Spaces of Quantum Systems

Theo Johnson-Freyd

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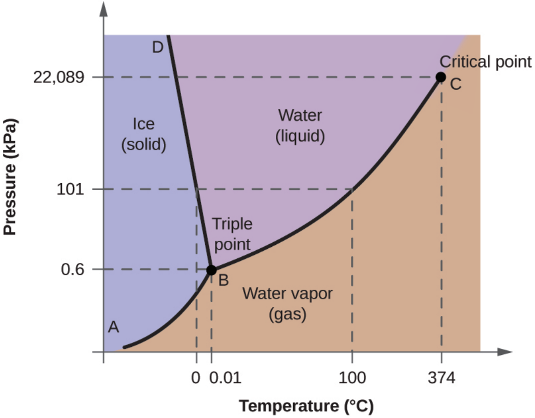
Thesis: Spaces of quantum systems, when correctly topologized, often have interesting homotopy types — often they are universal objects in higher algebra.

Outline:

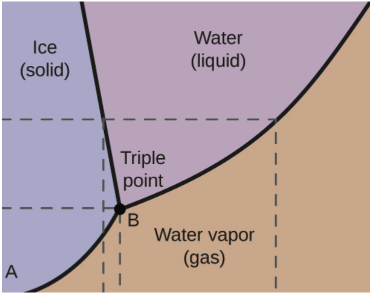
- Phases = homotopy
- Gapped topological matter
- Supersymmetric field theories

Phases = homotopy



The phase diagram for water.

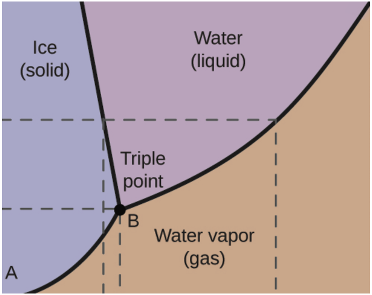
In grade school, we study the homotopy theory of spaces of physical systems. E.g.: {systems of water}.



In one topology,

$$\{\text{systems of water}\} \simeq *.$$

Indeed, it is perfectly reasonably to continuously vary Temperature and Pressure.



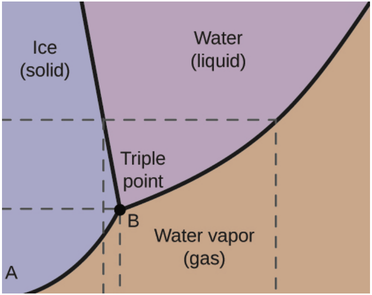
In a different topology, continuous paths cannot cross the phase transitions.

$$\pi_0\{\text{systems of water}\} = \{\text{phases of water}\} = \{\text{solid}, \text{liquid}, \text{gas}\}.$$

(The actual phase diagram is more complicated.)

At the phase transitions, the spectrum of the Hamiltonian undergoes a topological change: *gaps* between eigenvalues close and open. To classify phases of matter, we want a topology in which gaps cannot open and close.

Examples following this idea: ferromagnetic versus antiferromagnetic; conducting versus insulating; . . . .



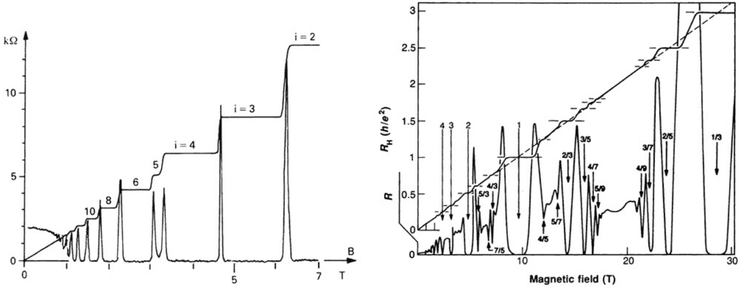
In the water example, there is no higher homotopy. In general, higher homotopy is also of physical interest.

If $\{\text{systems}\}$ *is topologized correctly:*

$$\pi_0 \operatorname{maps}(X, \{\text{systems}\}) = \{\text{phases of systems with an } X\text{-field}\},$$
$$\pi_0 \operatorname{maps}(BG, \{\text{systems}\}) = \{\text{phases of } G\text{-symmetric systems}\},$$

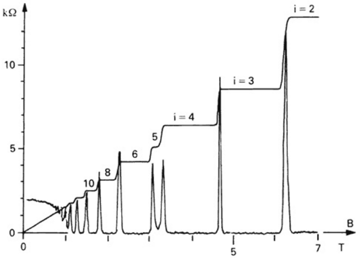
Example: Topological insulators are examples of G-symmetric systems, with $G = U(1) \times \mathbb{Z}_2^T$. They are in nontrivial in $\pi_0 \operatorname{maps}(BG, \{\text{systems}\})$ but trivial in $\pi_0\{\text{systems}\}$.

Gapped topological matter



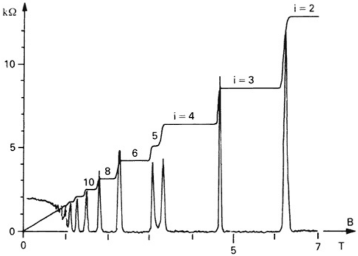
Hall resistance as a function of magnetic field. The plateaus show the IQHE (left) and FQHE (right).

In the late 20th century, more exotic *topological* phases were discovered, starting with the Integer and Fractional Quantum Hall Effects.



Goal: Classify *gapped topological phases of matter*. I.e. calculate the homotopy type of {gapped topological systems}.

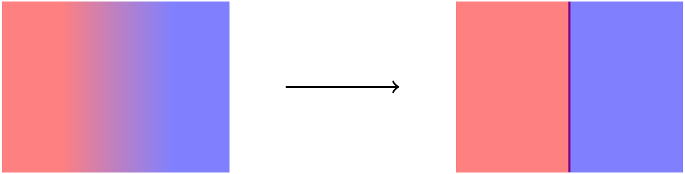
Physical definition: A phase of matter is *gapped topological* when there is a gap in the Hamiltonian, and the low-energy behaviour, i.e. the behaviour of the ground state, looks like a topological quantum field theory. For example, the ground state degeneracy should be independent of the geometry of space — it should depend only on the topology — and all excitations/operators/observables should have only topological dependence on their locations in spacetime.



Goal: Classify *gapped topological phases of matter.* I.e. calculate the homotopy type of {gapped topological systems}.

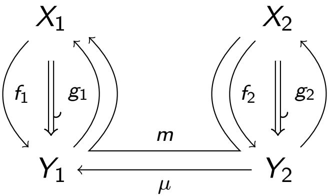
Warning: No complete definition of "gapped topological system." The problem is to make sense of "gapped" in a fully local way — in any finite region, the spectrum of the Hamiltonian depends on the boundary conditions.

Defining and topologizing this space will require a combination of analysis (e.g. PDE) and algebra (e.g. category theory).



Suppose you have two gapped topological phases X, Y and a way of "condensing" from X to Y. Run the procedure just on one half of the room. Now zoom out: you will produce an interface between X and Y.

Definition: The condensation procedure is *gapped topological* if the corresponding interface is.

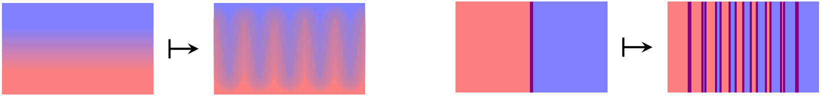


1-morphism in the Karoubi completion.

Theorem [1905.09566]: There is a *Karoubi completion* procedure for n-categories. The following n-categories are equivalent:
(1) The Karoubi completion of the n-categorical "one-point delooping" of $\mathbf{Vec}_{\mathbb{C}}$.
(2) The n-category of n-dimensional gapped topological phases that arise from the vacuum phase via gapped topological condensation.
(3) The n-category of fully dualizable $\mathbb{C}$-linear $(n-1)$-categories, i.e. the n-category of n-dimensional topological field theories.

Gapped topological phases may be *stacked* aka *layered*.

Definition: A phase X is *invertible* if it is invertible — up to gapped topological deformation — under stacking.

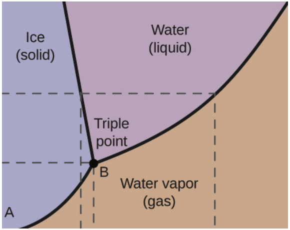


Some steps in verifying the Ω-spectrum axioms.

Physical Theorem [1712.07950]: The space of invertible gapped topological phases is naturally an Ω-spectrum $\mathcal{I}$, with spacetime dimension as the cohomological degree. If G is a group, then the set of *G-symmetry protected phases* is equal to $\widetilde{\mathrm{H}}^\bullet(BG;\mathcal{I})$, the (reduced) group cohomology of G with coefficients $\mathcal{I}$. Moreover, there are equivalences in low (conjecturally all) degrees:

$$\begin{aligned}\{\text{bosonic invertible phases}\} &\simeq I_{\mathbb{Z}}MSO\\ &= \{\text{oriented cobordism invariants}\}\\ \{\text{fermionic invertible phases}\} &\simeq I_{\mathbb{Z}}MSpin\\ &= \{\text{spin cobordism invariants}\}\end{aligned}$$

Supersymmetric field theories



If gaps may open and close,

$$\{\text{systems of water}\} \simeq *.$$

Conjecture: Contractibility always holds for bosonic systems, if gaps may open and close.

There is another source of topology: *supersymmetry*.

Example: The *index* of a supersymmetric quantum mechanics model, aka a *1D SQFT*, is the signed count

$$\#\{\text{bosonic ground states}\} - \#\{\text{fermionic group states}\}.$$

It is a deformation invariant because when a gap opens/closes, ground states are annihilated/created in pairs.

Theorem [folklore, c.f. Hohnhold–Stolz–Teichner]:
{SQM models} is naturally a symmetric ring Ω-spectrum. The cohomological degree is *not* the worldline dimension ($= 1$ for SQM models), but rather the *gravitational anomaly*. The spectrum structure comes from dynamicalizing (aka path integrating) a scalar susy multiplet. The index enhances to a *topological index*, which is an equivalence

$$\{\text{SQM models}\} \overset{\sim}{\to} KO = \text{oriented K-theory.}$$

Warning: Correctly defining and topologizing {SQM models} is not trivial, but it can be done rigorously in terms of Hilbert spaces and von Neumann algebras.

Expect the same in 2D. There is an *elliptic index*

$\{\text{2D SQFTs}\} \to MF_{\mathbb{Z}} =$ modular forms with integral q-expansion,

which is a deformation invariant.

Physical Theorem [1811.00589, 1902.10249, 1904.05788]: $\{\text{2D SQFTs}\}$ is naturally a symmetric ring Ω-spectrum. The cohomological degree is the gravitational anomaly $c_L - c_R$. The spectrum structure comes from dynamicalizing a scalar susy multiplet.

Warning: Defining and topologizing $\{\text{2D SQFTs}\}$ will be hard. It will require both analysis and category theory ("higher Hilbert spaces" and "higher von Neumann algebras").

Conjecture: The elliptic index enhances to a *topological elliptic index*, which is an equivalence

$$\{\text{2D SQFTs}\} \xrightarrow{\sim} \mathit{TMF} = \text{universal elliptic cohomology.}$$

Application: Predicts existence of SQFTs with certain indexes: for each $k \in \mathbb{N}$, there should exist a holomorphic SCFT with central charge $c = 12k$ and index $24/\gcd(k, 24)$. Predicts deformation invariants beyond the elliptic index.

Theorem [1811.00589]: The first prediction is true for $k \leq 5$.
Remark: "Holomorphic SCFTs" are completely well-defined in terms of vertex algebras.

Physical Theorem [1904.05788]: The second prediction is true.

Thank you!