Condensations and components

alternate title: In a fusion n-category, who should play the role of "the set of simples"?

UQSL 20. Feb 18, 2021. Theo Johnson-Freyd

These slides available at http://categorified.net/UQSL20.pdf.

Disclaimers:
- This talk will mostly focus on questions + conjectures.
- Much of this story is suggested in the works of Kong et al.
- Other parts are things I learned from David Reutter.
- I probably will run out of time.

A multi-fusion n-category is a monoidal n-category which is:

- "rigid": all $(<n)$-morphisms have adjoints.
  including objects; on both sides
- $\mathbb{C}$-linear and additive.
  i.e. contains $\oplus$s.
- Karoubi complete.
- "finite": strong finite-dimensionality conditions
  - vector spaces of n-mors are finite dim.
  - categories of $(n-1)$-mors are semisimple
  - etc.

Exactly how strong the conditions should be is not obvious.

If you drop monoidality (and adjoints for objects) you get "semisimple n-category"

<u>Conjecture:</u> Over $\mathbb{C}$, all reasonably strong finiteness conditions are equivalent

$e : X \circlearrowleft,\ e^2 = e.$ $\qquad$ $e = fg,\ gf = \mathrm{id}_Y$

A $\underset{n}{1}$-category is **Karoubi complete** if every idempotent has a splitting.

A condensation $X \twoheadrightarrow Y$ is $X \underset{f}{\overset{g}{\rightleftarrows}} Y$ & $\underbrace{gf \Rightarrow \mathrm{id}_Y}$.
categorifies "$gf = \mathrm{id}_Y$".

A condensation monad is $X \circlearrowleft e$ & $\underbrace{e^2 \Rightarrow e}$
categorifies "$e^2 = e$".
& (higher) associativity.

Thm [Gaiotto–JF]:

(1) $\mathrm{Kar}(\mathcal{C}) := \{$condensation monads, condensation bimodules$\}$ is the universal Kar-complete n-cat containing $\mathcal{C}$.

(2) If 1-mors in $\mathcal{C}$ have adjoints, then $\mathrm{Kar}(\mathcal{C}) \simeq \{$sep. unital monads, bimodules$\} =:$ (Douglas–Reutter 2-categorical Karoubi Completion.)

E.g: If $\mathcal{C}$ fusion, then $\Sigma\mathcal{C} := \mathrm{Kar}(B\mathcal{C}) = \left\{ \begin{matrix}\text{sep. alg.} \\ \text{objects}\end{matrix}, \begin{matrix}\text{sep. bim.} \\ \text{objects}\end{matrix} \right\} = \mathrm{Mod}(\mathcal{C})$

Pick a Kar-complete n-cat $\mathcal{X}$ and an object $X \in \mathcal{X}$.

Set $\Omega_X \mathcal{X} := \mathrm{End}_{\mathcal{X}}(X)$.

(1)
$$\begin{tikzcd}
\left\{\begin{array}{l}\text{pointed } (E_p) \\ \text{Kar-complete} \\ n\text{-cats}\end{array}\right\} \arrow[r, bend left, "\Omega"] \arrow[r, phantom, "\top"] & \left\{\begin{array}{l}\text{monoidal } (E_{p+1}) \\ \text{Kar-complete} \\ (n-1)\text{-cats}\end{array}\right\} \arrow[l, bend left, "\Sigma"]
\end{tikzcd}$$

(2) Counit of adjunction $\Omega\Sigma\mathcal{Y} \to \mathcal{Y}$ is equiv.

(3) Unit of adjunction $\Sigma\Omega\mathcal{X} \to \mathcal{X}$ is fully faithful.

Image of unit $= \{$Condensation descendants of $X\}$

$= \{$objects Y s.t. $\exists\ X \twoheadrightarrow Y\}$

In a 1-category, cond. descendants = direct summands.

In a (>1)-category, cond. descendants $\supsetneq$ direct summands.

↳ Kar-complete, additive, linear

An object $X \in \mathcal{X}$ is <u>indecomposable</u> if it is not a $\oplus$.

<u>Remark [Douglas-Reutter]</u>: X indec $\Leftrightarrow$ id_X indec.

If $\mathcal{X}$ semisimple, then X indec $\Leftrightarrow \dots \Leftrightarrow \underbrace{\Omega^n_X \mathcal{X}}_{\text{endo-}n\text{-morphisms of } X \in \mathcal{X}} = \mathbb{C}$

$\Leftrightarrow$ X is <u>simple</u>.

All the time, $\exists$ non-iso simple objects related by nonzero morphisms.

<u>Higher-categorical Schur's Lemma [Douglas-Reutter]</u>:

In a s.s. higher cat, the relation "$X \sim Y$ if $\hom(X,Y) \neq 0$" is an equivalence relation on simple objects.

<u>Pf</u>: If $f: X \to Y$ nonzero, combine w/ f^* to build $X \twoheadrightarrow Y$.

If $X \twoheadrightarrow Y \twoheadrightarrow Z$, then $X \twoheadrightarrow Z$, and condensations are nonzero.

Higher-categorical Schur's Lemma [Douglas-Reutter]:

In a s.s. higher cat, the relation "$X \sim Y$ if $\hom(X,Y) \neq 0$" is an equivalence relation on simple objects.

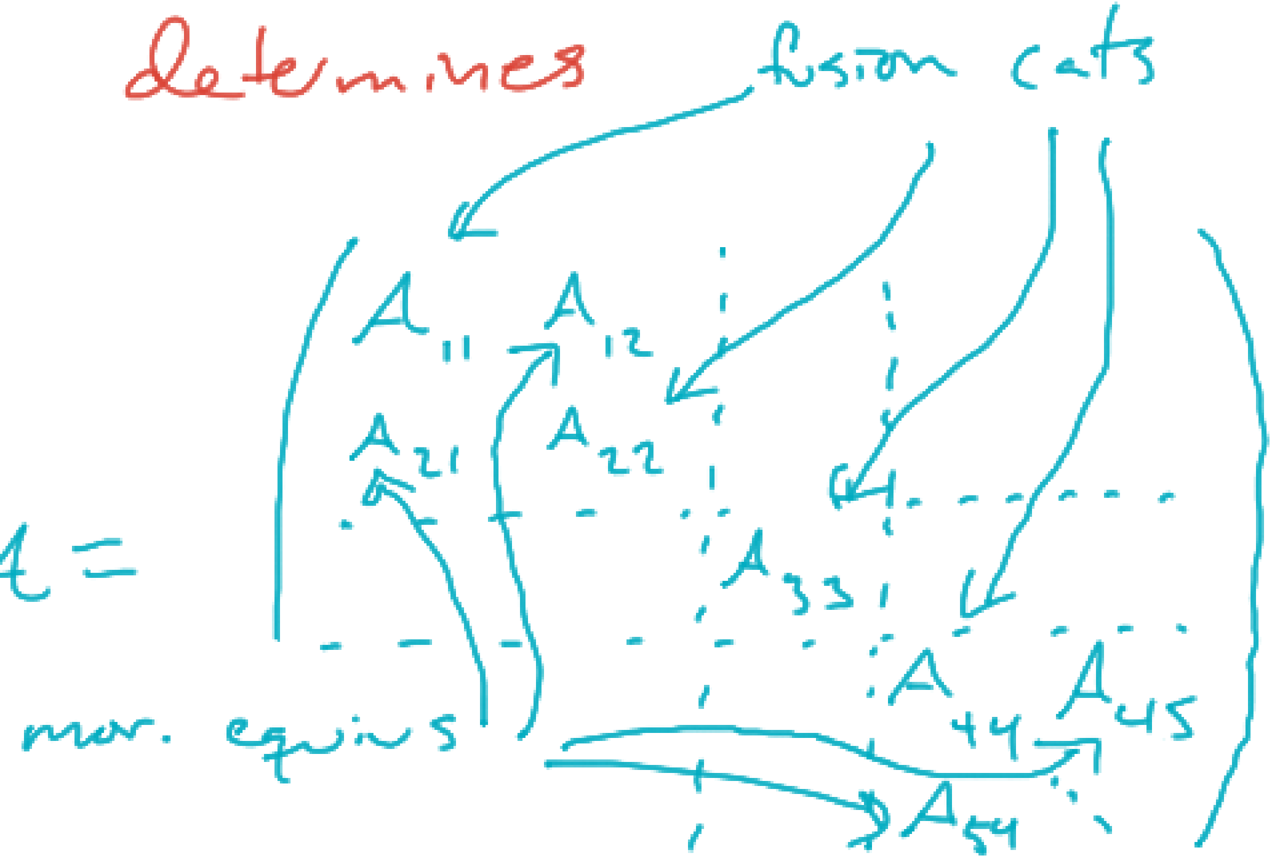
The components $\pi_0 \mathcal{C}$ of a semisimple n-category $\mathcal{C}$ are the equivalence classes of this relation.

All objects within a component are related by cond. descent.

Any object X within a component determines its component: it is $\Sigma \Omega_X \mathcal{C}$.

E.g.: If $\mathcal{A}$ is multifusion 1-cat, then $\pi_0 \Sigma \mathcal{A} =$ blocks of $\mathcal{A}$.

$\mathcal{A} =$



E.g.: If $\mathcal{C}$ is a fusion 1-cat, then $\pi_0 \Sigma \mathcal{C} = \{*\}$.

↑ simple $\mathbb{1}$

but simple objects = "quantum subgps".

E.g. of E.g.:

$$\Sigma \mathrm{Rep}(\mathbb{Z}/2) = \begin{tikzcd} \mathbb{1} \arrow[r, bend right, "\mathrm{Vec}"'] \arrow[loop below, "\mathrm{Rep}(\mathbb{Z}/2)"] & c \arrow[l, bend right, "\mathrm{Vec}"'] \arrow[loop below, "\mathrm{Vec}[\mathbb{Z}/2]"] \end{tikzcd}$$

$\mathrm{Rep}(\mathbb{Z}/2) = \{1, \text{electron}\}$

2 simple objects, one component.

Remark: $\mathrm{Rep}(\mathbb{Z}/2)$ is braided $\Rightarrow \Sigma \mathrm{Rep}(\mathbb{Z}/2)$ is fusion 2-cat.

For this $\otimes$ str, $c^2 \simeq c \oplus c = 2c$, whereas $\mathbb{1}^2 \simeq \mathbb{1}$.

Open question: If $\mathcal{A}$ is a fusion n-cat, does $\pi_0 \mathcal{A}$ have some "fusion coefficients"?

A multifusion n-cat $\mathcal{A}$ is fusion if $\mathbb{1}$ is simple, and connected fusion if $\pi_0 \mathcal{A} = \{*\}$.

$$\Omega : \left\{ \begin{matrix} \text{connected fusion} \\ n\text{-categories} \end{matrix} \right\} \xrightarrow{\sim} \left\{ \begin{matrix} \text{braided fusion} \\ (n-1)\text{-cats} \end{matrix} \right\} : \Sigma$$

Thm [JF]: This restricts to an equivalence

$$\left\{ \begin{matrix} \text{fusion } n\text{-cats} \\ \text{w/ trivial} \\ \text{Drinfeld} \\ \text{centre} \end{matrix} \right\} \simeq \left\{ \begin{matrix} \text{braided fusion} \\ (n-1)\text{-cats w/} \\ \text{nondeg. braiding} \end{matrix} \right\}$$

Slogan:

"$(n+1)$D top. orders"

=

"$(n-1)$-MTCs"

=

"Azumaya n-algebras"

In particular, if $Z(\mathcal{A}) = \text{trivial}$, then $\mathcal{A}$ is connected.

A fusion n-category $\mathcal{C}$ defines a "skein thy" in $(n+1)$D.

obj $\mathcal{C}$ = n-dim operators = 1-codim operators

obj $\Omega\mathcal{C}$ = $(n-1)$-dim operators = 2-codim operators

...

obj $\Omega^k\mathcal{C}$ = $(n-k)$-dim ops = $(k+1)$-codim ops.

...

a fusion 2-cat

E.g.: $\mathcal{Z}(\Sigma \mathrm{Rep}(\mathbb{Z}/2))$

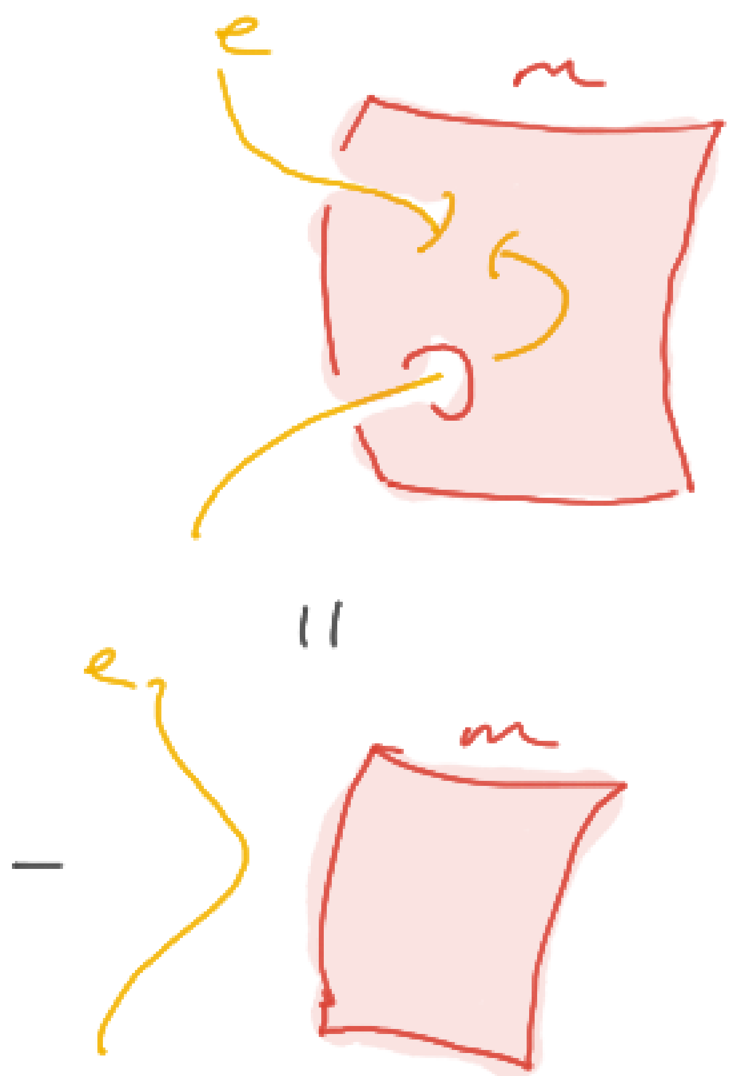
braided fusion 2-cat
= con. fusion 3-cat.

"magnetiz component"

= $\mathbb{1} \overset{\mathrm{Vec}}{\text{———}} c$

$1, e$ $\qquad$ $\mathrm{Vec}[\mathbb{Z}/2]$

m ——— $m \otimes c$



In general, a k-dim operator may be braided past a k-codim operator.
Morally, this gives a sort of "S-matrix"/pairing

$$\Omega^{n-k}\mathcal{C} \times \Omega^{k-1}\mathcal{C} \longrightarrow \text{"numbers"}$$

for any fusion n-cat $\mathcal{C}$.

The descendants of $\mathbb{1}$ are in the kernel of this pairing.

Open question / Conjecture: • Define an S-matrix

$$S^{(k)}: \pi_0 \Omega^{n-k}\mathcal{C} \times \pi_0 \Omega^{k-1}\mathcal{C} \longrightarrow \mathbb{C}.$$

• prove that if $\mathcal{C}$ is nondeg, then $S^{(k)}$ is nondeg (and in particular square!)

Special case: Let $\mathcal{B}$ be a braided fusion 1-cat with Müger centre $\mathcal{E}$. Then $Z(\Sigma\mathcal{B}) =: \mathcal{C}$ is a nondeg braided fusion 2-cat w/ $\Omega\mathcal{C} = \mathcal{E}$.

[Kong et al]
Classification $\Rightarrow$ $\pi_0\mathcal{C} = \{$c.c.s in gp reconstructed from $\mathcal{E}$ + (super) Tannakian duality$\}$

As a 2-cat,

$$\mathcal{C} \simeq \Sigma(\text{annular category of } \mathcal{B}).$$

Prove, w/o using classification, that

$$\left\{\begin{matrix}\text{blocks in the} \\ \text{annular cat}\end{matrix}\right\} = \left\{\begin{matrix}\text{c.c.s in the gp} \\ \text{reconstructed from } \mathcal{E}\end{matrix}\right\}.$$

Suppose $\mathcal{C}$ is symmetric monoidal. Then so is $\Sigma\mathcal{C}$, and so you can form a suspension tower $\Sigma^\bullet \mathcal{C}$.

"tower" = "loop spectrum of n-categories": [Scheimbauer]

a sequence $\mathcal{C}^\bullet$ where

* $\mathcal{C}^n$ is a (Kar-complete, additive, etc) pointed n-category
* $\mathcal{C}^{n-1} \simeq \Omega \mathcal{C}^n$ (data)

Then $\mathcal{C}^n$ is automatically sym monoidal.

Homotopy gps: Invertibles $(\mathcal{C}^\times)^\bullet = (\mathcal{C}^\bullet)^\times$ form a loop spectrum so you can ask for $\pi_{-n}(\mathcal{C}^\times) = \pi_0[(\mathcal{C}^n)^\times]$.

Homotopy sets: Or you can ask for $\pi_{-n}\mathcal{C} = \pi_0 \mathcal{C}^n$.

Conjecture [Freed–Hopkins]: There is a "universal" $\mathbb{R}$-linear tower $\mathcal{R}^\bullet$ s.t. $\mathcal{R}^\times \simeq \underbrace{I\mathbb{C}^\times}$ (equiv of loop spectra).

The "Pontryagin dual to spheres". Its characterizing property is that for any spectrum T,

$$\pi_0 \hom(T, I\mathbb{C}^\times) = \hom(\pi_0 T, \mathbb{C}^\times).$$

In particular, $\pi_{-n} I\mathbb{C}^\times = \hom(\underbrace{\pi_n^{\$}}, \mathbb{C}^\times)$.

It is not obvious that such a spectrum exists!

n^{th} stable homotopy gp of sheres, ie. $\lim_{k\to\infty} \pi_{n+k} S^k$.

Conjecture [JF]: The word "universal" means "separably closed": $\mathcal{R}^\bullet$ is semisimple, and for any semisimple tower $\mathcal{K}^\bullet$, $\exists\ \mathcal{K}^\bullet \to \mathcal{R}^\bullet$. (and $\mathcal{R}^\bullet$ is "etale contractible").

Examples:

$\pi_\bullet \mathcal{R}^\times$

- $\mathcal{R}^0 = \mathbb{C}$ [Fundamental thm of algebra] $\quad \mathbb{C}^\times$
- $\mathcal{R}^1 = \mathrm{SVec}_\mathbb{C}$ [Deligne's existence of super fibre functors] $\quad \{\mathbb{C}^{1|0}, \mathbb{C}^{0|1}\} = \mathbb{Z}/2$
- $\mathcal{R}^2 = \mathrm{SAlg}_\mathbb{C}$ [Hopkins - JF, unpublished] $\quad \{Cliff(0), Cliff(1)\} = \mathbb{Z}/2$
- $\mathcal{R}^3 \overset{?}{=}$ Freed - Scheimbauer - Teleman have an in-progress construction which probably builds this.

The unique map $\mathcal{R}^\times \to I\mathbb{C}^\times$ which is id in degree zero is an iso in degrees 1 and 2. This is because:

$\mathbb{C}^{0|1} \longleftrightarrow$ 1D spin TFT which sends $S^1_R \mapsto -1$.

$Cliff(1) \longleftrightarrow$ 2D spin TFT "$(-1)^{Arf}$".

Suppose you have constructed $\mathcal{R}^n$. How to build $\mathcal{R}^{n+1}$?

Start w/ $\Sigma\mathcal{R}^n$. Maybe add more components.

$$\mathrm{SVec}_{\mathbb{C}} = \Sigma\mathbb{C} \sqcup \text{fermionic component}$$

$$\mathrm{SAlg}_{\mathbb{C}} = \Sigma\,\mathrm{SVec}_{\mathbb{C}} \quad !$$

Conjecture [Hopkins–JF–Reutter]:

$$\pi_0\mathcal{R}^n \cong \hom(\pi_n O(\infty), \mathbb{C}^\times).$$

in particular,

$$\left\{\begin{matrix}\text{invertible}\\ \text{components}\end{matrix}\right\} = \begin{matrix}\text{Pontryagin dual to the}\\ \text{image of } J : \pi_n O(\infty) \to \pi_n^{\mathbb{S}}.\end{matrix}$$

$$\left\{\begin{matrix}\text{invertible cond.}\\ \text{descendants}\end{matrix}\right\} = \begin{matrix}\text{Pontryagin dual to the coker } J,\\ \text{aka ``the hard part'' of } \pi_n^{\mathbb{S}}.\end{matrix}$$