What the Hell is a Feynman Diagram?

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This note constitutes my prepared remarks for the eponymous talk I gave at Aarhus Universitet, IMF/CTQM, on 29 September 2009. Nothing in this note is particularly new — the talk was at an introductory seminar aimed at graduate students.

The goal of the talk is to introduce the notion of "Feynman Diagram" in a reasonably rigorous way, and to state some theorems proving that it is a good notion. I will organize the talk more-or-less via a "mathematician's history of mathematics," which is to say a false history, one that gives the impression that all ideas inevitably lead up to what we now know is the true and complete story. To quote Richard Feynman himself [4]:

> By the way, what I have just outlined is what I call a "physicist's history of physics," which is never correct. What I am telling you is a sort of conventionalized myth-story that the physicists tell to their students, and those students tell to their students, and is not necessarily related to the actual historical development, which I do not really know!

Thus, I will begin by describing why you might invent Feynman Diagrams. I'll then tell you about what the mathematicians have said about them. Time permitting, I'll finish with some speculation of my own.

1 Asymptotics for $\int_{\mathbb{R}^n}$

Feynman first invented his diagrams because he was trying to define an "integral" over an infinite-dimensional space, and he knew that such a space does not have an analytical Lebesgue measure. His earlier thesis work had suggested that he consider integrals of the form $\int \exp\big(\frac{i}{\hbar}f(x)\big)\,dx$, where x ranges over an infinite-dimensional space X, e.g. the space of paths in $\mathbb{R}^3$, or the space of vector-fields on $\mathbb{R}^4$, or.... So to figure out what the right answer should be — physicists never really compute anything; rather, they guess what the right answer should be and write it down and say "I computed this" when what they mean is "I defined this" — to figure out what the right answer should be, he looked at finite-dimensional integrals, where everything is analytically understood.

So, let's consider the integral $\int_{\mathbb{R}^n} \exp\big(\frac{i}{\hbar}f(x)\big)dx$, where dx is Lebesgue measure and $f : \mathbb{R}^n \to \mathbb{R}$ is smooth. We're most interested in the case when $\hbar$ is very small.

1 Lemma *The integral $\int_{\mathbb{R}^n} \exp\big(\frac{i}{\hbar}f(x)\big)dx$ converges conditionally if $f'(x)$ grows in all directions.*

2 Lemma *Let $\hbar$ be very small. Then $\int_{\mathbb{R}^n} \exp\big(\frac{i}{\hbar}f(x)\big)dx$ is dominated by small neighborhoods of critical points of f, where "small" depends on the value of $\hbar$.*

Let C be a compact subset of $\mathbb{R}^n$ that does not contain any critical points of f. Then

$$\lim_{\hbar\to 0}\int_C \exp\big(\frac{i}{\hbar}f(x)\big)dx = 0,$$

and in fact all asymptotics vanish, so that

$$\lim_{\hbar\to 0}\frac{\partial^n}{\partial\hbar^n}\int_C \exp\big(\frac{i}{\hbar}f(x)\big)dx = 0.$$

So, let's say that c is an isolated critical point of f, and $U \ni c$ a small neighborhood (say one that does not contain any other critical points). Then we can approximate f by its Taylor series (or at least by some Taylor polynomial, and we make U small enough that the approximation is good):

$$f(c+\xi) = \sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(c)\cdot\xi^{\otimes n} = \sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}_{i_1\dots i_n}(c)\,\xi^{i_1}\dots\xi^{i_n}$$

If we replace the word "small" by "formal", the above *defines* the function f on a formal neighborhood of c. The middle-hand-side is shorthand in which $f^{(n)}(c)$ is a symmetric n-linear functional $(T_cX)^{\otimes n}\to\mathbb{R}$ and $\cdot$ is the obvious pairing; the right-hand-side writes it in index notation.

Thus we are left with studying

$$\int_{\substack{\text{a small}\\\text{nbhd of } c}}\exp\left(\frac{i}{\hbar}f(x)\right)dx \approx \int_{\xi\in T_cX}\exp\left(\frac{i}{\hbar}\sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}_{i_1\dots i_n}(c)\,\xi^{i_1}\dots\xi^{i_n}\right)d\xi$$

$$= \exp\left(\frac{i}{\hbar}f(c)\right)\int_{\xi\in T_cX}\exp\left(\frac{i}{\hbar}\frac{1}{2}f^{(2)}_{i_1i_2}\xi^{i_1}\xi^{i_2}\right)\sum_{l=0}^{\infty}\frac{1}{l!}\left(\frac{i}{\hbar}\sum_{n=3}^{\infty}\frac{1}{n!}f^{(n)}_{i_1\dots i_n}(c)\,\xi^{i_1}\dots\xi^{i_n}\right)^l d\xi$$

Here we have used that c is a critical point to drop the linear term from the sum in the exponent. We have pulled the constant term out of the integral, and expanded exponential of the terms cubic and higher in Taylor series. This leaves us with a Gaussian integral.

But Gaussian integrals are easy. Recall that

$$\int_{\mathbb{R}}\exp\big(-\frac{1}{2}x^2\big)\,x^n\,dx = \begin{cases}\sqrt{2\pi}\,\frac{n!}{2^m\,m!}, & n=2m\text{ is even}\\ 0, & n\text{ is odd}\end{cases}$$

Easiest proof is to integrate by parts. More generally, we have the following (e.g. by diagonalizing a, changing coordiantes, and symmetrizing b):

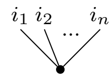
3 Lemma *Let a_{ij} be a positive-definite symmetric bilinear form, with inverse g^{ij}, so that $a_{ij}g^{jk} = g^{kj}a_{ji} = \delta_i^k$, and let $b_{i_1\dots i_n}$ be an arbitrary tensor. Define a* pairing *on the set $\{1,\dots,n\}$ to be an involutive assignment with no fixed points, so that if n is odd there are no pairings and if $n=2m$ then there are $n!/(2^m m!)$ pairings. The ordering on $\{1,\dots,n\}$ defines a standard representation of any given pairing π: the pairing breaks $\{1,\dots,n\}$ into blocks of size two, and we first sort each block alphabetically and then sort the blocks by first index. Thus, pairings correspond to permutations π with the properties that $p(2k-1)\le p(2l-1)$ if $1\le k\le l\le m=\frac{n}{2}$ and with $\pi(2k-1)\le\pi(2k)$ for $1\le k\le m$. Given this set-up, the Gaussian integral on the left-hand-side of the following assertion converges absolutely to the right-hand-side:*

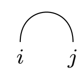
$$\int_{\mathbb{R}^N}\exp\left(-\frac{1}{2}a_{ij}\xi^i\xi^j\right)b_{i_1\cdots i_n}\xi^{i_1}\cdots\xi^{i_n}d\xi = (2\pi)^{N/2}\sqrt{\det g}\sum_{\text{pairings }p} b_{i_{p(1)}i_{p(2)}\cdots i_{p(n)}}g^{i_{p(1)}i_{p(2)}}\cdots g^{i_{p(n-1)}i_{p(n)}}$$

The determinant $\det g$ *is taken with respect to the volume form* $d\xi = |\,\mathrm{vol}_{j_1\dots j_N}|$ *(totally antisymmetric):* $\det g = g^{i_1j_1}\cdots g^{i_Nj_N}\,\mathrm{vol}_{i_1\dots i_N}\,\mathrm{vol}_{j_1\dots j_N}$.

The same formula works whenever a_{ij} is pure-imaginary (and symmetric nondegenerate), in which case the left-hand-side converges conditionally. Indeed, if a_{ij} is symmetric with positive-definite real part, then the convergence is absolute, and one can then take a limit to the pure-imaginary case.

We draw the tensor b as a vertex with n upward-pointing edges, and the bivector a^{-1} as an edge:

$b_{i_1\cdots i_n} =$ 

$(a_{ij})^{-1} =$ 

a pairing ($n = 4$): 

Each graph is a picture of how to contract the tensors. We will say more about "a picture of how to contract tensors" later; for now, I trust that you believe that you fully understand finite-dimensional vector spaces.

One can rewrite the Gaussian integral formula in the symmetric case as:

$$\int_{\mathbb{R}^N} \frac{1}{n!} b_{i_1\cdots i_n} x^{i_1}\cdots x^{i_n} \exp\left(-\frac{1}{2}a_{ij}x^i x^j\right) d^N x = \\ = \begin{cases} 0, & n \text{ odd} \\ \dfrac{\sqrt{\det(2\pi a^{-1})}}{2^k k!}\, b_{i_1\cdots i_n}(a^{-1})^{i_1 i_2}\cdots(a^{-1})^{i_{n-1}i_n}, & n = 2k \end{cases}$$

The $n!$ and $2^k k!$ terms count the number of symmetries of the corresponding diagrams. More generally, each summand in the expanded-out sum

$$\sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{i}{\hbar}\sum_{n=3}^{\infty}\frac{1}{n!} f^{(n)}_{i_1\cdots i_n}\Big|_c x^{i_1}\cdots x^{i_n}\right)^m = 1 + \frac{i}{\hbar}\left(\frac{1}{3!} f^{(3)}_{ijk}\Big|_c x^i x^j x^k + \frac{1}{4!} f^{(4)}_{ijkl}\Big|_c x^i x^j x^k x^l + \dots\right) \\ + \left(\frac{i}{\hbar}\right)^2 \left(\frac{1}{2!3!^2}\left(f^{(3)}_{ijk}\Big|_c x^i x^j x^k\right)^2 + \dots\right) \\ + \dots$$

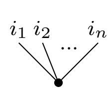
corresponds to some collection of vertices (the 1 corresponds to the empty collection), the power on $i\hbar$ counts the number of vertices, and the factorial prefactor counts the number of symmetries of each collection.

Thus, we are lead to the following description of the integral we are most interested in:

4 Theorem *Let f be a smooth function $\mathbb{R}^N \to \mathbb{R}$ such that f' grows in all directions. Assume that f has finitely many critical points, and that at each critical point $c \in \mathbb{R}^N$ the second derivative $f^{(2)}_{ij}\Big|_c$ is invertible. Then the following formula gives the correct asymptotics as $\hbar \to 0$:*

$$\int_{\mathbb{R}^N} \exp\left(\frac{-f(x)}{i\hbar}\right)\frac{d^N x}{(i\hbar 2\pi)^{N/2}} \approx \sum_{\substack{\text{critical}\\ \text{points } c}} e^{\frac{i}{\hbar}f(c)}\sqrt{\det\left((f^{(2)}|_c)^{-1}\right)} \sum_{\text{diagrams }\Gamma}\frac{(i\hbar)^{\chi(\Gamma)}\mathcal{F}(\Gamma)}{\operatorname{Aut}\Gamma}$$

The sum ranges over equivalence classes of graphs or "diagrams" Γ. We define $\chi(\Gamma)$ to be the number of edges minus the number of vertices, and Aut Γ *to be the number of symmetries of* Γ. *We evaluate $\mathcal{F}(\Gamma)$ via the following* Feynman rules*"*



$= -\, f^{(n)}_{i_1\cdots i_n}\Big|_c, \quad n \geq 3$

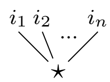
$= \left(f^{(2)}_{ij}\Big|_c\right)^{-1}$

The sign of the square root is chosen like this. Define the index $\eta(c)$ *at* c *to be the number of negative eigenvalues of* f*. Then we have* $\sqrt{\det\left((f^{(2)}|_c)^{-1}\right)} = i^{\eta(c)}\sqrt{\left|\det\left((f^{(2)}|_c)^{-1}\right)\right|}$.

More generally, if $g(x) : \mathbb{R}^N \to \mathbb{R}$ *is another smooth function that does not grow too quickly, then:*

$$\int_{\mathbb{R}^N} \frac{1}{i\hbar} g(x) \exp\left(\frac{-f(x)}{i\hbar}\right) \frac{d^N x}{(i\hbar 2\pi)^{N/2}} \approx \sum_c e^{\frac{i}{\hbar}f(c)} \sqrt{\det\left((f^{(2)}|_c)^{-1}\right)} \sum_\Gamma \frac{(i\hbar)^{\chi(\Gamma)}\mathcal{F}(\Gamma)}{\operatorname{Aut}\Gamma}$$

where now the sum ranges over diagrams with precisely one marked vertex, and we introduce the Feynman rule:



$$= g^{(n)}_{i_1\cdots i_n}\Big|_c, \quad n \text{ arbitrary}$$

We have written $\frac{i}{\hbar}f(x) = -f(x)/(i\hbar)$ to make the integrand look more like the positive-definite Gaussian case.

A few remarks are in order. First of all, the sum almost never converges. Indeed, try to use it for a cubic $f(x)$, and $N = 1$. Then a simple counting exercise shows that the sum of diagrams grows as

$$\sum_{n=0}^{\infty} \frac{(6n)!}{(2n)!\,(3n)!} z^n$$

where $z = i\hbar\alpha$, and α is a non-zero real number depending on the coefficients of f. But

$$\frac{(6n)!}{(2n)!\,(3n)!} \sim n!$$

and so the sum has zero radius of convergence.

Nevertheless, we can take $\hbar$ to be a formal variable, and then make sense of the right-hand-side as a formal power series. A *formal variable* is one for which all power series converge. The algebraic geometers say that a space is determined by the functions on it. So take your favorite smooth space (e.g. a smooth variety; "smooth" for me just means that you can differentiate functions), and take a point p in it. We define the *formal neighborhood of* p in terms of the functions on the neighborhood. Well, take all functions near p, and identify functions whose first k derivatives agree — this gives the *k-jet space at* p. Now take the inverse-limit of all the k-jet spaces; this gives the *∞-jet space at* p, and this space is the space of functions on the *formal neighborhood of* p. In the algebraic setting, what I've done is modeled my space near p as the polynomial ring $\mathbb{K}[x]$, and then completed it to the formal power series ring $\mathbb{K}[[x]]$.

(Warning: if your space is not affine, then jets do not transform as tensors. One can show that under change-of-coordinates, the right-hand-side transforms as the contraction of jets — i.e. it satisfies a u-substitution formula. Thus one can write a similar theorem for integrals over manifolds, but care must be taken to get the correct covariance.)

Thus, the right-hand-side *defines* the left-hand-side when $\hbar$ is formal. Indeed, (by analytic continuation, if you want), up to choosing a branch of the square root, the right-hand-side defines the left-hand-side even when f, g are complex, grow too fast, etc. Moreover, $\hbar$ is never without its i: we can just work with $i\hbar$ as the name of the variable.

Finally, let me say some words about what the physicists actually do. First of all, physicists interpret the diagrams as pictures of how particles can interact, because the integral on the left-hand-side for them is an integral over all possible histories of a collection of particles. Then they think of the function $g(x)$ as being an "observable", like a scattering amplitude for certain "incoming" and "outgoing" momenta. Well, say g is a monomial of degree k, and you take the vertex $\star$ and write it at ∞ (so the chalk board is now a sphere); then the picture looks like particles interacting. Now imagine varying g. Well, any given value of g when contracted with the diagram gives a number, and this is is linear in g. But g is a k-tensor, i.e. an element of $(\mathbb{R}^N)^{\otimes k}$, i.e. (a linear combination of) products of vectors, and instead of writing g we could just label all the external lines by their corresponding vectors. By varying those we get all the data we'd get by varying g in general (by linearity), and so the open diagrams really just give "scattering amplitudes".

Second, let's say that our space $\mathbb{R}^N$ naturally splits as two spaces $V \oplus W$, and for now let's assume that $f^{(2)}$ does not mix these, so that $f^{(2)} \cdot x^2 = f_V^{(2)} \cdot v^2 + f_W^{(2)} \cdot w^2$. So it is a block matrix, and its inverse is still in blocks. On the other hand, every vertex also splits, via $S^n(V \oplus W) \cong \bigoplus_{k=0}^n S^k V \otimes S^{n-k} W$. So we can just as well think of our diagrams as having edges colored by different vector spaces, and still understand the contractions. Also, everyone knows that a vector space is not isomorphic to its dual space, although our symmetric nondegenerate pairing $f^{(2)}$ defines such an isomorphism. But if $f^{(2)}$ has a block off-diagonal form $f^{(2)} = \left(\begin{smallmatrix} 0 & a \\ a^T & 0 \end{smallmatrix}\right)$, then we can think of $\mathbb{R}^N$ as $V \oplus V^*$ for some V, and write our edges with arrows.

Finally, like Feynman physicists are mostly interested in infinite-dimensional integrals, where all hell breaks loose. Indeed, if we replace $\mathbb{R}^N$ by an infinite-dimensional vector space V, then bilinear pairings like $f^{(2)}$ never have inverses. Or, anyway, not inverses that contract with arbitrary vertices; for example, the circle with one bilinear vertex should give the value $-\dim V$. Making sense of such ∞s occupies reams of the Feynman Diagram literature. Here's one situation in which these ∞s can be dealt with:

5 Theorem (JF [5]) *Consider mechanics on $\mathbb{R}^{d_+ + d_-}$, by which I mean choose a (constant) semi-Riemannian metric, an arbitrary (smooth) magnetic potential, and an arbitrary (smooth) electric potential. Provided that the classical mechanics is well-behaved (so that a Hamilton-Jacobi principal function exists), then we can use the right-hand-side of the integrals in theorem 4 to define a path integral (at least, an integral over all paths that are in formal neighborhoods of the classical paths) — every diagram in the sum converges. This path integral is a formal quantization of the classical mechanics, in that it gives a formal solution to Schrödinger's equation.*

Since this is not a physics talk, I won't elaborate on this result — it seems to be missing from the literature, but I will be posting a proof soon (I'm in the final-edit, spell-check stage of the writing). For the physicists in the audience, I'll mention that even in the case of mechanics on a manifold, infinities enter into the diagrams.

2 In categorical language

2.1 Individual diagrams

So far I haven't really said what is a Diagram, other than that it is a labeled picture that tells you how to contract tensors. In fact, mathematicians didn't really describe the diagrams rigorously until around 1990, largely, it seems, because of the difficulty in printing them. Credit usually goes to Penrose [9] for first using diagrams to describe tensor contractions — Penrose was interested in the representation theory of super-SL(2).

Anyway, as a good mathematician, one might wonder: what if I draw some pathological picture? Can I still interpret it? It seems obvious, but then you remember how hard the Jordan Curve Theorem is. Never fear: Joyal and Street [7] give the answer in the affirmative. Indeed, they say the following:

6 Definition *A (finite, generalized)* graph *is a Hausdorff space* Γ *with a finite subset* Γ_0 *such that* $\Gamma_1 = \Gamma \smallsetminus \Gamma_0$ *has finitely many connected components, each of which is a one-dimensional manifold. I.e. a (finite, generalized) graph is a finite one-dimensional CW complex.*

A graph Γ *is* polarized *if it has a height function* $f : \Gamma \to \mathbb{R}$ *that is monotonic on each connected component of* Γ_1*. This rules out self-edges and circles. Also, this defines at each vertex the "incoming" and "outgoing" edges, and we demand that at each vertex each such collection is totally ordered.*

A valuation *on a polarized graph is a labeling by a monoidal category, i.e. each edge is labeled by an object, and the vertices are labeled by morphisms between the correct tensor products.*

Then they prove:

7 Theorem (Joyal and Street) *If the category is symmetric, then any valuation allows us to evaluate the graph, and this evaluation is invariant under polarized isotopy.*

Thus, we can use Feynman diagrams to calculate with super-vector spaces, with representations of your favorite group, etc.

Other versions exist too — in fact Joyal and Street first proved the planar version:

8 Theorem

(Joyal and Street) *A graph* Γ *is* planar *if it is equipped with an embedding* $\Gamma \to \mathbb{R}^2$*, and if* Γ *is polarized we demand that the polarization match the projection onto the second component. Also, we demand that the ordering at each vertex matches the left-to-write ordering. Then we can evaluate in any monoidal category.*

(Joyal and Street) *Similar statement for graphs in* $\mathbb{R}^3$ *and braided categories.*

(Joyal and Street II.) *Similar statements for non-polarized graphs and categories with duals.*

(Reshetikhin and Turaev [10]) *Similar statements for framed graphs and ribbon categories.*

In all of these results, the correct statement has the following form: there is a category of colored ADJECTIVE diagrams, and it is the free ADJECTIVE category generated by its colors.

These results are useful for:

Computation Physicists, Penrose, Cvitanovič [2] (classifying simple Lie algebras and their representations), Jones and others (skein-theoretic descriptions of subfactors), Etingof and others (describing Quantum Groups, c.f. [3]; also [6])

Low-dimensional topology HOMFLY, Reshetikhin and Turaev, Spin Foam models for Quantum Gravity, TQFT.

In fact, theorem 8 suggests other types of categories, e.g. spherical categories (more or less: planar, but left trace and right trace are the same). Also suggests that we should define (strict) higher categories as the things which allow us to make similar statements with higher-dimensional CW complexes. (Compare [8].)

But so what? If all you want to do is contract tensors in finite-dimensional vector spaces, then maybe your response to all these theorems is that really you should have used a combinatorial interpretation all along. For example:

9 Definition *A* graph *is a finite set* Γ_0 *and a symmetric matrix* $\Gamma_1 : \Gamma_0 \times \Gamma_0 \to \mathbb{N}$.

This is OK, although it gets the wrong definition of "symmetry of a graph" without some tweaking. And perhaps we should have directed graphs:

10 Definition *A* directed graph *is a finite set* Γ_0 *and a matrix* $\Gamma_1 : \Gamma_0 \times \Gamma_0 \to \mathbb{N}$.

This is better, but it's still a little hard to figure out how to incorporate the labelings.

But I claim that you probably don't just want to contract finite-dimensional tensors. Indeed, perhaps you care about fermions; then you really need projections (with transverse crossings) of your graph to a plane, so that you can deal with signs. Or perhaps you care about infinite-dimensional spaces. Then spaces don't have duals, and tensor products start to become very interesting, so having polarized graphs is a prerequisite.

2.2 The sum

There are different definitions of diagram — CW complexes, finite sets with the right combinatorial data, etc. We mentioned the theorems that say that they have good interpretations. But what about the sum of diagrams?

We were a bit brief when we said "the sum over all diagrams". There are way too many CW complexes, for instance. Really we should sum over all equivalence classes of diagrams. But even this isn't *really* a set, although for any given Euler characteristic it is a finite collection. Also, what's with the dividing by symmetry factors?

An answer is given by Baez and Dolan [1]. They were trying to understand the following: Addition makes sense in terms of sets (to add two piles of rocks, push them together); subtraction makes sense in terms of sets (to subtract a pile from another pile, try to take some rocks out of the second pile until you get a third pile the size of the first); multiplication makes sense in terms of sets (put down a pile the same size as the first pile for each rock in the second pile, and push everything together). What about division? Well, if G is a finite group that acts freely on a set X, then the space of orbits has size X/G (this is not as strange as it sounds: we each take a rock so that they are all the same size, then each take another, etc.; the action should be a group action to make sure the division is fair, i.e. symmetrical, and should be free to make sure we don't end up trying to share a rock). What if the action is not free? For example, $\mathbb{Z}/2$ acts on $\{0, 1, 2, 3, 4\}$ by $j \mapsto 4 - j$; can we say that the quotient has $2\frac{1}{2}$ points in it?

In fact, we can, for the right kind of quotient. Indeed, we can construct a *groupoid* with objects $\{0, 1, 2, 3, 4\}$, identity morphisms, and a nontrivial morphism $i \to j$ if and only if the nontrivial element of $\mathbb{Z}/2$ maps i to j. It's clear ow to generalize this construction to define an "action groupoid" for any group action. Anyway, in our example, the groupoid we end up with is equivalent as a groupoid to the groupoid $\{0, 1, 2\}$ with only one nontrivial morphism, connecting 2 to itself.

Then Baez and Dolan give a definition of "cardinality of a groupoid" that is invariant under equivalence of groupoids and assigns the value of $2\frac{1}{2}$ to our example. (Their definition is only really good when the groupoid is equivalent to a finite groupoid.) It agrees with the earlier "Euler characteristic of an orbifold." His definition is precisely the one in our sum of diagrams:

11 Definition *Let* $\mathscr{G}$ *be a groupoid, and for each object* $x \in \mathscr{G}$*, write* $[x]$ *for its isomoprhism class, and* $\mathrm{Aut}(x)$ *for the set of morphisms* $x \to x$*. Then* $|\mathrm{Aut}(x)|$ *is constant on equivalence classes. Define* $|\mathscr{G}| = \sum |\mathrm{Aut}(x)|^{-1}$*, where the sum ranges over isomorphism classes in* $\mathscr{G}$*.*

The point is that a groupoid is determined up to equivalence by a collection of isomorphism classes and a choice of a group for each isomorphism class.

Then it's clear how to write down an "integral" over a groupoid $\mathscr{G}$. Let $f : \mathscr{G} \to \mathbb{R}$ be a *class function*, meaning that it is constant on isomorphism classes of $\mathscr{G}$. Then, at least when $\mathscr{G}$ is (equivalent to) a finite groupoid, define

$$\sum_{\mathscr{G}} f = \sum_{[x] \text{ an isomorphism class in } \mathscr{G}} \frac{f(x)}{|\operatorname{Aut} x|}$$

This is good enough for us, but I should mention Weinstein's results [11] extending this to a good notion of integration $\int_{\mathscr{G}}$ when $\mathscr{G}$ is (equivalent to) a compact Lie groupoid; as always happens when moving from sums to integrals, Weinstein needs extra data corresponding to a measure, and defines what the correct notion is.

Anyway, this means that it doesn't matter what definition of "diagram" you take. Whatever definition you use, there should be a natural notion of "isomorphism of diagrams". And as long as your groupoid of diagrams is equivalent as a category to my groupoid of diagrams, then our sums will agree.

2.3 Speculation

There are various other things I should say. For example, one can give a direct diagrammatic proof that the sum-of-diagrams definition of an integral satisfies the Fubini theorem, the u-substitution formula, etc. Then in any sufficiently nice (say: abelian, closed symmetric monoidal) category be able to come up with a notion of formal transcendental calculus. I think I know what to do when my category is $\mathbb{K}$-linear for $\mathbb{K}$ a field of characteristic 0 (and I have ideas about what to do in non-zero characteristic, but I'm not there yet).

Here's what I can do. A *polynomial* from V to W is a reasonably-well-understood thing. Indeed, the homogeneous polynomials of degree n are the symmetric maps $V^{\otimes n} \to W$, and a polynomial should be a (direct) sum of symmetric maps. Then a *formal function* should be an element of the adic completion of the ring of polynomials. More succinctly, there is a perfectly good space $\mathcal{S}^\bullet V$ — the "symmetric algebra generated by V" — and a formal function from V to W is a linear map $\mathcal{S}^\bullet V \to W$.

Let g be a formal function from V to W. Then a version of the sum-of-diagrams assigns a value to expressions like

$$\frac{\displaystyle\int_V \exp\left(-\frac{1}{2}a^{-1}v^2\right) g(v)\, dv}{\displaystyle\int_V \exp\left(-\frac{1}{2}a^{-1}v^2\right)\, dv}$$

where a ranges over $\mathcal{S}^2 V$; the expression gives a formal function from $\mathcal{S}^2 V$ to W. It's expressions like the above that are what physicists usually are most interested in calculating.

In the abstract setting, it's less clear to me how to define expressions like $\int_V \exp\big(\frac{-1}{\hbar} f(v)\big) dv$. Among other problems is that the notion of "inverse of a bilinear form" is not a linear map. If I have an individual bilinear map, then it's ok — I can tell what its inverse is. So I think I should restrict my attention to the "concrete" case, where I have a good notion of what an "element" of an object is. Let's write $\mathbb{K}$ for the unit object in our category ($\operatorname{Hom}(\mathbb{K},\mathbb{K})$ is an algebra, and in most cases is isomorphic to your ground field.) Then if $f : V \rightsquigarrow \mathbb{K}$ is a formal function which starts in degree 2, and with invertible second derivative at 0 — then up to defining $\sqrt{\det f^{(2)}}$ I know what to do, because I can just expand $\exp f$ in Taylor series. Better, if f is a polynomial, then I know how to evaluate $f(v)$ for elements $v \in V$, and then I can specify whether v is a critical point of f. Also, in the particular case of supervector spaces, everything is understood — if V is a finite-dimensional superspace, then $\mathcal{S}^\bullet V$ is finite dimensional.

But anyway, what should happen is that you should take all ways of getting from $\mathbb{K}$ to $\mathbb{K}$ by contracting components of f (well, components starting in degree 3, and the inverse to the degree-2 component), you should give it a groupoid structure, and then you should integrate over this groupoid. If you want to talk about $\int \exp(f)\, g$, then you should take all ways to get from $\mathbb{K}$ to W by contracting components of f with components of g.

Indeed, going backwards, let's say that f is a formal function from V to W and g is a formal function from W to X. What's to be made of the composition $g \circ f$? Well, you can use the Faa di Bruno formula to calculate the derivatives of $g \circ f$ when f and g are smooth, and it agrees with the naive power-series composition, of course, but actually the Faa di Bruno formula has a good combinatorial interpretation: $g \circ f$ consists of all ways to get from V to X by contracting a components of f with a component of g.

Finally, let's say that you extend your Feynman rules, and write a dot for $-f^{(0)}(c)$. Then $\chi(\bullet) = -1$, and

$$\exp\left(\frac{-1}{i\hbar} f(c)\right) = \sum_{\Gamma = \bullet \cdots \bullet} \frac{(i\hbar)^{\chi(\Gamma)} \mathcal{F}(\Gamma)}{\operatorname{Aut}\Gamma}.$$

I would like a way to understand the determinant as arising from allowing bivalent vertices into the diagrams. I almost can, but there are a few issues.

The start, of course, is to write $\det = \exp \operatorname{tr} \log$, and to think about the log of the whole expression. Well,

$$\log\left(\sum_{\Gamma} \frac{(i\hbar)^{\chi(\Gamma)} \mathcal{F}(\Gamma)}{\operatorname{Aut}\Gamma}\right) = \sum_{\Gamma \text{ connected}} \frac{(i\hbar)^{\chi(\Gamma)} \mathcal{F}(\Gamma)}{\operatorname{Aut}\Gamma}$$

which is immediate (we used the opposite relation when interpreting $\exp(\sum_{n=3}^{\infty} f^{(n)} x^n/n!)$ as a sum of vertices). So it's tempting to consider an expression like

$$\sum_{\Gamma \text{ connected}} \frac{(i\hbar)^{\chi(\Gamma)} \mathcal{F}(\Gamma)}{\operatorname{Aut}\Gamma}$$

except to allow arbitrary vertices, and try to use a relation like

$$\frac{1}{1-x} = 1 + x + x^2 + \dots$$

to get the $(f^{(2)})^{-1}$ out of the bivalent vertex, but I'm not sure how. The $2n$ in:

$$\log\sqrt{\det(1-A)} = \operatorname{tr}\sum_{n=1}^{\infty} \frac{A^n}{2n}$$

is very attractive, though — $2n$ being the number of symmetries of a necklace of length n. This is the type of tool used in proving Fubini and u-sub formulas, and showing that the integral is well-behaved under infinitesimal perturbations of the parameters. But getting all the way to the whole integral I don't know how to do.

Anyway, I think I'll stop here.

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