+n})$, where each $P$ is assigned to some curve in the pants decomposition. The letter $b$ is the parameter that runs throughout the story. The $\alpha$s are the tuple $\alpha = (\alpha_1, \dots, \alpha_n)$, that we had when asking for an expectation value. And $q$ is the gluing coordinates of the curve. **Kolya:** It is a square because the infinitesimal representation is actually a representation of Vir times its dual. The $P$s are highest weight vectors.

**Jorg:** Perhaps it is more clear if I emphasize: the description for mathematicians' conformal block doesn't depend on the pants decomposition. But we would like to get our hands on an explicit basis.

So to understand this definition, and why it can be turned into something manageable, we may consider an example. Let $C = \mathbb{P}^1 \smallsetminus \{z_1, z_2, z_3\}$, and choose coordinates $t_i$ around each point. I invite you to do the following exercise: show that with the help of the above equations, then if you are looking at the value $F(L_{-n} v_3 \otimes v_2 \otimes v_1)$, then you can use the above rules to rewrite it as $\sum_k \chi_{k,2} F(v_3 \otimes L_k v_2 \otimes v_1) + \sum \chi_{k,1} F(v_3 \otimes v_2 \otimes L_k v_1)$. You think of the representation as looking like a tower, where $L_{-n}$ goes up and $L_k$ goes down. Then the above equations are rules for how to shuffle around the Virasoro operators.

What it amounts to is: you can calculate $F(\mathcal{L}_{-I_3} v_{\alpha_3} \otimes \mathcal{L}_{-I_2} v_{\alpha_2} \otimes \mathcal{L}_{-I_1} v_{\alpha_1})$, where $L_{-I} = L_{-n_1} \dots L_{-n_k}$ is a monomial, can be written as $\mathcal{N}(\alpha_3, \alpha_2, \alpha_1)\, \mathcal{G}(I_3, I_2, I_1, \alpha_3, \alpha_2, \alpha_1)$, and $\mathcal{G}$ is defined completely by (CWI).

Then we have:

$$F_c(v_1 \otimes \cdots \otimes v_n) = \sum_I F_{\mathcal{C}_1}(v_1 \otimes v_m \otimes q_\gamma^{L_0} \mathcal{L}_{-I} v_{p_\gamma}) \times F_{\mathcal{C}_2}(\mathcal{L}_{-I}^\vee v_{p_\gamma} \otimes v_{m+1} \otimes \cdots \otimes v_n)$$

and what we're summing over are the elements of a basis of a representation with index $P_\gamma$, so that $\langle \mathcal{L}_I v_p, \mathcal{L}_{I'}^\vee v_p \rangle = \delta_{II'}$. Here we have taken a Riemann surface $C$ and cut it along $\gamma$, with our procedure as above.

**Kolya:** Why do you call this a "gluing construction"? I would call it a degeneracy construction. **Jorg:** Well, the difference between a hole and a puncture isn't that much. **Kolya:** But a puncture is an algebraic thing, and a hole is very different. It seems that what you are proving is a uniqueness theorem. **Jorg:** I see what you're saying. I should have a better answer. If you have a puncture, then the Hilbert space at that "boundary" consists of states actually placed at the punctures, and to get to a hole you'd have to do an infinite-time evolution, which cannot be done. But in this gluing procedure, I'm saying that the infinite-time evolution sort of cancels on each side.

So using this iteratively, getting down to the pants decomposition, you get a construction of $\mathcal{F}(P, b, \alpha, q)$ as a power series in $q_1, \ldots, q_{3g-3+n}$. The main issue in conformal field theory is to calculate the correct normalization factors $\mathcal{N}(\alpha_1, \alpha_2, \alpha_3)$. This cannot come just from (CWI).

To summarize, we have

$$\left\langle \prod e^{2\alpha_r \phi(z_r, \bar{z}_r)} \right\rangle = \int \mathrm{d}\mu(P)\, |\mathcal{F}|^2$$

$$\mathcal{F}() = \prod_t \mathcal{N}_t \sum_{\nu \in \mathbb{Z}^{3g+3+m}} \mathcal{G}_\nu(P, b, \alpha)\, q^{\Delta_{p_1} + n_1} \ldots q^{\Delta_{p_{3g+3+m}} + n_{3g+3+m}}$$

The extra physical requirements that we need to invoke are: (a) $\langle \ldots \rangle$ should be single-valued and real-analytic on $\mathcal{M}_{g,n}$, and (b) it should be independent of the pants decomposition. In physics literature, these are the *modular invariance* or *crossing symmetry*.

So far, our description is only valued in some asymptotic region of the moduli space, or in some cases in a neighborhood of the boundary where the series do converge. To make sense of the physical requirements, we need to continue all over the moduli space. And we need to compare the analytic continuations of the power series we write down in different places. For them to agree is highly nontrivial, and will depend sensitively on the normalization factors. Generically the analytic continuations are completely different.

Instead, modular invariance becomes possible due to the existence of relations like:

$$\mathcal{F}_{\sigma_2}(P_2, b, \alpha, q_2) = \int_{\mathbb{R}_+^{3g-3+m}} \mathrm{d}\mu(P_1)\, f_{\sigma_2,\sigma_1}(P_2, P_1, b, \alpha)\, \mathcal{F}_{\sigma_1}(P_1, b, \alpha, q_1) \tag{F}$$

And now you see this is very similar to the formulas from before, when we talked about the dependence of the quantization on the pair of pants decomposition. Ultimately I want to say that these formulae are all the same, because the $\mathcal{F}$s will be identified with the $\Theta$s. Of course, precise forms of these relations will depend on the normalization factors $\mathcal{N}$.

I will finish today by just flashing one result:

**Claim:** *There exists a canonical of $\mathcal{N}(\alpha_1, \alpha_2, \alpha_3)$ such that the integral transform (F) describes a unitary operator on $\mathrm{L}^2(\mathbb{R}_+^{3g-3+m})$ (with standard Lebesgue measure).*

This is a key result for many things. What it says is that there is on the space of conformal blocks a natural Hilbert space structure with respect to which these become unitary representations. This is the basis of a future harmonic analysis related to the Virasoro algebra.

# Friday, October 21

I begin by briefly recalling what we were discussing last time. It is a crash course in non-rational conformal field theroy, which is quite a bit more subtle than other field theories, but nevertheless is quite manageable. The main objects to understand are the *correlation functions*:

$$\left\langle \prod_{r=1}^{n} e^{2\alpha_r \phi()} \right\rangle_C = \int_{\mathbb{R}_+^{3g-3+m}} \mathrm{d}\mu(P) \big|\mathcal{F}(P, b, \alpha, q)\big|^2$$

$$\mathcal{F}(\dots) = \prod_t \mathcal{N}_t(P_3', P_2', P_1') \sum_{\nu \in \mathbb{Z}^{3g-3+m}} \mathcal{G}_\nu(P, b, \alpha)\, q_1^{\Delta_{P_1} + n_1} \cdots q_{3g-3+m}^{\Delta_{P_{3g-g}} + n}$$

The $\mathcal{F}$ are the *conformal blocks*, and have a power-series expansion as above. The $\mathcal{G}$ are uniquely defined by the conformal invariance. The $\mathcal{N}$ are constrained, but not determined, by modularity. See, we have our Riemann surface, which we decompose into *trinions*, i.e. pairs of pants, and the $t$ run over these trinions, and we parameterize the $r$th puncture as $\alpha_r = \frac{Q}{2} + iP_r$.

**Harold:** Can you explain more where the integral at the top comes from? **Jorg:** Physically: the Hilbert space is a direct integral. Mathematically, for each pair of pants we had a basis of conformal blocks, and these are related schematically by $\mathcal{F}_{\sigma_2}(P_2, q) = \int \mathrm{d}P_1\, F_{\sigma_2 \sigma_2}(P_2, P_1)\, \mathcal{F}_{\sigma_2}(P_1, q)$. If you are familiar with CFT, this is the story. If not, I will skip it.

So if we require moreover that the above expression can be real-analytically continued and single valued on all of Teichmüller space, then that constrains the data further.

We stopped last time with the following result:

**Claim:** *There exists a canonical choice of $|\mathcal{N}(P_3, P_2, P_1)|^2$ such that $\langle \dots \rangle$ is modular invariant.*

By "canonical" I mean that it doesn't depend on any choices we haven't already introduced. Namely, it is one of the kernels $F$, and those $F$s come from Virasoro algebra, i.e. the harmonic analysis of $\mathrm{Diff}(S^1)$.

## 2.1 Lioville = quantum $\mathcal{M}$

Recall that we were discussing two a priori completely unrelated topics: Quantization of $\mathcal{M}_0^{\mathbb{R}}$ on the one hand, and Lioville theory on the other hand. Then the relation is that $\Theta_\sigma(q,\lambda) = \mathcal{F}_\sigma(P,q)$, where $P = 2\pi b\lambda$, and choosing $\mathcal{N}(P_3,P_2,P_1) = |\mathcal{N}(P_3,P_2,P_1)|$. See, what's canonically determined is the norm of $\mathcal{N}$, and we decide to make $\mathcal{N}$ equal to its norm.

**Kevin:** Doesn't $\mathcal{N}$ depend on the cft we're working with? **Jorg:** Absolutely. See, $|\mathcal{N}(\dots)|^2 = \langle e^{2\alpha_3\phi}e^{2\alpha_2\phi}e^{2\alpha_1\phi}\rangle$. You have conformal blocks, but it's known that there can be multiple ways to combine them into a CFT. But there is one canonical choice: you get a CFT determined just by the chiral algebra.

**Kolya:** Is there a heuristic explanation of this somewhat mystical correspondence? **Jorg:** It is a very deep question. You can ask for a direct correspondence, and this is an open project. But here's an idea. You have a classical correspondence between moduli space of Riemann surfaces and Virasoro algebra (the algebra of vector fields). The nice way of proving this relation should be by proving some sort of "reduction commutes with quantization" theorem. Then you could prove that it is the same to classically reduce to moduli space, and then quantize, or to go the other way. If you can make this diagram commute, it should prove this theorem. **Kolya:** The path integral formulation of the theory is supposed to do this? **Jorg:** Some physicists would say that formally you can see this from the path integral. But for a long time, the status of Liouville as a qft was quite controversial, and that's why we needed to define it without that.

So we conclude, basically, a complete equivalence between Liouville theory and quantum $\mathcal{M}_0^{\mathbb{R}}$.

To make a whole story, we have been discussing not just the stratum $\mathcal{M}_0$ of holomorphic opers. We have also been discussing the stratum $\mathcal{M}_{3g-3+n}$, the stratum with the maximal number of apparent singularities, and then we used the coordinates of the apparent singularities $(\omega_k,\kappa_k)_{k=1,\dots,3g-3+n}$, and these were Darboux coordinates.

So we can ask: does Liouville theory say anything about this stratum? We consider $n$ primary fields as before, and add something extra:

$$\left\langle \prod_{r=1}^{n} e^{2\alpha_r\phi(z_r,\bar z_r)} \prod_{k=1}^{3g-3+n} e^{-b^{-1}\phi(w_k,\bar w_k)} \right\rangle = \sum_{\epsilon} \int \mathrm{d}\mu(P)\, \big|\mathcal{F}(P,\epsilon;b;\alpha;q,w)\big|^2$$

We here are taking some very special fields $\mathcal{F}$, and they are so special that they satisfy a second-order pde, and they do this because of Virasoro representation theory. For genus zero, the equations are:

$$\left( b^2\frac{\partial^2}{\partial w_k^2} + \sum_r \left( \frac{\Delta_r}{(w_k-z_r)^2} + \frac{1}{w_k-z_r}\frac{\partial}{\partial z_r}\right) + \sum_{k'\neq k}\left( \frac{-2+3b^{-2}}{4(w_k-w_{k'})^2} + \frac{1}{w_k-w_{k'}}\frac{\partial}{\partial w_{k'}}\right)\right)\mathcal{F} = 0$$

So from this, we see that $\mathcal{F}$ correspond to wave functions for quantized $\mathcal{M}_{\text{flat}}^{d=3g-3+n}$. **Ed:** What do you mean by "wave function"? Solution to some equation? Eigenfunction for some hamiltonian?

Element of some Hilbert space? **Jorg:** I mean, $\mathcal{F}(w,\dots) = \phi(w,\dots)$. Quantizing the conditions give constraints on local functions, and I mean that $\mathcal{F}$ satisfies these constraints.

## 3 (Quantum) Hitchin integrable system and Liouville theory

Recall, we proposed a magic diamond, with Hitchin at the top, quantum Hitchin at the right, flat connections at the left, and Liouville at the bottom. We've so far only talked about the SW edge of this diamond. We'd like to say something about the rest.

Consider *Higgs pairs* $(\mathcal{E},\Theta)$ on $C$, where $\mathcal{E}$ is a holomorphic vector bundle and $\Theta \in \mathrm{H}^0(C,\mathrm{End}(\mathcal{E})\otimes\mathcal{K})$. Let $\mathcal{M}_{\mathrm{H}}$ be the moduli space of such pairs, subject to the obvious gauge relations. Then we note that $\mathrm{H}^0(C,\mathrm{End}(\mathcal{E})\otimes\mathcal{K}) = \mathrm{T}^*\,\mathrm{Bun}_{C,\mathcal{E}}$, and this suggests a symplectic structure $\Omega_I$ on $\mathcal{M}_{\mathrm{H}}$.

The integrability: we expand

$$(\mathcal{E},\Theta) \longrightarrow \mathrm{tr}(\Theta^2) = \sum_{r=1}^{3g-3+n} \vartheta_r(y)\,H_r$$

Then Hitchin's theorem says:

1. $\{H_r,H_s\}_I = 0$
2. subspace defined by $H_r = E_r\ \forall r$, given a tuple $E = (E_1,\dots,E_{3g-3+n})$, are abelian varieties (tori) for generic $E$.

This gives the generic picture for algebraic integrable systems. We have a base $\mathcal{B}$, parameterized by $E_r$, and over each point we have a torus, which only degenerates at positive codimensional subspaces.

In case of punctures, we allow regular singularities $\Theta \sim \frac{\Theta'_r}{y-z_r}$. We will find it useful to assume that $\Theta'_r = \begin{pmatrix} j_r & 0 \\ p_r & -j_r \end{pmatrix}$ in some neighborhood $U_r \ni z_r$, and furthermore we will replace the gauge group by the positive Borel in that neighborhood.

For $g = 0$, we may write in $U_0$ that $\Theta_- = \sum_r \frac{\Theta_r}{y-z_r}$, where $\Theta_r = \begin{pmatrix} 1 & -x_r \\ & 1 \end{pmatrix}\begin{pmatrix} j_r & 0 \\ p_r & -j_r \end{pmatrix}\begin{pmatrix} 1 & x_r \\ & 1 \end{pmatrix} = \begin{pmatrix} j_r - x_r P_r & 2j_r x_r - x_r^2 p_r \\ p_r & -j_r + x_r p_r \end{pmatrix}$. These coordinates — $x$s are coordinates for $\mathrm{Bun}_G$, and $p$ for the cotangent direction, and indeed these coordinates $(x,p)$ are Darboux coordinates for $\mathcal{M}_{\mathrm{H}}$.

Then there are Hamiltonians for $\mathrm{tr}(\Theta^2) = \sum\bigl(\frac{-j_r^2}{(y-z_r)^2} + \frac{H_r}{y-z_r}\bigr)$, and using this we see that $H_r = \sum_{s\neq r} \frac{\Theta_r^a \Theta_s^a}{z_r - z_s}$, where $a$ is a summed index ranging over the basis of (2).

### 3.1 Separation of variables (following Sklyanin)

We consider the spectral curve $\Sigma = \{(v, y), v^2 - \operatorname{tr}\Theta^2 = 0\}$. This is a double cover of the curve we're working with, and it conveniently encodes the hamiltonians by letting $H_r$ by the moduli of $\Sigma$.

We then introduce new coordinates for $\mathcal{M}_{\mathrm{H}}$. Writing $\Theta = \left(\begin{smallmatrix}\Theta^0 & \Theta^+ \\ \Theta^- & -\Theta^0\end{smallmatrix}\right)$, we set

$$\Theta^- = \sum \frac{P_r}{y - z_r} = u \frac{\prod_{k=1}^{n-3}(y - w_k)}{\prod_r^n (y - z_r)}; \qquad \kappa_k = \Theta^0(w_k)$$

and taken together, one can show that $(w_k, \kappa_k)$ for $k = 1, \dots, n-3$ are a new system of Darboux coordinates for $\mathcal{M}_{\mathrm{H}}$.

Note: $\operatorname{tr}\Theta^0\big|_{y=0} = (\Theta^0)^2\big|_{y=w} = \kappa_k^2$, and so $\kappa_k^2 - \operatorname{tr}(\Theta(w_k)^2) = 0$. We will write $\operatorname{tr}(\Theta(w_k)) = t(y)\big|_{w_k} = \sum_r \left(\frac{-j_r}{(w_k - z_r)^2} + \frac{H_r}{w_k - z_r}\right)$.

### 3.2 Quantization

We turn now to the quantization of the (genus-zero) Hitchin. The first part is easy. We have Darboux coordiantes $(x_r, p_r)$, and we will make them noncommutative variables, and let's skip this step by simply making $p_r \rightsquigarrow \frac{\partial}{\partial x_r}$. Then the $H_r$s become differential operators $\hat{H}_r$, second order in $x_r$. We consider representations on wave-functions $\Psi(x, \bar{x})$ — there are different choices, there are different quantizations one can study — and we don't ask $\Psi$ to be holomorphic, but rather allow it to depend on both $x$ and $\bar{x}$, so we will also have $\frac{\partial}{\partial \bar{x}_r}$.

The aim is to solve the eigenvalue problem

$$H_r \Psi_E(x, \bar{x}) = E_r \Psi_E(x, \bar{x})$$

and the complex conjugate equation

$$\bar{H}_r \Psi_E(x, \bar{x}) = \bar{E}_r \Psi_E(x, \bar{x}).$$

As a first step, we do the quantum SOV. You should think of the above representation as $\mathrm{T}^* \mathrm{Bun}_G$, where the representation corresponds to the zero section of $\mathrm{Bun}_G$. And now we will do a Fourier transform and instead work where the variables on $\mathrm{T}^*$ are represented simply. So we consider:

$$\Phi(p, \bar{p}) = \prod_r |p_r|^{2j+2} \int \mathrm{d}x_r \mathrm{d}\bar{x}_r \, e^{p_r x_r - \bar{p}_r \bar{x}_r} \, \Psi(x, \bar{x})$$

Then we do a change of variables via

$$\Theta^- = \sum \frac{P_r}{y - z_r} = u \frac{\prod (y - w_r)}{\prod (y - z_r)}.$$

Then what Sklyanin shows is that the eigenvalues equations above correspond to

$$\left(\partial_{w_k}^2 + t(w_k)\right)\Phi(w,\bar w) = 0, \qquad t(y) = \sum \left(\frac{j_r(j_r+1)}{(y-z_r)^2} + \frac{H_r}{y-z_r}\right) \qquad \text{and complex conjugate}$$

And what Ed has nicely explained is that the separation of variables / Fourier transform exactly realizes the geometric Langlands correspondence for $\mathrm{Bun}_G$.

The separation of variables has done just that: we can consider each variable on its own, and we've turned a multi-dimensional problem into a one-dimensional one. This can then be solved in factorized form:

$$\Phi(w,\bar w) = \prod_k \chi(w_k, \bar w_k)$$

where $\chi(y,\bar y)$ solves $(\partial_y^2 + t(y))\chi(y,\bar y)$ and complex conjugate.

So this shows, I can for each collection of $H$s, solve this equation — well, there are two solutions — and combine them somehow and throw them into the Fourier transform. But what is not automatic are the *quantization conditions*. Not all solutions will be acceptable for a mathematician. Instead, it is natural to propose that $\Psi_E(x,\bar x)$ be single valued, and this happens it suffices for $\chi_E(x,\bar x)$ is single valued.

**Ed:** It's not even clear what integration means for non-single-valued functions. What you can try to do is to integrate over cycles valued in the dual local system, which might not exist, but you can try to do that. **Jorg:** Ah, that's a much better reason than the physical ones.

This can only be realized for a discrete subset $\mathcal{S} \subseteq \mathcal{B}$. Recall that $\mathcal{B}$ was the space of values $E = (E_1, \dots, E_r)$.

This suggests the following problem: describe $\mathcal{S} \subseteq \mathcal{B}$. Our proposal, which is a variant of Nekrasov–Shatashvili, is that there exists a function $\mathcal{Y}(\lambda, q)$, for the Yang–Yang function, such that:

$$E_r = \frac{\partial}{\partial q_r}\mathcal{Y}(\lambda,q)\Big|_{\lambda=\lambda_c}$$

where $\lambda_c$ ("$c$" for "critical") are solutions of:

$$\Im(\lambda_c) = 0$$
$$\frac{\partial}{\partial \lambda_r}\Re\big(\mathcal{Y}(\lambda,q)\big)\Big|_{\lambda=\lambda_c} = 2\pi n_r, \quad n_r \in \mathbb{Z}$$

This is a fairly precise proposal, and it's not clear, except through analogy with NS, where it comes from. In NS, they connect their problem to gauge theory, and then calculate the corresponding integrable system, by relating integrable systems to the Seiberg–Witten theory of the gauge theory. They don't have anything like our variables $x, p$ and the separation of variables that's present in our approach. Nevertheless they have a good argument, if you know gauge theory, and they can use it to predict exact results for certain models for which the spectrum was not known, and they give correct results. A post-doc of mine and myself have checked their proposal directly for Toda.

### 3.3 Relation to Liouville theory

The end of these lectures are supposed to describe how this proposal can be derived from — I shouldn't say "derived from", but "strongly hinted at by" — Liouville theory.

Consider something we've seen before:

$$\left\langle \prod_{r=1}^{n} e^{2\alpha_r \phi(z_r,\bar z_r)} \prod_{k=1}^{n-3} e^{-b\phi(w_k,\bar w_k)} \right\rangle$$

This is almost what we wrote when considering the other stratum, but there we had $b^{-1}$.

This is a solution to

$$\left[ b^2 \frac{\partial^2}{\partial w_k^2} + \sum_r \left( \frac{\Delta_r}{(w_k - z_r)^2} + \frac{1}{w_k - z_r}\frac{\partial}{\partial z_r} \right) + \sum_{k\neq k'} \left( \frac{b^2}{\dots} + \dots \right) \right] \langle \cdots \rangle = 0$$

And this shows that $\langle \cdots \rangle = \sum_\epsilon \int \mathrm{d}\mu(P)\, |\mathcal{F}(P,\epsilon,\dots,q)|^2$.

Then we can study this in the classical limit $b \to 0$, where the integral is peaked by a saddle point. One finds that in the limit we get an expression of the following form:

$$\langle \dots \rangle \sim \prod_{k=1}^{n-3} \chi(w_k,\bar w_k)\, e^{-\frac{1}{b^2} S(\lambda_s,q)},$$

where $\chi(y,\bar w)$ is a solution to

$$\big(\partial_y^2 + t(y)\big)\chi = 0, \qquad t(y) = \sum_r \left( \frac{\delta_r}{(y - z_r)^2} + \frac{H_r}{y - z_r} \right)$$

The point is that that $b^2$ terms drop out.

The $H_r$s are determined by

$$H_r = -\frac{\partial}{\partial q_r} S(\lambda, q),$$

and the $\lambda_s \in \mathbb{R}$ are determined by solutions to

$$\frac{\partial}{\partial \lambda_r} \Re\big(S(\lambda,q)\big)\Big|_{\lambda=\lambda_s} = 0.$$

So theses are the conditions in the proposal above.

Furthermore, the term $\prod_{k=1}^{n-3} \chi(w_k,\bar w_k)$ is simply an wave function $\phi(w,\bar w)$.

**Ed:** What is $\delta_r$? **Jorg:** I have dome some rescalings, and $\Delta_r = b^2\delta_r$. In the classical limit, since I'm considering conformal dimensions that diverge, the terms in the second line were dominated and go away, and the terms in the first line after multiplying by $b^2$ give these coefficients.

This suggests to identify $\mathcal{Y}(\lambda, q) \equiv S(\lambda, q)$, which was previously identified as the generating function for the change of variables $(\lambda, \tau) \to (q, H)$. This was the picture that led me to identify these theories independently of, and a little earlier, than Nekrasov and Shatashvili (who arrived at this identification using gauge theory arguments).

**Mina:** And other states? **Jorg:** So far, what I've analyzed is one distinguished state. It is very distinguished — you might call it ground state. But we'd like other states, of course. **Mina:** Using heavy branes? **Jorg:** Of course. **Ed:** One state where? **Jorg:** In the space of quantizations of Hitchin. Out of the space of opers, we are picking one point. **Ed:** These is the unique solution? **Jorg:** Using uniformization theory, I can show that this is the state determined by uniformization. But there are other solutions.

### 3.4 Conclusion and invitation

To conclude, we'll put on the blackboard the nice unified picture emerging from this. We'll split it into two pictures:

- A diamond with:

  **N** classical Hitchin (SOV)

  **W** meromorphic opers

  **E** quantum Hitchin (SOV)

  **S** Lioville

  This is "$\mathcal{M}_{\mathrm{H}}$ in separation of variables".

- **N** classical Hitchin, thought of as $\mathrm{T}^* \mathrm{Bun}_G$

  **W** flat connections $(\mathcal{E}, \nabla')$, $\partial_y + M(y)$.

  **E** quantum Hitchin, quantum $\mathrm{T}^* \mathrm{Bun}_G$.

  **S** WZW at the level $k$.

  In this second diagram, the arrow from WZW to quantum Hitchin is a "critical level limit" $k \to -h^\vee$.

The Lioville theory has a remarkable property, that it is self-dual. It is the same as the theory with $\epsilon_1, \epsilon_2$ exchanged. The SOV correspondence between Lioville and WZW — it has a counterpart on the quantum theory — it breaks the self-duality. So there is another diamond, where at the bottom instead we have WZW with level $\check{k}$. If I did this with an arbitrary $\mathfrak{g}$, then I would suppose to replace Liouville with $\mathfrak{g}$-Toda. Here I have $-\frac{1}{k+2} = b^2 = -\check{k} + 2$.

So the vision of how to get Langlands correspondence is: I start with Liouville, and on the one hand, I get the eigenfunctions for q-Hitchin. But on the other hand, I could use the Liouville self-duality, which corresponds to duality between $(\text{Toda})_{\mathfrak{g},k}$ and $(\text{Toda})_{{}^L\mathfrak{g},\check{k}}$. Then this picture says that exchanging the two $\epsilon$s is the same as replacing $\mathfrak{g}$ with ${}^L\mathfrak{g}$ in the Toda theory.

**Ed:** We know this in dimension 2. But the diamond in general shouldn't close for arbitrary $\mathfrak{g}$. **Jorg:** No, I should have two different classical Hitchins, related by T-duality. This is not a final answer, certainly, but an invitation to work in this direction.


theojf@math.berkeley.edu